Math 461 Test 1, Spring 2008

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (16 points) An electric circuit is as below. The probability that the i-th switch is on is equal to $\frac{1}{i+1}$, $i = 1, 2, 3, 4$. Assume that all switches function independently. (a) Find the probability that electric current can flow from A to B. (b) Given that electric current can flow from A to B, find the probability that the second switch is on.

Solution Let E be the event that electric current can flow from A to B. For $i = 1, 2, 3, 4$, let F_i be the event that the i-th switch is on. Then

$$E = F_1 \cap (F_2 \cup (F_3 \cap F_4)).$$

Thus (a)

$$\begin{aligned} P(E) &= P(F_1)P(F_2 \cup (F_3 \cap F_4)) = P(F_1)\left(P(F_2) + P(F_3 \cap F_4) - P(F_2 \cap F_3 \cap F_4)\right) \\ &= P(F_1)\left(P(F_2) + P(F_3)P(F_4) - P(F_2)P(F_3)P(F_4)\right) = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\frac{1}{5} - \frac{1}{3}\frac{1}{4}\frac{1}{5}\right) \\ &= \frac{11}{60}. \end{aligned}$$

(b)

$$P(F_2|E) = \frac{P(F_2 \cap E)}{P(E)} = \frac{P(F_1 \cap F_2)}{P(E)} = \frac{10}{11}.$$

2. (14 points) Suppose that box I contains 2 white and 4 red balls and that box II contains 1 white and 1 red ball. A ball is randomly selected from box I and put into box II, and a ball is then randomly selected from box II. Find (a) the probability that the ball selected from box II is white; (b) the probability that the transferred ball was white given that a white ball is selected from box II.

Solution. Let W_1 be the event that the ball selected from box I is white, W_2 the event that the ball selected from box II is white, R_1 the event that the ball selected from box I is red. Then (a)

$$P(W_2) = P(W_1)P(W_2|W_1) + P(R_1)P(W_2|R_1) = \frac{2}{6}\frac{2}{3} + \frac{4}{6}\frac{1}{3} = \frac{4}{9}.$$

(b)

$$P(W_1|W_2) = \frac{P(W_1)P(W_2|W_1)}{P(W_2)} = \frac{1}{2}.$$

3. (15 points) a 13 card poker hand is randomly drawn from an ordinary deck of 52 cards. Find the probability that the hand contains (a) all 4 aces; (b) all 4 aces and all 4 kings; (c) all 4 aces, all 4 kings and all 4 queens; (d) all 4 cards of at least 1 of the 13 denominations.

Solution. Let E_1 be the event that the hand contains all 4 aces, E_2 the event that the hand contains all 4 twos, ..., E_{10} the event that the hand contains all 4 tens, E_{11} the event that the hand contains all 4 jacks, $E12$ the event that the hand contains all 4 queens and E_{13} the event that the hand contains all 4 kings.

(a)

$$P(E_1) = \frac{\binom{48}{9}}{\binom{52}{13}}.$$

(b)

$$P(E_1 \cap E_{13}) = \frac{\binom{44}{5}}{\binom{52}{13}}.$$

(c)

$$P(E_1 \cap E_{12} \cap E_{13}) = \frac{\binom{40}{1}}{\binom{52}{13}}.$$

(d)

$$P(\cup_{i=1}^{13} E_i) = \binom{13}{1}\frac{\binom{48}{9}}{\binom{52}{13}} - \binom{13}{2}\frac{\binom{44}{5}}{\binom{52}{13}} + \binom{13}{3}\frac{\binom{40}{1}}{\binom{52}{13}}.$$

4. (15 points) A and B play a series of games. Each game is independently won by A with probability $\frac{2}{3}$ and by B with probability $\frac{1}{3}$. They stop when the total number of wins of one of the players is two greater than that of the other player. Find the probability that a total of 4 games were played.

Solution.

$$\begin{aligned} P(\text{exactly four games are played}) &= P(ABAA) + P(BAAA) + P(ABBB) + P(BABB) \\ &= 2(\frac{2}{3})^3\frac{1}{3} + 2\frac{2}{3}(\frac{1}{3})^3 = \frac{20}{81}. \end{aligned}$$

5. (10 points) 10 people, including A and B, are randomly arranged in a line. Find the probability that there is exactly one person between A and B.

Solution.

$$\frac{16 \cdot 8!}{10!} = \frac{8}{45}.$$

6. (15 points) From a group of 7 freshman, 6 sophomores, 5 juniors and 5 seniors a committee of size 4 is randomly selected. (a) Find the probability that the committee consists of 1 from

each class. (b) Find the probability that the committee consists of people all from the same class.

Solution. (a)

$$\frac{7 \cdot 6 \cdot 5 \cdot 5}{\binom{23}{4}}.$$

(b)

$$\frac{\binom{7}{4}}{\binom{23}{4}} + \frac{\binom{6}{4}}{\binom{23}{4}} + 2\frac{\binom{5}{4}}{\binom{23}{4}}.$$

7. (15 points) Two teams play a series of games and the series is finished as soon as one of the teams wins 4 games. Suppose that each team has probability 1/2 of winning each game, independent of the outcomes of other games. Find the expected number of games played.

Solution. Let X be the number of games played in a match. Then

$$P(X = i) = 2\binom{i-1}{i-4}\left(\frac{1}{2}\right)^i, \qquad i = 4, \ldots, 7.$$

Hence, $E[X] = 2\sum_{i=4}^{7} i\binom{i-1}{i-4}\left(\frac{1}{2}\right)^i = \frac{93}{16}$