Twelfth Homework Set — Solutions

Chapter 7

Problem 39 We have

$$\begin{aligned}
\mathrm{Cov}\,(Y_n, Y_n) &= \mathrm{Var}\,(Y_n) = 3\sigma^2, \\
\mathrm{Cov}\,(Y_n, Y_{n+1}) &= \mathrm{Cov}\,(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\
&= \mathrm{Cov}\,(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) \\
&= \mathrm{Var}\,(X_{n+1} + X_{n+2}) = 2\sigma^2, \\
\mathrm{Cov}\,(Y_n, Y_{n+2}) &= \mathrm{Cov}\,(X_n + X_{n+1} + X_{n+2}, X_{n+2} + X_{n+3} + X_{n+4}) \\
&= \mathrm{Cov}\,(X_{n+2}, X_{n+2}) = \mathrm{Var}\,(X_{n+2}) = \sigma^2, \quad \text{and} \\
\mathrm{Cov}\,(Y_n, Y_{n+j}) &= 0 \quad \text{if } j \geq 3.
\end{aligned}$$

Problem 41 the number of carp is a hypergeometric random variable, so that we have

$$E[X] = \frac{20 \cdot 30}{100} = 6,$$

and

$$\mathrm{Var}\,(X) = \frac{20 \cdot 80}{99} \cdot \frac{3}{10} \cdot \frac{7}{10} = \frac{112}{33}.$$

Problem 42 (a) Let X_i be one if the i-th pair consists of a man and a women, and zero otherwise. Then the sum $X_1 + \cdots + X_{10}$ is the number of pairs that consist of a man and a woman.
We have $E[X_i] = P\{X_i = 1\} = 2 \cdot \frac{10 \cdot 10}{20 \cdot 19} = \frac{10}{19}$, so that

$$E[X_1 + \cdots + X_{10}] = \frac{100}{19}.$$

Now, we have $\mathrm{Var}\,(X_i) = E[X_i^2] - E[X_i]^2 = \frac{10}{19} - \frac{100}{361} = \frac{90}{361}$, and $\mathrm{Cov}\,(X_i, X_j) = E[X_i X_j] - E[X_i]\,E[X_j] = \frac{10}{19} \cdot \frac{9}{17} - \frac{100}{361} = \frac{10}{6137}$ if $i \neq j$, so that

$$\mathrm{Var}\,(X_1 + \cdots + X_{10}) = \frac{900}{361} + 10 \cdot 9 \cdot \frac{10}{6137} = \frac{16200}{6137} = 2.6397.$$

(b) Let Y_i be one if the i-th couple are paired together. $E[Y_i] = P\{Y_i = 1\} = \frac{2 \cdot 10 \cdot 18!}{20!} = \frac{1}{19}$, so that

$$E[Y_1 + \cdots + Y_{10}] = \frac{10}{19}.$$

We have $\mathrm{Var}\,(Y_i) = E\left[Y_i^2\right] - E\left[Y_i\right]^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E\left[Y_iY_j\right] = \frac{8\binom{10}{2}\cdot 16!}{20!} = \frac{1}{323}$, so that $\mathrm{Cov}\,(Y_i, Y_j) = \frac{1}{323} - \frac{1}{361} = \frac{2}{6137}$, so that

$$\mathrm{Var}\,(Y_1 + \cdots + Y_{10}) = \frac{180}{361} + 90 \cdot \frac{2}{6137} = \frac{3240}{6137}.$$

Problem 50 We have

$$f_Y y = \int_0^\infty \frac{e^{-\frac{x}{y}} - y}{y} dx = e^{-y}$$

for $y > 0$, so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{e^{-\frac{x}{y}}}{y} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Now, we have

$$E\left[X^2|Y\right] = \int_0^\infty \frac{x^2}{y} e^{-\frac{x}{y}} dx = 2y^2.$$

Problem 51 We have

$$f_Y(y) = \int_0^y \frac{e^{-y}}{y} dx = e^{-y},$$

so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0, y) \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that

$$E\left[X^3|Y = y\right] = \int_0^y \frac{x^3}{y} dx = \frac{y^3}{4}.$$

Problem 56 Let Y_i be one if the elevator stops at the i-th floor, for $i = 1, \ldots, N$. Let $Y = Y_1 + \cdots + Y_{10}$. Let X be the number of passengers, i.e., X is Poisson with parameter 10. We have $E\left[Y_i = 1|X = k\right] = 1 - \left(\frac{N-1}{N}\right)^k$, so that

$$E\left[Y|X = k\right] = N\left(1 - \left(\frac{N-1}{N}\right)^k\right).$$

We have

$$\begin{aligned} E[Y] &= E[E[Y|X]] = E\left[N\left(1-\left(\frac{N-1}{N}\right)^k\right)\right] \\ &= N - N\sum_{k=0}^{\infty}\left(\frac{N-1}{N}\right)^k \frac{10^k}{k!}e^{-10} \\ &= N(1-e^{-\frac{10}{N}}). \end{aligned}$$

Problem 57 By Example 4d in Section 7.4, we have

$$E\left[\sum_{i=1}^{N} X_i\right] = E[N]\,E[X_1] = 12.5.$$