Second Homework Set — Solutions

Chapter 2

Problem 2 Let $X = \{1, 2, 3, 4, 5\}$. The sample space S consists of all finite sequences $(x_1, \ldots, x_n)$, where $x_1, \ldots, x_{n-1} \in X$ and $x_n = 6$, as well as infinite sequences with values in X.

E_n is the set of all sequences of length n in S, and $(\cup_{n=1}^{\infty} E_n)^c$ is the set of all infinite sequences in S, i.e., the set of infinite sequences with values in X.

Problem 3

$$
\begin{aligned}
E =& \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),\\
& (3,2),(3,4),(3,6),(4,1),(4,3),(4,5),\\
& (5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}\\
F =& \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\
& (2,1),(3,1),(4,1),(5,1),(6,1)\}\\
G =& \{(1,4),(2,3),(3,2),(4,1)\}
\end{aligned}
$$

- Meaning of EF: The sum is odd *and* at least one of the dice lands on 1.
- Meaning of $E \cup F$: The sum is odd *or* at least one the dice lands on 1.
- Meaning of FG: At least one of the dice lands on 1 and the sum equals 5, i.e., $FG = \{(1,4),(4,1)\}$.
- Meaning of EF^c: The sum is odd and none of the dice lands on 1.
- Meaning of EFG: The sum is odd and one of the dice lands on 1 and the sum is five, i.e., $EFG = FG$.

Problem 5 (a) There are 2^5 possible outcomes (two outcomes per component, use generalized counting principle).

(b) Represent an outcome as a binary number, e.g., 00101 means that components 3 and 5 work, and components 1, 2, and 4 do not work. Then

$$
\begin{aligned}
W =& \{11000, 11001, 11010, 11011, 11100, 11101, 11110, 11111,\\
& 00110, 00111, 01110, 01111, 10110, 10111, 10101\}.
\end{aligned}
$$

(c) There are $2^3 = 8$ outcomes in A.

(d) $AW = \{11000, 11100\}$.

Problem 6 (a) $S = \{(0, g), (0, f), (0, s), (1, g), (1, f), (1, s)\}$.

(b) $A = \{(0, s), (1, s)\}$.

(c) $B = \{(0, g), (0, f), (0, s)\}$.

(d) $B^c \cup A = \{(1, g), (1, f), (1, s), (0, s)\}$.

Problem 7 (a) There are 6^{15} outcomes in the sample space.

(b) There are 3^{15} outcomes without any blue-collar workers, so that there are $6^{15} - 3^{15}$ outcomes with at least one blue-collar worker.

(c) If there are no independents, then for each player, there are 4 outcomes, so that there are 4^{15} outcomes altogether.

Problem 8 Suppose that A, B are mutually exclusive events for which $P(A) = 0.3$ and $P(B) = 0.5$.

(a) $P(\text{either } A \text{ or } B) = P(A) + P(B) = 0.8$ because A and B are mutually exclusive.

(b) $P(A \text{ occurs but } B \text{ does not}) = 0.3$.

(c) $P(A \cap B) = 0$.

Problem 9 Let A be the event that a randomly chosen customer carries an Amex card, and let V be the event that a randomly chosen customer carries a Visa card. Then $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(AB) = 0.24 + 0.61 - 0.11 = 0.74$. Hence, 74 percent of all customers carry an acceptable credit card.

Problem 12 Let E be the event that a randomly chosen student is taking Spanish, let F be the event that a randomly chosen student is taking French, and let D be the event that a randomly chosen student is taking German.

Note that

$$
\begin{aligned}
P(E) &= \frac{28}{100} = \frac{7}{25} \\
P(F) &= \frac{26}{100} = \frac{13}{50} \\
P(D) &= \frac{16}{100} = \frac{4}{25} \\
P(EF) &= \frac{12}{100} = \frac{3}{25} \\
P(ED) &= \frac{4}{100} = \frac{1}{25} \\
P(FD) &= \frac{6}{100} = \frac{3}{50} \\
P(EFD) &= \frac{2}{100} = \frac{1}{50}
\end{aligned}
$$

(a) The probability of a student not being in any of these classes is

$$
\begin{aligned}
1 - P(E \cup F \cup D) =& 1 - (P(E) + P(F) + P(D) \\
& - P(EF) - P(ED) - P(FD) + P(EFD)) \\
=& 1 - \frac{28 + 26 + 16 - 12 - 4 - 6 + 2}{100} = 1 - \frac{1}{2} = \frac{1}{2}.
\end{aligned}
$$

(b) We have

$$
\begin{aligned}
P(\text{only } E) &= P(E) - P(EF) - P(ED) + P(EFD) = 0.14 \\
P(\text{only } F) &= P(F) - P(EF) - P(FD) + P(EFD) = 0.1 \\
P(\text{only } D) &= P(D) - P(ED) - P(FD) + P(EFD) = 0.08
\end{aligned}
$$

Hence, $P(\text{exactly one language class}) = 0.14 + 0.1 + 0.08 = 0.32$.

(c) Since fifty students are not taking any of the courses, the probability that neither of two randomly picked students is $\frac{\binom{50}{2}}{\binom{100}{2}} = \frac{49}{198}$, so that the probability of at least one of them taking a language class is $1 - \frac{49}{198} = \frac{149}{198}$.

Problem 17 There are $64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57$ ways of arranging 8 castles on a chess board. Of these, there are $64 \cdot 49 \cdot 36 \cdot 25 \cdot 16 \cdot 9 \cdot 4 \cdot 1 = \prod_{i=1}^{8} i^2$ in which none of the rooks can capture any of the others. So the answer is

$$\frac{\prod_{i=1}^{8} i^2}{64 \cdot 63 \cdot 62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57}.$$

Problem 18

$$\frac{2 \cdot 4 \cdot 16}{52 \cdot 51}.$$

Problem 20 Let A be the event that you are dealt a blackjack, and let B be the event that the dealer is dealt a blackjack.

Then

$$\begin{aligned} P(A) = P(B) &= \frac{2 \cdot 4 \cdot 16}{52 \cdot 51} \\ P(AB) &= \frac{4 \cdot 4 \cdot 16 \cdot 3 \cdot 15}{52 \cdot 51 \cdot 50 \cdot 49} \\ P(A \cup B) &= P(A) + P(B) - P(AB) = 0.0948. \end{aligned}$$

Hence, then probability that neither you nor the dealer is dealt a blackjack is $1 - P(A \cup B) = 0.9052$.

Problem 21

(a) $P(1) = \frac{4}{20} = \frac{1}{5}$, $P(2) = \frac{8}{20} = \frac{2}{5}$, $P(3) = \frac{5}{20} = \frac{1}{4}$, $P(4) = \frac{2}{20} = \frac{1}{10}$, and $P(5) = \frac{1}{20}$.

(b) There are 48 children altogether, so that $P(1) = \frac{4}{48} = \frac{1}{12}$, $P(2) = \frac{2 \cdot 8}{48} = \frac{1}{3}$, $P(3) = \frac{3 \cdot 5}{48} = \frac{5}{16}$, $P(4) = \frac{4 \cdot 2}{48} = \frac{1}{6}$, and $P(5) = \frac{5}{48}$.

Problem 25 Let E_n be the event that a sum of 5 occurs on the nth roll, and no sum of 5 or 7 occurs on the first $n-1$ rolls. There are 36 outcomes of a single roll, and four of them give a sum of 5, while 6 of them give a sum of 7. Hence,

$$P(E_n) = \left(\frac{26}{36}\right)^{n-1} \frac{4}{36} = \left(\frac{13}{18}\right)^{n-1} \frac{1}{9}.$$

A sum of 5 occurs before a sum of 7 precisely if the events E_n occurs for some n. Since E_n and E_m are disjoint if $n \neq m$, the desired probability is

$$\sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1} \cdot \frac{1}{9} = \frac{1}{9} \cdot \frac{1}{1 - \frac{13}{18}} = \frac{1}{9} \frac{18}{5} = \frac{2}{5}.$$

Problem 27

$$P(A \text{ wins in one move}) = \frac{3}{10}$$
$$P(A \text{ wins in three moves}) = \frac{7}{10}\cdot\frac{6}{9}\cdot\frac{3}{8} = \frac{7}{40}$$
$$P(A \text{ wins in five moves}) = \frac{7}{10}\cdot\frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6} = \frac{1}{12}$$
$$P(A \text{ wins in seven moves}) = \frac{7}{10}\cdot\frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4} = \frac{1}{40}$$
$$P(A \text{ wins}) = \frac{3}{10}+\frac{7}{40}+\frac{1}{12}+\frac{1}{40} = \frac{7}{12}$$

Problem 28 (a) Without replacement:

$$P(\text{all three balls are the same color}) = \frac{\binom{5}{3}+\binom{6}{3}+\binom{8}{3}}{\binom{19}{3}}$$

With replacement:

$$P(\text{all three balls are the same color}) = \left(\frac{5}{19}\right)^3+\left(\frac{6}{19}\right)^3+\left(\frac{8}{19}\right)^3$$

(b) Without replacement:

$$P(\text{all three balls are of different colors}) = \frac{\binom{5}{1}\cdot\binom{6}{1}\cdot\binom{8}{1}}{\binom{19}{3}}$$

With replacement:

$$P(\text{all three balls are of different colors}) = 3!\cdot\frac{5}{19}\cdot\frac{6}{19}\cdot\frac{8}{19}$$

Problem 32 There are $(b+g)!$ ways to line up the children. There are $g\cdot(b+g-1)!$ arrangements with a girl in the ith position. The desired probability is $\frac{g(b+g-1)!}{(b+g)!} = \frac{g}{b+g}$.