Eleventh Homework Set — Solutions
Chapter 6

Problem 11 Let A be the number of people buying an ordinary set, B the number of people buying a plasma set, and C the number of people who are just browsing. Then $P\{A=2,B=1,C=2\} = \frac{5!}{2!1!2!}0.45^2 \cdot 0.15 \cdot 0.4^2 = 0.1458$.

Problem 13 Let X be uniform on $(-15,15)$, and let Y be uniform on $(-30,30)$. Nobody waits longer than five minutes if $|Y-X|<5$.

$$\begin{aligned}
P\{|Y-X|<5\} &= P\{-5<Y-X<5\} \\
&= P\{X-5<Y<X+5\} \\
&= \int_{-15}^{15}\int_{x-5}^{x+5}\frac{1}{30\cdot 60}dydx \\
&= \frac{30\cdot 10}{30\cdot 60} = \frac{1}{6}.
\end{aligned}$$

The probability that the man arrives first is $P\{X<Y\} = \frac{1}{2}$ by symmetry.

Problem 14 Let X,Y be uniform random variables on $(0,L)$. Let $Z=|Y-X|$. We want to find $E[Z]$. First, find $F_Z(a)$, for $a \geq 0$. We have $F_Z(a) = P\{Z \leq a\} = P\{|Y-X| \leq a\} = P\{-a \leq Y-X \leq a\} = \frac{2aL-a^2}{L^2}$. using geometric considerations. Hence, $f_Z(x) = \frac{2L-2x}{L^2}$ if $0 \leq a \leq L$. Hence,

$$\begin{aligned}
E[Z] &= \int_0^L x\cdot\frac{2L-2x}{L^2}dx \\
&= \frac{2}{L^2}\left(\frac{Lx^2}{2}-\frac{x^3}{3}\right)|_0^L \\
&= \frac{L}{3}.
\end{aligned}$$

Problem 18 Let X be uniform on $(0,\frac{L}{2})$ and let Y be uniform on $(\frac{L}{2},L)$. We want to find $P\left\{Y-X>\frac{L}{3}\right\}$.

$$P\left\{Y-X>\frac{L}{3}\right\}=P\left\{Y<\frac{L}{2}+\frac{L}{3},X<Y-\frac{L}{3}\right\}+P\left\{Y>\frac{L}{2}+\frac{L}{3}\right\}$$
$$=\int_{\frac{L}{2}}^{\frac{5L}{6}}\int_0^{y-\frac{L}{3}}\frac{4}{L^2}dxdy+\int_{\frac{5L}{6}}^{\frac{L}{2}}\frac{2}{L}dy$$
$$=\frac{4}{9}+\frac{1}{3}=\frac{7}{9}.$$

Problem 20 If the joint density function of X and Y is

$$f(x,y)=\begin{cases}xe^{-(x+y)} & x>0,y>0\\ 0 & \text{otherwise,}\end{cases}$$

then $f(x,y)=f_X(x)f_Y(y)$, where $f_X(x)=xe^{-x}$ for $x>0$, and $f_Y(y)=e^{-y}$ for $y>0$ (0 otherwise), so that X and Y are independent.

If

$$f(x,y)=\begin{cases}2 & 0<x<y,0<y<1\\ 0 & \text{otherwise,}\end{cases}$$

then X and Y are not independent because the nonzero values of f are located in a triangular domain.

Problem 21

(a) Check: $\int_0^1\int_0^{1-y}24xydxdy=\int_0^1 12(1-y)^2ydy=12\int_0^1 y-2y^2+y^3dy=6y^2-8y^3+3y^4|_0^1=6-8+3=1$.

(b) First, find $f_X(x)=\int_0^{1-x}24xydy=12x(1-x)^2$. Now, $E[X]=\int_0^1 12x^2(1-x)^2dx=4x^2-6x^3+\frac{12}{5}x^5|_0^1=4-6+\frac{12}{5}=\frac{2}{5}$.

(c) $E[Y]=E[X]=\frac{2}{5}$ by symmetry.

Problem 22 Let X and Y be jointly continuous with density function

$$f(x,y)=\begin{cases}x+y & 0<x<1,0<y<1\\ 0 & \text{otherwise.}\end{cases}$$

(a) X and Y are not independent, since $f(x,y)$ is clearly not a product of functions of x and y.

(b) $f_X(x)=\int_0^1 x+ydy=x+\frac{y^2}{2}|_0^1=x+\frac{1}{2}$.

(c) $P\{X+Y<1\} = \int_0^1 \int_0^{1-y} x + y dx dy = \int_0^1 \frac{(1-y)^2}{2} + y(1-y) dy = \frac{1}{2}\int_0^1 1 - y^2 dy = \frac{1}{2}(1-\frac{1}{3}) = \frac{1}{3}$.

Problem 23 Let X and Y be jointly distributed with density function

$$f(x,y) = \begin{cases} 12xy(1-x) & 0<x<1, 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

First, compute $f_X(x) = \int_0^1 12xy(1-x)dy = 6x(1-x)$ and $f_Y(y) = \int_0^1 12xy(1-x)dy = 2y$.

(a) Clearly, $f(x,y) = f_X(x) f_Y(y)$, so that X and Y are independent.

(b) $E[X] = \int_0^1 6x^2(1-x)dx = 2x^3 - \frac{3}{2}x^4|_0^1 = \frac{1}{2}$.

(c) $E[Y] = \int_0^1 2y^2 dy = \frac{2}{3}y^3|_0^1 = \frac{2}{3}$.

(d) First, find $E[X^2] = \int_0^1 6x^3(1-x)dx = \frac{3}{2}x^4 - \frac{6}{5}x^5|_0^1 = \frac{3}{10}$. Now, $\mathrm{Var}(X) = E[X^2] - EX^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$.

(e) First, find $E[Y^2] = \int_0^1 2y^3 dy = \frac{1}{2}y^4|_0^1 = \frac{1}{2}$. Now, $\mathrm{Var}(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$.

Problem 27 Let X be uniform on $(0,1)$, and let Y be exponential with $\lambda = 1$.

(a) If $Z = X+Y$, then

$$\begin{aligned} f_Z(a) &= \int_{-\infty}^{\infty} f_Y(a-x) f_X(x) dx = \int_0^1 f_Y(a-x)dx \\ &= \begin{cases} 0 & a<0 \\ \int_0^a e^{x-a}dx & 0<a<1 \\ \int_0^1 e^{x-a}dx & 1<a \end{cases} \\ &= \begin{cases} 0 & a<0 \\ 1-e^{-a} & 0<a<1 \\ e^{-a}(e-1) & 1<a \end{cases} \end{aligned}$$

(b) If $Z = \frac{X}{Y}$, we first find the cumulative distribution function of Z. Note that $F_Z(a) = 0$ if $a \leq 0$. Assume that $a > 0$.

$$\begin{aligned} F_Z(a) &= P\{Z \leq a\} = P\{X \leq aY\} \\ &= \int_0^1 \int_{\frac{x}{a}}^{\infty} f_Y(y) dy dx \\ &= a\left(1 - e^{-\frac{1}{a}}\right). \end{aligned}$$

Now, we have

$$f_Z(a) = \frac{d}{da} F_Z(a) = \begin{cases} 0 & a \leq 0 \\ 1 - e^{-\frac{1}{a}}\left(1 + \frac{1}{a}\right) & 0 < a. \end{cases}$$

Problem 28 Let X_1, X_2 be exponential random variables with parameter λ_1, λ_2. Let $Z = \frac{X_1}{X_2}$. Note that $F_Z(a) = 0$ if $a \leq 0$. Compute $F_Z(a)$ for $a > 0$:

$$\begin{aligned} F_Z(a) &= P\{Z \leq a\} = P\{X_1 \leq aX_2\} \\ &= \lambda_1 \lambda_2 \int_0^{\infty} \int_0^{ay} e^{-\lambda_1 x - \lambda_2 y} dx dy \\ &= \frac{\lambda_1 a}{\lambda_1 a + \lambda_2}, \end{aligned}$$

so that

$$f_Z(a) = \frac{d}{da} F(a) = \frac{\lambda_1}{\lambda_1 a + \lambda_2} - \frac{\lambda_1^2 a}{(a\lambda_1 + \lambda_2)^2}.$$

Finally, we have

$$P\{X_1 < X_2\} = P\{Z < 1\} = F_Z(1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Problem 31 Let X_1 be the number of accidents in the next month, X_2 the number of accidents in the month after that, and X_3 the number of accidents in the third month. It makes sense to think of X_1, X_2, and X_3 as independent Poisson random variables with parameter $\lambda = 2.2$.

Let $X = X_1$, $Y = X_1 + X_2$, and $Z = X_1 + X_2 + X_3$. Then X, Y, and Z are Poisson with parameter $2.2, 4.4$, and 6.6, respectively.

(a) $P\{X > 2\} = 1 - e^{-2.2}\left(1 + 2.2 + \frac{2.2^2}{2}\right) = 0.3773.$

(b) $P\{Y > 4\} = 1 - e^{-4.4}\left(1 + 4.4 + \frac{4.4^2}{2} + \frac{4.4^3}{3!} + \frac{4.4^4}{4!}\right) = 0.4488.$

(c) $P\{Z > 5\} = 1 - e^{-6.6}\left(1 + 6.6 + \frac{6.6^2}{2} + \frac{6.6^3}{3!} + \frac{6.6^4}{4!} + \frac{6.6^5}{5!}\right) = 0.6453.$

Problem 32 Let X_1, X_2 be independent normal random variables with $\mu = 2200$ and $\sigma^2 = 230^2$, representing the gross sales over this week and next week, respectively. Then $X = X_1 + X_2$ is normal with mean 4400 and variance $2 \cdot 230^2 = 105800$.

(a) $P\{X > 5000\} = P\left\{\frac{X-4400}{\sqrt{105800}} > \frac{600}{\sqrt{105800}}\right\} = 1 - \Phi(1.84) = 1 - 0.9671 = 0.0329.$

(b) Let $p = P\{X_1 > 2000\} = P\left\{\frac{X_1-2200}{230} > \frac{-200}{230}\right\} = 1 - \Phi\left(-\frac{20}{23}\right) = \Phi(0.87) = 0.8078.$

Let N be the number of weeks (out of three) in which the sales exceed \$2000. Then N is binomial with parameters $(p, 3)$, so that $P\{N \geq 2\} = p^3 + 3p^2(1-p) = 0.9034.$

Problem 34 Let X be the number of women who never eat breakfast, and let Y be the number of men who never eat breakfast. Let $Z = X + Y$. By DeMoivre-Laplace, X is approximated by a normal random variable with mean $200 \cdot 0.236 = 47.2$ and variance $47.2 \cdot 0.764 = 36.061$, and Y is normal with mean $200 \cdot 0.252 = 50.4$ and variance $50.4 \cdot 0.748 = 37.699$.

Let $Z_1 = X + Y$ and $Z_2 = X - Y$. Then Z_1 is normal with mean 97.6 and variance $36.061 + 37.699 = 73.76$, and Z_2 is normal with mean -3.2 and variance 73.76.

(a) $P\{Z_1 \geq 110\} = P\{Z_1 > 109.5\} = P\left\{\frac{Z_1-97.6}{\sqrt{73.76}} > \frac{11.9}{\sqrt{73.76}}\right\} = 1 - \Phi(1.39) = 1 - 0.9177 = 0.0823.$

(b) $P\{X \geq Y\} = P\{X - Y \geq 0\} = P\{Z_2 \geq 0\} = P\{Z_2 > -0.5\} = P\left\{\frac{Z_2+3.2}{\sqrt{73.76}} > \frac{2.7}{\sqrt{73.76}}\right\} = 1 - \Phi(0.31) = 0.3783.$

Problem 39 (a) $P\{X = i, Y = j\} = \frac{1}{5i}$ for $i = 1, \ldots, 5$ and $j = 1, \ldots, i$, 0 otherwise.

$P\{X=i,Y=j\}$	Y=1	Y=2	Y=3	Y=4	Y=5	$P\{X=i\}$
X=1	$\frac{1}{5}$	0	0	0	0	$\frac{1}{5}$
X=2	$\frac{1}{10}$	$\frac{1}{10}$	0	0	0	$\frac{1}{5}$
X=3	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	0	0	$\frac{1}{5}$
X=4	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	0	$\frac{1}{5}$
X=5	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{5}$
$P\{Y=j\}$	$\frac{137}{300}$	$\frac{77}{300}$	$\frac{47}{300}$	$\frac{9}{100}$	$\frac{1}{25}$	1

(b) $P\{X=i|Y=j\} = \frac{\frac{1}{5i}}{\sum_{k=i}^{5}\frac{1}{5k}}$

$P\{X=i|Y=j\}$	Y=1	Y=2	Y=3	Y=4	Y=5
$X=1$	$\frac{60}{137}$	0	0	0	0
$X=2$	$\frac{30}{137}$	$\frac{30}{77}$	0	0	0
$X=3$	$\frac{20}{137}$	$\frac{20}{77}$	$\frac{20}{47}$	0	0
$X=4$	$\frac{15}{137}$	$\frac{15}{77}$	$\frac{15}{47}$	$\frac{5}{9}$	0
$X=5$	$\frac{12}{137}$	$\frac{12}{77}$	$\frac{12}{47}$	$\frac{4}{9}$	1

(c) No.

Problem 41

$p(i,j)$	$j=1$	$j=2$	
$i=1$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$
$i=2$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{5}{8}$
	$\frac{1}{4}$	$\frac{3}{4}$	1

(a)

$P\{X=i|Y=j\}$	$j=1$	$j=2$
$i=1$	$\frac{1}{2}$	$\frac{1}{3}$
$i=2$	$\frac{1}{2}$	$\frac{2}{3}$

(b) No.

(c) $P\{XY \le 3\} = 1 - p(2,2) = \frac{1}{2}$
$P\{X+Y > 2\} = 1 - p(1,1) = \frac{7}{8}$
$P\left\{\frac{X}{Y} > 1\right\} = p(2,1) = \frac{1}{8}$

Problem 42 Let X and Y be jointly continuous with density function $f(x,y) = xe^{-x(y+1)}$ for $x>0, y>0$. Note that $f_X(x) = \int_0^\infty f(x,y)dy = e^{-x}$ for $x>0$, and $f_Y(y) = \int_0^\infty f(x,y)dx = \frac{1}{(y+1)^2}$ for $y>0$.

(a) $f_{X|Y}(x|y) = (y+1)^2 xe^{-x(y+1)}$ for $x>0, y>0$, 0 otherwise, and $f_{Y|X}(y|x) = xe^{-xy}$ for $x>0, y>0$.

(b) Let $Z = XY$. Find $F_Z(a) = P\{XY < a\} = \int_0^\infty \int_0^{\frac{a}{x}} xe^{-x(y+1)}dydx = 1 - e^{-a}$ for $a > 0$. Hence, $f_Z(a) = \frac{d}{da}F_Z(a) = e^{-a}$ for $a > 0$, 0 otherwise.

Problem 43 Let X and Y be jointly continuous with density function

$$f(x, y) = c(x^2 - y^2)e^{-x}$$

for $0 \leq x < \infty, -x \leq y \leq x$. For $x > 0$, we have

$$f_X(x) = \int_{-x}^{x} c(x^2 - y^2)e^{-x}dy = \frac{4c}{3}x^3e^{-x}.$$

Hence, $f_{Y|X}(y|x) = \frac{3}{4}\frac{x^2-y^2}{x^3}$ for $-x < y < x$, 0 otherwise. We conclude that

$$F_{Y|X}(y|x) = \begin{cases} 0 & y \leq -x \\ \frac{3}{4}\int_{-x}^{y} \frac{x^2-y^2}{x^3}dy = \frac{1}{4}\left(\frac{y(3x^2-y^2)}{x^3} + 2\right) & -x < y < x \\ 1 & x \leq y. \end{cases}$$

Problem 49 Let $X_1, \ldots, X_5$ be independent exponential random variables with parameter λ.

(a)

$$\begin{aligned} P\{\min(X_1, \ldots, X_5) \leq a\} &= 1 - P\{\min(X_1, \ldots, X_5) > a\} \\ &= 1 - P\{X_1 > a, \ldots, X_5 > a\} \\ &= 1 - P\{X_1 > a\} \cdots P\{X_5 > a\} \\ &= \begin{cases} 1 - (e^{-\lambda a})^5 & a > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} P\{\max(X_1, \ldots, X_5) \leq a\} &= P\{X_1 \leq a, \ldots, X_5 \leq a\} \\ &= P\{X_1 \leq a\} \cdots P\{X_5 \leq a\} \\ &= \begin{cases} (1 - e^{-\lambda a})^5 & a > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$