Third Homework Set — Solutions

Chapter 2

Problem 37 (a) There are $\binom{10}{5}$ selections for the final exam. The number of selections that allow the student to solve all problems is $\binom{7}{5}$, so that the desired probability is $\frac{\binom{7}{5}}{\binom{10}{5}} = 0.08333$.

(b) There are $\binom{7}{4} \cdot \binom{3}{1}$ selections that'll let the student solve exactly four problems, so that the probability of solving at least four problems is $\frac{\binom{7}{5}+\binom{7}{4}\cdot\binom{3}{1}}{\binom{10}{5}} = \frac{1}{2}$.

Problem 43 (a) There are $n!$ ways to arrange n people in a line. There are $2(n-1)!$ ways to arrange them such that A and B are next to each other. Hence, the probability of A and B being next to each other is $\frac{2(n-1)!}{n!} = \frac{2}{n}$.

(b) If $n = 2$, then A and B will always be next to each other. Now, assume that $n > 3$. After A picks a seat, there are $n-1$ seats left, two of which are next to A, so that the desired probability is $\frac{2}{n-1}$.

Problem 50 The probability that you have five spades and your partner has the remaining eight spades is

$$\frac{\binom{13}{5}\binom{39}{8}\binom{8}{8}\binom{31}{5}}{\binom{52}{13,13,26}} = 2.6084 \cdot 10^{-6}.$$

Problem 53 Let E_i be the event that the i-th couple sit together, for $j = 1, \ldots, 4$. Then $P(E_i) = \frac{2}{8} = \frac{1}{4}$ (Problem 43(a)). Moreover, if $i < j$, then $P(E_iE_j) = \frac{2^2 \cdot 6!}{8!}$. Similarly, if $i < j < k$, then $P(E_iE_jE_k) = \frac{2^3 \cdot 5!}{8!}$. Finally, we have $P(E_1E_2E_3E_4) = \frac{2^4 \cdot 4!}{8!}$. Using inclusion-exclusion, we obtain

$$\begin{aligned}
P(E_1 \cup E_2 \cup E_3 \cup E_4) &= \sum_{i=1}^{4} P(E_i) - \sum_{i<j} P(E_iE_j) + \sum_{i<j<k} P(E_iE_jE_k) - P(E_1E_2E_3E_4) \\
&= 4 \cdot \frac{1}{4} - \binom{4}{2}\frac{2^2 \cdot 6!}{8!} + \binom{4}{3}\frac{2^3 \cdot 5!}{8!} - \frac{2^4 \cdot 4!}{8!} \\
&= 1 - \frac{3}{7} + \frac{2}{21} - \frac{1}{105} = \frac{23}{35}.
\end{aligned}$$

Hence, the probability that no husband sits next to his wife is $1 - \frac{23}{35} = \frac{12}{35}$.

Problem 54 Let S, H, C, and D be the event that spades are missing, hearts are missing, etc. Then

$$\begin{aligned} P(S \cup H \cup C \cup D) =& P(S) + P(H) + P(C) + P(D) \\ & - P(SH) - P(SC) - P(SD) - P(HC) - P(HD) - P(CD) \\ & + P(SHC) + P(SHD) + P(SCD) + P(HCD) \\ & - P(SHCD) \\ =& 4 \cdot \frac{\binom{39}{13}}{\binom{52}{13}} - 6 \cdot \frac{\binom{26}{13}}{\binom{52}{13}} + 4 \cdot \frac{1}{\binom{52}{13}} - 0 \\ =& 0.0511. \end{aligned}$$

Chapter 3

Problem 1 Let E be the event that at least one die lands on six, and let F be the event that the dice land of different numbers. Then $P(EF) = 2 \cdot \frac{1}{6} \cdot 56 = \frac{5}{18}$ and $P(F) = \frac{30}{36} = \frac{5}{6}$. Hence,

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{5}{18}}{\frac{5}{6}} = \frac{1}{3}.$$

Problem 5

$$\frac{6 \cdot 5 \cdot 9 \cdot 8}{15 \cdot 14 \cdot 13 \cdot 12} = \frac{6}{91}$$

Problem 6 Let A be the event that the sample drawn contains exactly three white balls. Let B be the event that the first and third ball drawn are white.

Without replacement

$P(A) = 4\frac{8 \cdot 7 \cdot 6 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 9}$ and $P(AB) = 2\frac{8 \cdot 7 \cdot 6 \cdot 4}{12 \cdot 11 \cdot 10 \cdot 9}$, hence $P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2}$.

With replacement

$P(A) = \binom{4}{3}\left(\frac{2}{3}\right)^3 \frac{1}{3}$ and $P(AB) = 2\left(\frac{2}{3}\right)^3 \frac{1}{3}$, hence $P(B|A) = \frac{P(AB)}{P(A)} = \frac{1}{2}$.

Problem 9 Let E_i be the event that the ball drawn from the i-th urn is white, for $i = 1, 2, 3$. Let F be the event that exactly two white balls were drawn. Then

$$\begin{aligned} P(E_1|F) &= \frac{E_1 F}{P(F)} \\ &= \frac{P(E_1E_2E_3^c) + P(E_1E_2^cE_3)}{P(E_1E_2E_3^c) + P(E_1E_2^cE_3) + P(E_1^cE_2E_3)} \\ &= \frac{\frac{2\cdot 8\cdot 3}{6\cdot 12\cdot 4} + \frac{2\cdot 4\cdot 1}{6\cdot 12\cdot 4}}{\frac{2\cdot 8\cdot 3}{6\cdot 12\cdot 4} + \frac{2\cdot 4\cdot 1}{6\cdot 12\cdot 4} + \frac{4\cdot 8\cdot 1}{6\cdot 12\cdot 4}} \\ &= \frac{7}{11}. \end{aligned}$$

Problem 10 For $i = 1, 2, 3$, let E_i be the event that the i-th card is a spade. Then

$$P(E_1E_2E_3) = \frac{13}{52}\frac{12}{51}\frac{11}{50}$$

and

$$P(E_2E_3) = P(E_1E_2E_3) + P(E_1^cE_2E_3) = \frac{13}{52}\frac{12}{51}\frac{11}{50} + \frac{39}{52}\frac{13}{51}\frac{12}{50}.$$

Thus

$$P(E_1|E_2E_3) = \frac{P(E_1E_2E_3)}{P(E_2E_3)} = \frac{11}{50}.$$

Problem 20 (a) $P(F|C) = \frac{P(FC)}{P(C)} = \frac{.02}{.05} = \frac{2}{5}$.

(b) $P(C|F) = \frac{P(FC)}{P(F)} = \frac{.02}{.52} = \frac{1}{26}$.

Problem 23 (a) Let W be the event that the ball selected urn II is white, E be the event that the transfered ball is white and F be the event that the transfered ball is red, then

$$P(W) = P(E)P(W|E) + P(F)P(W|F) = \frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3} = \frac{4}{9}.$$

(b)

$$P(E|W) = \frac{P(EW)}{P(W)} = \frac{P(E)P(W|E)}{P(W)} = \frac{1}{2}.$$

Problem 30 Let B and W be the events that the marble is black and white respectively, and let B_i be the event that box i is chosen. Then

$$P(B) = P(B_1)P(B|B_1) + P(B_2)P(B|B_2) = \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{2}{3} = \frac{2}{12},$$

$$P(B_1|W) = \frac{P(B_1W)}{P(W)} = \frac{P(B_1)P(W|B_1)}{P(W)} = \frac{\frac{1}{2}\frac{1}{2}}{\frac{5}{12}} = \frac{3}{5}.$$

Problem 47 (a)

$$P(\text{all white}) = \frac{1}{6}(\frac{5}{15}+\frac{5}{15}\frac{4}{14}+\frac{5}{15}\frac{4}{14}\frac{3}{13}+\frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}+\frac{5}{15}\frac{4}{14}\frac{3}{13}\frac{2}{12}\frac{1}{11}).$$

(b)

$$P(3|\text{all white}) = \frac{\frac{1}{6}\frac{5}{15}\frac{4}{14}\frac{3}{13}}{P(\text{all white})}.$$

Problem 51 Let R be the event that she receives a job offer, S be the event that event of a strong recommendation, M the event of a moderate recommendation and W the event of a weak recommendation.

(a)

$$P(R) = P(S)P(R|S) + P(M)P(R|M) + P(W)P(R|W)$$
$$= (.8)(.7) + (.4)(.2) + (.1)(.1) = .65.$$

(b)

$$P(S|R) = \frac{P(SR)}{P(R)} = \frac{P(S)P(R|S)}{P(R)} = \frac{(.8)(.7)}{.65} = \frac{56}{65}.$$

Similarly

$$P(M|R) = \frac{8}{65}, \quad P(W|R) = \frac{1}{65}.$$

Problem 56

$$P(\text{new}) = \sum_{i=1}^{m} p_i P(\text{new}|\text{the } n\text{-th coupon is of type i}) = \sum_{i=1}^{m} p_i(1-p_i)^{n-1}.$$