Solutions to Math 461 Test 1, Spring 2009

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (15 points) An electric circuit is as below. The probability that the i-th switch is on is equal to $\frac{1}{i+1}$, $i = 1, 2, 3, 4$. Assume that all switches function independently. (a) Find the probability that electric current can flow from A to B. (b) given that electric current can flow from A to B, find the probability that the second switch is on.

Solution Let E be the event that electric current can flow from A to B. For $i = 1, 2, 3, 4$, let F_i be the event that the i-th switch is on. Then

$$E = F_1 \cap (F_2 \cup (F_3 \cap F_4)).$$

Thus (a)

$$\begin{aligned} P(E) &= P(F_1)P(F_2 \cup (F_3 \cap F_4)) = P(F_1)\left(P(F_2) + P(F_3 \cap F_4) - P(F_2 \cap F_3 \cap F_4)\right) \\ &= P(F_1)\left(P(F_2) + P(F_3)P(F_4) - P(F_2)P(F_3)P(F_4)\right) = \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\frac{1}{5} - \frac{1}{3}\frac{1}{4}\frac{1}{5}\right) \\ &= \frac{11}{60}. \end{aligned}$$

(b)

$$P(F_2|E) = \frac{P(F_2 \cap E)}{P(E)} = \frac{P(F_1 \cap F_2)}{P(E)} = \frac{10}{11}.$$

2. (14 points) Suppose that box I contains 2 white and 4 red balls and that box II contains 1 white and 1 red ball. A ball is randomly selected from box I and put into box II, and a ball is then randomly selected from box II. Find (a) the probability that the ball selected from box II is white; (b) the probability that the transferred ball was white given that a white ball is selected from box II.

Solution. Let W_1 be the event that the ball selected from box I is white, W_2 the event that the ball selected from box II is white, R_1 the event that the ball selected from box I is red. Then (a)

$$P(W_2) = P(W_1)P(W_2|W_1) + P(R_1)P(W_2|R_1) = \frac{2}{6}\frac{2}{3} + \frac{4}{6}\frac{1}{3} = \frac{4}{9}.$$

(b)

$$P(W_1|W_2) = \frac{P(W_1)P(W_2|W_1)}{P(W_2)} = \frac{1}{2}.$$

3. (15 points) A 9 card poker hand is randomly drawn from an ordinary deck of 52 cards. Find the probability that the hand contains (a) all 4 aces; (b) all 4 aces and all 4 kings; (c) all 4 cards of at least 1 of the 13 denominations.

Solution. Let E_1 be the event that the hand contains all 4 aces, E_2 the event that the hand contains all 4 twos, ..., E_{10} the event that the hand contains all 4 tens, E_{11} the event that the hand contains all 4 jacks, $E12$ the event that the hand contains all 4 queens and E_{13} the event that the hand contains all 4 kings.

(a)

$$P(E_1) = \frac{\binom{48}{5}}{\binom{52}{9}}.$$

(b)

$$P(E_1 \cap E_{13}) = \frac{\binom{44}{1}}{\binom{52}{9}}.$$

(c)

$$P(\cup_{i=1}^{13} E_i) = \binom{13}{1} \frac{\binom{48}{5}}{\binom{52}{9}} - \binom{13}{2} \frac{\binom{44}{1}}{\binom{52}{9}}.$$

4. (10 points) Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/3, & 0 \le x < 1, \\ x/2, & 1 \le x < 2, \\ 1, & 2 \le x. \end{cases}$$

Find (a) $P(1/2 \le X \le 3/2)$; (b) $P(1/2 \le X \le 1)$; (c) $P(1/2 \le X < 1)$; (d) $P(1 \le X \le 3/2)$; (e) $P(1 < X < 2)$.

Solution. a. $P(\frac{1}{2} \le X \le \frac{3}{2}) = F(\frac{3}{2}) - F(\frac{1}{2}-) = \frac{3}{4} - \frac{1}{6}$.
b. $P(\frac{1}{2} \le X \le 1 = F(1) - F(\frac{1}{2}-) = \frac{1}{2} - \frac{1}{6}$.
c. $P(\frac{1}{2} \le X < 1) = F(1-) - F(\frac{1}{2}-) = \frac{1}{3} - \frac{1}{6}$.
d. $P(1 \le X \le \frac{3}{2}) = F(\frac{3}{2}) - F(1-) = \frac{3}{4} - \frac{1}{3}$.
e. $P(1 < X < 2) = F(2-) - F(1) = 1 - \frac{1}{2}$.

5. (15 points) A and B paly a series of games. Each game is independently won by A with probability 2/3 and by B with probability 1/3. They stop when the total number of wins of one of the players is two greater than that of the other player. The player with the greater number of wins is declared the match winner. Find the probability that A is the match winner.

Solution. A can win the match after an even number of games. For $k = 1, 2, \ldots$, let E_k be the event that A wins the match after exactly $2k$ games. In order for E_k to occur, each of the $k-1$ pairs of games have to be split between the two, and the last pair is won by A, so $P(E_k) = (\frac{4}{9})^{k-1}\frac{4}{9} = \frac{4}{9})^k$. Thus the probability that A is the match winner is

$$\sum_{k=1}^{\infty}(\frac{4}{9})^k = \frac{4}{9}\frac{1}{1-\frac{4}{9}} = \frac{4}{5}.$$

6. (15 points) From a group of 7 freshman, 6 sophomores, 5 juniors and 5 seniors a committee of size 4 is randomly selected. (a) Find the probability that the committee consists of 1 from each class. (b) Find the probability that the committee consists of people all from the same class.

Solution. (a)

$$\frac{7 \cdot 6 \cdot 5 \cdot 5}{\binom{23}{4}}.$$

(b)

$$\frac{\binom{7}{4}}{\binom{23}{4}} + \frac{\binom{6}{4}}{\binom{23}{4}} + 2\frac{\binom{5}{4}}{\binom{23}{4}}.$$

7. (16 points) Two teams play a series of games and the series is finished as soon as one of the teams wins 4 games. Suppose that each team has probability 1/2 of winning each game, independent of the outcomes of other games. Let X be the number games played in the series. Find the mass function and the expectation of X.

Solution. X can only take the values $4, 5, 6, 7$. For $i = 4, \ldots, 7$,

$$P(X = i) = 2\binom{i-1}{3}\left(\frac{1}{2}\right)^i.$$

Hence, $E[X] = 2\sum_{i=4}^{7} i\binom{i-1}{3}\left(\frac{1}{2}\right)^i = \frac{93}{16}$