Tenth Homework Set — Solutions

Chapter 6

Problem 2 (a)

$P\{X_1 = i, X_2 = j\}$	$j = 0$	$j = 1$	$P\{X_1 = i\}$
$i = 0$	$\frac{8}{13}\frac{7}{12} = \frac{14}{39}$	$\frac{8}{13}\frac{5}{12} = \frac{10}{39}$	$\frac{24}{39}$
$i = 1$	$\frac{5}{13}\frac{8}{12} = \frac{10}{39}$	$\frac{5}{13}\frac{4}{12} = \frac{5}{39}$	$\frac{15}{39}$
$P\{X_2 = j\}$	$\frac{24}{39}$	$\frac{15}{39}$	1

(b)

$$P\{X_1 = 0, X_2 = 0, X_3 = 0\} = \frac{8}{13}\frac{7}{12}\frac{6}{11} = \frac{28}{143}$$
$$P\{X_1 = 0, X_2 = 0, X_3 = 1\} = \frac{8}{13}\frac{7}{12}\frac{5}{11} = \frac{70}{429}$$
$$P\{X_1 = 0, X_2 = 1, X_3 = 0\} = \frac{8}{13}\frac{5}{12}\frac{7}{11} = \frac{70}{429}$$
$$P\{X_1 = 1, X_2 = 0, X_3 = 0\} = \frac{5}{13}\frac{8}{12}\frac{7}{11} = \frac{70}{429}$$
$$P\{X_1 = 0, X_2 = 1, X_3 = 1\} = \frac{8}{13}\frac{5}{12}\frac{4}{11} = \frac{40}{429}$$
$$P\{X_1 = 1, X_2 = 0, X_3 = 1\} = \frac{5}{13}\frac{8}{12}\frac{4}{11} = \frac{40}{429}$$
$$P\{X_1 = 1, X_2 = 1, X_3 = 0\} = \frac{5}{13}\frac{4}{12}\frac{8}{11} = \frac{40}{429}$$
$$P\{X_1 = 1, X_2 = 1, X_3 = 1\} = \frac{5}{13}\frac{4}{12}\frac{3}{11} = \frac{5}{143}$$

Problem 7 $P\{X_1 = i, X_2 = j\} = p^2(1-p)^{i+j}$

Problem 8 X, Y are jointly continuous with probability density function

$$f(x,y) = \begin{cases} c(y^2 - x^2)e^{-y} & -y \le x \le y, 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

(a) Note that

$$\int\int_{R^2} f(x,y) = \int_0^\infty \int_{-y}^{y} c(y^2 - x^2)e^{-y}dxdy = 8c,$$

so that $c = \frac{1}{8}$.

(b)

$$f_X(x) = \frac{1}{8}\int_{|x|}^{\infty}(y^2 - x^2)e^{-y}dy = \frac{(|x|+1)e^{-|x|}}{4}$$

$$f_Y(y) = \frac{1}{8}\int_{-y}^{y}(y^2 - x^2)e^{-y}dx = \frac{1}{6}y^3e^{-y} \qquad \text{for } y > 0$$

(c) $E[X] = 0$ by symmetry.

Problem 9 Let X, Y be jointly continuous with joint density function $f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$ for $0 < x < 1$, $0 < y < 2$.

(a)

$$\int_0^1\int_0^2 x^2 + \frac{xy}{2}dydx = \int_0^1 2x^2 + xdx = \frac{7}{6}$$

(b)

$$f_X(x) = \frac{6}{7}x(2x+1) \qquad \text{for } 0 < x < 1$$

(c)

$$P\{X > Y\} = \int_0^1\int_0^x f(x,y)dydx = \frac{15}{56}$$

(d)

$$P\left\{Y > \frac{1}{2}|X < \frac{1}{2}\right\} = \frac{P\left\{X < \frac{1}{2}, Y > \frac{1}{2}\right\}}{P\left\{X < \frac{1}{2}\right\}} = \frac{\int_{\frac{1}{2}}^2\int_0^{\frac{1}{2}} f(x,y)dxdy}{\int_0^{\frac{1}{2}} f_X(x)dx} = 0.8625$$

(e)

$$E[X] = \int_0^1 xf_X(x)dx = \frac{5}{7}$$

(f)

$$E[Y] = \int_0^2 y\int_0^1 f(x,y)dxdy = \frac{8}{7}$$

Problem 10 Let X, Y be jointly distributed with density function $f(x,y) = e^{-(x+y)}$ for $0 \le x < \infty$, $0 \le y < \infty$.

(a) $P\{X < Y\} = \frac{1}{2}$ by symmetry

(b) $P\{X < a\} = \int_0^a \int_0^\infty e^{-(x+y)} dydx = 1 - e^{-a}$

Problem 11 Let A be the number of people buying an ordinary set, B the number of people buying a plasma set, and C the number of people who are just browsing. Then $P\{A = 2, B = 1, C = 2\} = \frac{5!}{2!1!2!} 0.45^2 \cdot 0.15 \cdot 0.4^2 = 0.1458$.

Problem 13 Let X be uniform on $(-15, 15)$, and let Y be uniform on $(-30, 30)$. Nobody waits longer than five minutes if $|Y - X| < 5$.

$$\begin{aligned} P\{|Y - X| < 5\} &= P\{-5 < Y - X < 5\} \\ &= P\{X - 5 < Y < X + 5\} \\ &= \int_{-15}^{15} \int_{x-5}^{x+5} \frac{1}{30 \cdot 60} dydx \\ &= \frac{30 \cdot 10}{30 \cdot 60} = \frac{1}{6}. \end{aligned}$$

The probability that the man arrives first is $P\{X < Y\} = \frac{1}{2}$ by symmetry.

Problem 14 Let X, Y be uniform random variables on $(0, L)$. Let $Z = |Y - X|$. We want to find $E[Z]$. First, find $F_Z(a)$, for $a \geq 0$. We have $F_Z(a) = P\{Z \leq a\} = P\{|Y - X| \leq a\} = P\{-a \leq Y - X \leq a\} = \frac{2aL - a^2}{L^2}$. using geometric considerations. Hence, $f_Z(x) = \frac{2L - 2x}{L^2}$ if $0 \leq a \leq L$. Hence,

$$\begin{aligned} E[Z] &= \int_0^L x \cdot \frac{2L - 2x}{L^2} dx \\ &= \frac{2}{L^2} \left(\frac{Lx^2}{2} - \frac{x^3}{3} \right) |_0^L \\ &= \frac{L}{3}. \end{aligned}$$

Problem 18 Let X be uniform on $(0, \frac{L}{2})$ and let Y be uniform on $(\frac{L}{2}, L)$. We want to find $P\left\{Y - X > \frac{L}{3}\right\}$.

$$P\left\{Y - X > \frac{L}{3}\right\} = P\left\{Y < \frac{L}{2} + \frac{L}{3}, X < Y - \frac{L}{3}\right\} + P\left\{Y > \frac{L}{2} + \frac{L}{3}\right\}$$
$$= \int_{\frac{L}{2}}^{\frac{5L}{6}} \int_0^{y-\frac{L}{3}} \frac{4}{L^2} dx dy + \int_{\frac{5L}{6}}^{\frac{L}{2}} \frac{2}{L} dy$$
$$= \frac{4}{9} + \frac{1}{3} = \frac{7}{9}.$$

Problem 20 If the joint density function of X and Y is

$$f(x,y) = \begin{cases} xe^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise,} \end{cases}$$

then $f(x,y) = f_X(x) f_Y(y)$, where $f_X(x) = xe^{-x}$ for $x > 0$, and $f_Y(y) = e^{-y}$ for $y > 0$ (0 otherwise), so that X and Y are independent.

If

$$f(x,y) = \begin{cases} 2 & 0 < x < y, 0 < y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

then X and Y are not independent because the nonzero values of f are located in a triangular domain.

Problem 21 (a) Check: $\int_0^1 \int_0^{1-y} 24xy dx dy = \int_0^1 12(1-y)^2 y dy = 12 \int_0^1 y - 2y^2 + y^3 dy = 6y^2 - 8y^3 + 3y^4 |_0^1 = 6 - 8 + 3 = 1$.

(b) First, find $f_X(x) = \int_0^{1-x} 24xy dy = 12x(1-x)^2$. Now, $E[X] = \int_0^1 12x^2(1-x)^2 dx = 4x^2 - 6x^3 + \frac{12}{5}x^5 |_0^1 = 4 - 6 + \frac{12}{5} = \frac{2}{5}$.

(c) $E[Y] = E[X] = \frac{2}{5}$ by symmetry.

Problem 22 Let X and Y be jointly continuous with density function

$$f(x,y) = \begin{cases} x + y & 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) X and Y are not independent, since $f(x,y)$ is clearly not a product of functions of x and y.

(b) $f_X(x) = \int_0^1 x + y dy = x + \frac{y^2}{2} |_0^1 = x + \frac{1}{2}$.

(c) $P\{X+Y<1\} = \int_0^1 \int_0^{1-y} x + y dx dy = \int_0^1 \frac{(1-y)^2}{2} + y(1-y) dy = \frac{1}{2}\int_0^1 1 - y^2 dy = \frac{1}{2}(1-\frac{1}{3}) = \frac{1}{3}$.

Problem 23 Let X and Y be jointly distributed with density function

$$f(x,y) = \begin{cases} 12xy(1-x) & 0<x<1, 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

First, compute $f_X(x) = \int_0^1 12xy(1-x)dy = 6x(1-x)$ and $f_Y(y) = \int_0^1 12xy(1-x)dy = 2y$.

(a) Clearly, $f(x,y) = f_X(x)f_Y(y)$, so that X and Y are independent.

(b) $E[X] = \int_0^1 6x^2(1-x)dx = 2x^3 - \frac{3}{2}x^4|_0^1 = \frac{1}{2}$.

(c) $E[Y] = \int_0^1 2y^2 dy = \frac{2}{3}y^3|_0^1 = \frac{2}{3}$.

(d) First, find $E[X^2] = \int_0^1 6x^3(1-x)dx = \frac{3}{2}x^4 - \frac{6}{5}x^5|_0^1 = \frac{3}{10}$. Now, $\text{Var}(X) = E[X^2] - EX^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$.

(e) First, find $E[Y^2] = \int_0^1 2y^3 dy = \frac{1}{2}y^4|_0^1 = \frac{1}{2}$. Now, $\text{Var}(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$.