Math 561 Final, Spring 2009

Show all work to qualify for full credits

1. (17 points) Suppose that $X_1, X_2, \ldots,$ are independent and identically distributed random variables with

$$P(X_1 = n) = P(X_1 = -n) = \frac{c}{n^2 \log n}, \quad n = 3, 4, \ldots$$

for some constant $c > 0$. For any $n \geq 1$, define $S_n = X_1 + \cdots + X_n$. Show that

$$\limsup_{n \to \infty} \frac{S_n}{n} = \infty, \quad \liminf_{n \to \infty} \frac{S_n}{n} = -\infty.$$

2. (17 points) Let $X_1, X_2, \cdots$ be independent random variables with $EX_i = 0$ and $\mathrm{Var}(X_i) \leq C < \infty$. Put $S_n = X_1 + \cdots + X_n$ and $D_n = \max_{n^2 \leq k < (n+1)^2} |S_k - S_{n^2}|$. (a) Use the Borel-Cantelli lemma to show that

$$\lim_{n \to \infty} \frac{S_{n^2}}{n^2} = 0$$

almost surely. (b) Use the Borel-Cantelli lemma to show that

$$\lim_{n \to \infty} \frac{D_n}{n^2} = 0$$

almost surely. (c) Use the results of (a) and (b) to show that

$$\lim_{n \to \infty} \frac{S_n}{n} = 0$$

almost surely.

3. (16 points) Suppose that $\{X_n : n \geq 1\}$ is a sequence of real-valued random variables converging weakly to a real-valued random variable X, and that $\{Y_n : n \geq 1\}$ is a sequence of nonnegative random variables converging weakly a positive constant c. Show that the sequence $\{X_n Y_n : n \geq 1\}$ converges weakly to cX.

4. (16 points) Suppose that $X_1, X_2, \ldots$ are independent and identically distributed nonnegative random variables with $EX_1 = 1$, $\mathrm{Var}(X_1) = 1$. Let $S_n = X_1 + \cdots + X_n$. Show that $2(\sqrt{S_n} - \sqrt{n})$ converges weakly to a standard normal random variable as $n \to \infty$.

5. (17 points) Let X_n, $n \geq 1$, be independent nonnegative random variables with $EX_n = 1$ for each $n \geq 1$. For each n, define $a_n = E(\sqrt{X_n})$,

$$M_n = X_1 X_2 \cdots X_n$$

and

$$N_n = \frac{\sqrt{X_1}}{a_1} \frac{\sqrt{X_2}}{a_2} \cdots \frac{\sqrt{X_n}}{a_n}.$$

(a) Show that M_n and N_n are nonnegative martingales. (b) Show that if $\prod_{n=1}^{\infty} a_n > 0$, then $\sup_n EN_n^2 < \infty$. (c) Show that, under the assumption that $\prod_{n=1}^{\infty} a_n > 0$, the martingale M_n converges in L^1 as $n \to \infty$.

6. (17 points) Let S_n be a one-dimensional simple symmetric random walk starting at 0. For any positive integer a, let $T = \inf\{n : S_n \notin (-a, a)\}$. (a) Show that $E(T) = a^2$. (b) Find constants b and c so that $Y_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale. (c) Find $E(T^2)$.