Seventh Homework Set — Solutions
Chapter 5

Problem 1 (a) We have $1 = \int_{-1}^{1} c(1-x^2)dx = cx\left(1-\frac{x^2}{3}\right)|_{-1}^{1} = \frac{4}{3}c$, so that $c = \frac{3}{4}$.

(b) We have $\int_{-1}^{x} f(y)dy = \frac{3}{4}y\left(1-\frac{y^2}{3}\right)|_{-1}^{x} = \frac{1}{2} + \frac{3}{4}x\left(1-\frac{x^2}{3}\right)$ if $-1 \leq x \leq 1$. Hence,

$$F(x) = \begin{cases} 0 & x < -1, \\ \frac{1}{2} + \frac{3}{4}x\left(1-\frac{x^2}{3}\right) & -1 \leq x \leq 1, \\ 1 & x > 1. \end{cases}$$

Problem 2 Determine C: $\int_0^\infty xe^{-\frac{x}{2}}dx = -2xe^{-\frac{x}{2}}|_0^\infty + \int_0^\infty 2e^{-\frac{x}{2}}dx = (-2x-4)e^{-\frac{x}{2}}|_0^\infty = 4$, so that $C = \frac{1}{4}$.

Now, we have $P\{X \geq 5\} = \int_5^\infty \frac{1}{4}xe^{-\frac{x}{2}} = -(\frac{x}{2}+1)e^{-\frac{x}{2}}|_5^\infty = \frac{7}{2}e^{-\frac{5}{2}}$

Problem 4 (a) $P\{X > 20\} = \int_{20}^\infty \frac{10}{x^2}dx = -\frac{10}{x}|_{20}^\infty = \frac{1}{2}$.

(b)

$$F(x) = \begin{cases} 0 & x < 10 \\ 1 - \frac{10}{x} & x \geq 10 \end{cases}$$

(c) Let's assume that lifetimes of the six devices are independent of each other. Let $p = 1 - F(15)$. Then the desired probability is

$$\sum_{i=3}^{6} \binom{6}{i} p^i (1-p)^{6-i}.$$

Problem 5 We want to find C such that $F(C) \geq 0.99$. We have $F(C) = \int_0^C 5(1-x)^4 dx = -(1-x)^5|_0^C = 1-(1-C)^5$. We want $1-(1-C)^5 \geq 0.99$, i.e., $(1-C)^5 \leq 0.01$, hence $C \geq 1-(0.01)^{0.2}$.

Problem 6 (a)

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} xf(x)dx = \frac{1}{4}\int_0^\infty x^2 e^{-\frac{x}{2}}dx \\ &= \frac{1}{4}\left(-2x^2 - 8x - 16\right)e^{-\frac{x}{2}}|_0^\infty = 4 \end{aligned}$$

(b) $E[X] = \int_{-1}^{1} c(1-x^2)x dx = 0$ by symmetry

(c) $E[X] = \int_5^\infty x \frac{5}{x^2} dx = \int_5^\infty \frac{5}{x} = \infty$

Problem 10 (a) Let X be uniform on $[0, 60]$. Then

$$\begin{aligned}
&P(\text{passenger goes to } A) \\
&= P\{5 \le X < 15\} + P\{20 \le X < 30\} P\{35 \le X < 45\} \\
&\quad + P\{50 \le X < 60\} \\
&= \frac{2}{3}.
\end{aligned}$$

(b) Same as above.

Problem 12 If service stations are located in A, B, and the center, then the distance between two service stations is 50 miles, so that the expected distance from a service station at the time of a breakdown is

$$\frac{1}{50}\left(\int_0^{25} x dx + \int_{25}^{50} (50-x) dx\right) = \frac{1}{50}\left(\frac{25^2}{2} + 25 \cdot 50 - \frac{50^2}{2} + \frac{25^2}{2}\right) = 12.5.$$

If the service stations are located at mile 25, 50, and 75, then the expected distance from a station at the time of a breakdown is

$$\begin{aligned}
&\frac{1}{50}\left(\int_0^{25} x dx + \int_{25}^{37.5} (x-25) x dx + \int_{37.5}^{50} (50-x) dx\right) \\
&= \frac{1}{50}\left(\frac{25^2}{2} + 2\frac{12.5^2}{2}\right) = 9.375.
\end{aligned}$$

The second strategy is more efficient.

Problem 13 (a) $P\{X > 10\} = \frac{2}{3}$

(b) $P\{X > 25 | X > 15\} = \frac{P\{X>25\}}{P\{X>15\}} = \frac{\frac{5}{30}}{\frac{15}{30}} = \frac{1}{3}$.