Twelfth Homework Set — Solutions

Chapter 7

Problem 5 If (X, Y) is the location of the accident, then X and Y are uniform random variables on $(-\frac{3}{2}, \frac{3}{2})$. Let $D = |X| + |Y|$. Then

$$\begin{aligned} E[D] &= E[|X|] + E[|Y|] = 2E[|X|] \\ &= 2\int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{|x|}{3} dx = \frac{4}{3}\int_{0}^{\frac{3}{2}} x dx \\ &= \frac{4}{3} \cdot \frac{9}{8} = \frac{3}{2}. \end{aligned}$$

Problem 6 Let X_i be the outcome of the i-th roll of the die, for $i = 1, \ldots, 10$, and note that $E[X_i] = \frac{7}{2}$. Let $X = X_1 + \cdots + X_{10}$. Now $E[X] = E[X_1] + \cdots + E[X_{10}] = 10E[X_1] = 35$.

Problem 7 (a) Let X_i be one if both A and B choose the i-th object, for $i = 1, \ldots, 10$. Then $E[X_i] = P\{X_i = 1\} = \left(\frac{3}{10}\right)^2 = \frac{9}{100}$. Now, the expected number of objects chosen by both A and B is $E[X_1] + \cdots + E[X_{10}] = 10E[X_1] = 0.9$.

(b) Let Y_i be one if neither A nor B choose the i-th object. Then $E[Y_i] = P\{Y_i = 1\} = \left(\frac{7}{10}\right)^2 = \frac{49}{100}$, so that $E[Y_1 + \cdots + Y_{10}] = 10E[Y_1] = 4.9$.

(c) Let Z_i be one if either A or B (but not both) chooses the i-th object. Then $E[Z_i] = P\{Z_i = 1\} = 2\frac{3}{10}\frac{7}{10} = \frac{21}{50}$. Now, $E[Z_1 + \cdots + Z_{10}] = 10E[Z_1] = \frac{21}{5} = 4.2$.

Problem 8 Following the hint, let X_1 be one if the i-th arrival sits at a previously unoccupied table. Then $E[X_i] = P\{X_i = 1\} = (1-p)^{i-1}$, so that

$$E[X_1 + \cdots + X_N] = \sum_{i=1}^{N} (1-p)^{i-1} = \frac{1-(1-p)^N}{1-(1-p)} = \frac{1-(1-p)^N}{p}.$$

Problem 11 Let X_i be one if the i-th outcome differs from the $(i-1)$-th outcome, for $i = 2, \ldots, n$. We have $E[X_i] = P\{X_i = 1\} = 2p(1-p)$, so that $E[X_2 + \cdots + X_n] = 2(n-1)p(1-p)$.

Problem 18 Let X_i be one if the i-th card is a match, for $i = 1, \ldots, 13$, and let $X = X_1 + \cdots + X_{52}$. Then $P\{X_i = 1\} = \frac{1}{13}$, so that $E[X] = 52E[X_1] = \frac{52}{13} = 4$.

Problem 19 (a) If X is the number of insects caught before a type 1 catch, then $(X+1)$ is geometric with parameter P_1, so that $E[X] = \frac{1}{P_1} - 1$.

(b) Let Y_i be one if an insect of type i is caught before an insect of type 1, for $i = 2, \ldots, r$. Then $Y = Y_2 + \cdots + Y_r$ is the number of insects caught before an insect of type 1. We have $E[Y_i] = P\{Y_i = 1\} = \frac{P_i}{P_i + P_1}$, so that

$$E[Y] = \sum_{i=2}^{r} \frac{P_i}{P_i + P_1}.$$

Problem 21 (a) Let X be the number of days of the year that are birthdays of exactly 3 people. For $i = 1, \ldots, 365$, let $X_i = 1$ if the i-day is the birthday of eaxctly 3 people and $X_i = 0$ otherwise. Then $X = \sum_{i=1}^{365} X_i$. Since for each i,

$$EX_i = P(X_i = 1) = \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97},$$

we get that

$$EX = 365 \binom{100}{3} \left(\frac{1}{365}\right)^3 \left(\frac{364}{365}\right)^{97}.$$

(b) Let Y be the number of distinct birthdays. For $i = 1, \ldots, 365$, let $Y_i = 1$ if the i-day is someone's birthday and $Y_i = 0$ otherwise. Then $Y = \sum_{i=1}^{365} Y_i$. Since for each i,

$$EY_i = P(Y_i = 1) = 1 - P(Y_i = 0) = 1 - \left(\frac{364}{365}\right)^{100},$$

we get that

$$EY = 365 \left[1 - \left(\frac{364}{365}\right)^{100}\right].$$

Problem 30 Note that $E[X^2] = E[Y^2] = \mathrm{Var}(X) + E[X]^2 = \sigma^2 + \mu^2$. Now we conclude that

$$E\left[(X-Y)^2\right] = E\left[X^2\right] - 2E[X]E[Y] + E\left[Y^2\right] = 2\sigma^2,$$

using the fact that X and Y are independent.

Problem 31 Let X_i be the outcome of the i-th roll of the die, for $i = 1, \ldots, 10$. Then $\mathrm{Var}(X_i) = \frac{35}{12}$, so that

$$\mathrm{Var}(X_1 + \cdots + X_{10}) = 10 \cdot \frac{35}{12} = \frac{175}{6}.$$

Problem 33 (a)

$$E\left[(2+X)^2\right] = 4 + 4E[X] + E\left[X^2\right] = 8 + \mathrm{Var}(X) + E[X]^2 = 14.$$

(b)

$$\mathrm{Var}(4+3X) = 9\mathrm{Var}(X) = 45.$$

Problem 38 We have

$$\begin{aligned} E[XY] &= \int_0^\infty \int_0^x 2ye^{-2x} dy dx = \int_0^\infty x^2 e^{-2x} dx = \frac{1}{4}, \\ E[X] &= \int_0^\infty \int_0^x 2e^{-2x} dy dx = \int_0^\infty 2xe^{-2x} dx = \frac{1}{2}, \quad \text{and} \\ E[Y] &= \int_0^\infty \int_0^x \frac{2y}{x} e^{-2x} dy dx = \int_0^\infty xe^{-2x} dx = \frac{1}{4}. \end{aligned}$$

Hence,

$$\mathrm{Cov}(X,Y) = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

Problem 39 We have

$$\begin{aligned} \mathrm{Cov}(Y_n, Y_n) &= \mathrm{Var}(Y_n) = 3\sigma^2, \\ \mathrm{Cov}(Y_n, Y_{n+1}) &= \mathrm{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\ &= \mathrm{Cov}(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) \\ &= \mathrm{Var}(X_{n+1} + X_{n+2}) = 2\sigma^2, \\ \mathrm{Cov}(Y_n, Y_{n+2}) &= \mathrm{Cov}(X_n + X_{n+1} + X_{n+2}, X_{n+2} + X_{n+3} + X_{n+4}) \\ &= \mathrm{Cov}(X_{n+2}, X_{n+2}) = \mathrm{Var}(X_{n+2}) = \sigma^2, \quad \text{and} \\ \mathrm{Cov}(Y_n, Y_{n+j}) &= 0 \quad \text{if } j \geq 3. \end{aligned}$$

Problem 41 the number of carp is a hypergeometric random variable, so that we have

$$E[X] = \frac{20 \cdot 30}{100} = 6,$$

and

$$\text{Var}(X) = \frac{20 \cdot 80}{99} \cdot \frac{3}{10} \cdot \frac{7}{10} = \frac{112}{33}.$$

Problem 42 (a) Let X_i be one if the i-th pair consists of a man and a women, and zero otherwise. Then the sum $X_1 + \cdots + X_{10}$ is the number of pairs that consist of a man and a woman.

We have $E[X_i] = P\{X_i = 1\} = 2 \cdot \frac{10 \cdot 10}{20 \cdot 19} = \frac{10}{19}$, so that

$$E[X_1 + \cdots + X_{10}] = \frac{100}{19}.$$

Now, we have $\text{Var}(X_i) = E[X_i^2] - E[X_i]^2 = \frac{10}{19} - \frac{100}{361} = \frac{90}{361}$, and $\text{Cov}(X_i, X_j) = E[X_i X_j] - E[X_i] E[X_j] = \frac{10}{19} \cdot \frac{9}{17} - \frac{100}{361} = \frac{10}{6137}$ if $i \neq j$, so that

$$\text{Var}(X_1 + \cdots + X_{10}) = \frac{900}{361} + 10 \cdot 9 \cdot \frac{10}{6137} = \frac{16200}{6137} = 2.6397.$$

(b) Let Y_i be one if the i-th couple are paired together. $E[Y_i] = P\{Y_i = 1\} = \frac{2 \cdot 10 \cdot 18!}{20!} = \frac{1}{19}$, so that

$$E[Y_1 + \cdots + Y_{10}] = \frac{10}{19}.$$

We have $\text{Var}(Y_i) = E[Y_i^2] - E[Y_i]^2 = \frac{1}{19} - \frac{1}{361} = \frac{18}{361}$ and $E[Y_i Y_j] = \frac{8\binom{10}{2} \cdot 16!}{20!} = \frac{1}{323}$, so that $\text{Cov}(Y_i, Y_j) = \frac{1}{323} - \frac{1}{361} = \frac{2}{6137}$, so that

$$\text{Var}(Y_1 + \cdots + Y_{10}) = \frac{180}{361} + 90 \cdot \frac{2}{6137} = \frac{3240}{6137}.$$

Problem 50 We have

$$f_Y y = \int_0^\infty \frac{e^{-\frac{x}{y}} - y}{y} dx = e^{-y}$$

for $y > 0$, so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{e^{-\frac{x}{y}}}{y} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Now, we have

$$E\left[X^2|Y\right] = \int_0^\infty \frac{x^2}{y} e^{-\frac{x}{y}} dx = 2y^2.$$

Problem 51 We have

$$f_Y(y) = \int_0^y \frac{e^{-y}}{y} dx = e^{-y},$$

so that

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0, y) \\ 0 & \text{otherwise.} \end{cases}$$

We conclude that

$$E\left[X^3|Y = y\right] = \int_0^y \frac{x^3}{y} dx = \frac{y^3}{4}.$$

Problem 56 Let Y_i be one if the elevator stops at the i-th floor, for $i = 1, \ldots, N$. Let $Y = Y_1 + \cdots + Y_{10}$. Let X be the number of passengers, i.e., X is Poisson with parameter 10. We have $E\left[Y_i = 1|X = k\right] = 1 - \left(\frac{N-1}{N}\right)^k$, so that

$$E\left[Y|X = k\right] = N\left(1 - \left(\frac{N-1}{N}\right)^k\right).$$

We have

$$\begin{aligned} E\left[Y\right] = E\left[E\left[Y|X\right]\right] &= E\left[N\left(1 - \left(\frac{N-1}{N}\right)^k\right)\right] \\ &= N - N\sum_{k=0}^{\infty}\left(\frac{N-1}{N}\right)^k \frac{10^k}{k!} e^{-10} \\ &= N(1 - e^{-\frac{10}{N}}). \end{aligned}$$

Problem 57 By Example 4d in Section 7.4, we have

$$E\left[\sum_{i=1}^N X_i\right] = E\left[N\right] E\left[X_1\right] = 12.5.$$

Problem 75 X is a random variable with moment generating function $M_X(t) = \exp\{2e^t - 2\} = \exp\{2(e^t-1)\}$, i.e., X is Poisson with parameter $\lambda = 2$.

Y is a random variable with moment generating function $M_Y(t) = (\frac{3}{4}e^t + \frac{1}{4})^{10}$, i.e., Y is binomial with parameters $(10, \frac{3}{4})$.

(a)

$$\begin{aligned}
P\{X+Y=2\} &= P\{X=0\}P\{Y=2\} + P\{X=1\}P\{Y=1\} \\
&+ P\{X=2\}P\{Y=0\} \\
&= e^{-2}\cdot\binom{10}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^8 + 2e^{-2}\cdot 10\frac{3}{4}\left(\frac{1}{4}\right)^9 \\
&+ 2e^{-2}\cdot\left(\frac{1}{4}\right)^{10} \\
&= e^{-2}\left(\frac{1}{4}\right)^{10}(405+60+2) = \frac{467}{4^{10}e^2}.
\end{aligned}$$

(b)

$$\begin{aligned}
P\{XY=0\} &= P\{X=0\} + P\{Y=0\} - P\{X=0\}P\{Y=0\} \\
&= e^{-2} + \frac{1}{4^{10}} - e^{-2}\frac{1}{4^{10}} = \frac{4^{10}+e^2-1}{4^{10}e^2}.
\end{aligned}$$

(c)

$$\begin{aligned}
E[XY] &= E[X]\cdot E[Y] \quad \text{by independence} \\
&= 2\cdot 7.5 \\
&= 15.
\end{aligned}$$

Chapter 8

Problem 1 $P(0 \le X \le 40) = 1 - P(|X-20| > 20) \ge 1 - \frac{20}{400} = \frac{19}{20}$.

Problem 2 (a) $P(X \ge 85) \le \frac{75}{85} = \frac{15}{17}$.

(b) $P(65 \le X \le 85) = 1 - P(|X-75| > 10) \ge 1 - \frac{25}{100} = \frac{3}{4}$.

(c) Since

$$P\left(|\sum_{i=1}^{n}\frac{X_i}{n} - 75| > 5\right) \le \frac{25}{25n},$$

we need $n = 10$.

Problem 4 (a) $P(\sum_{i=1}^{20} X_i > 15) \leq \frac{20}{15}$.

(b)

$$\begin{aligned} P(\sum_{i=1}^{20} X_i > 15) &= P(\sum_{i=1}^{20} X_i > 15.5) \approx P(Z > \frac{15.5-20}{\sqrt{20}} \\ &= P(Z > -1.006) \approx .8428. \end{aligned}$$

Problem 5 Let X_i be the i-th roundoff erroe, then $E(\sum_{i=1}^{50} X_i) = 0$ and $\text{Var}(\sum_{i=1}^{50} X_i) = \frac{50}{12}$. Hence by the central limit theorem

$$P\left(|\sum_{i=1}^{50} X_i| > 3\right) \approx P(|Z| > \frac{3}{\sqrt{12/50}}) = 2P(Z > 1.47) = .1416.$$

Problem 8 If we let X_i be the lifetime of the i-th light bulb and R_i be the time to replace the i-th light bulb, then the desired probability is

$$P\left(\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i \leq 550\right).$$

It follows from the central limit theorem that $\sum_{i=1}^{100} X_i$ is approximately a normal random variable with mean 500 and variance 2500 and that $\sum_{i=1}^{99} R_i$ is approximately a normal random variable with mean 24.75 and variance 99/48, therefore $\sum_{i=1}^{100} X_i + \sum_{i=1}^{99} R_i$ is approximately a normal random variable with mean 524.75 and variance 2502.02. Consequently the desired probability is equal to

$$P(Z \leq \frac{550-524.75}{\sqrt{2502.02}} = P(Z \leq .505) = .693.$$