Stable random variables and infinitely divisible random variables

In this lecture I will give a brief introduction to stable random variables and infinitely divisible random variables. I am not going to present proofs of the results. For details of the proof, one can refer to [1] or [2].

Recall that, for any $\alpha \in (0, 2]$, a random variable is called a symmetric α-stable random variable if its characteristic function is given by

$$\phi(t) = e^{-c|t|^\alpha}, \quad t \in \mathbb{R}$$

for some $c > 0$. A symmetric 2-stable random variable is simply a symmetric normal random variable.

Definition 0.1 *A random variable X is called a stable random variable (or said to have a stable distribution) if for every positive integer $n > 1$, and $X_1, \ldots, X_n$ independent with the same distribution as X, there exists constants $a_n > 0$ and b_n such that $(X_1 + X_2 + \cdots + X_n - b_n)/a_n$ has the same distribution as X, or equivalently, $X_1 + \cdots + X_n$ has the same distribution as $a_n X + b_n$.*

If we use ϕ to denote the characteristic function of X, then X is stable if and only if for every positive integer $n > 1$, there exist constants $a_n > 0$ and b_n such that

$$\phi^n(t) = e^{b_n t}\phi(a_n t), \quad t \in \mathbb{R}.$$

The following result explains the role of stable random variables play in limit theorems.

Theorem 0.2 *A random variable X is the weak limit of some normalized sums*

$$\frac{X_1 + \cdots + X_n - B_n}{A_n}$$

of some independent identically distributed random variables $X_1, X_2, \ldots$ with $A_n > 0$ and $B_n \in \mathbb{R}$ if and only if X is stable.

The following result gives the general form for the characteristic function of a stable random variable.

Theorem 0.3 *Suppose that X is a stable random variable. Then either has a normal distribution, or there exist $\alpha \in (0, 2)$ and constants $m_1 \geq 0$, $m_2 \geq 0$ and $\beta \in \mathbb{R}$ such that the logarithm of the characteristic function ϕ of X is given by*

$$\begin{aligned}\log \phi(t) &= i\beta t + m_1 \int_0^\infty \left(e^{itx} - 1 - itx1_{\{x<1\}}\right)\frac{dx}{x^{1+\alpha}} \\ &\quad + m_1 \int_{-\infty}^0 \left(e^{itx} - 1 - itx1_{\{x>-1\}}\right)\frac{dx}{x^{1+\alpha}}\end{aligned}$$

The constant α is called the stability index (or stability exponent) of the stable random variable X. The stability index of a normal random variable is 2.

Theorem 0.4 *$\phi(t) = \exp(\psi(t))$ is the characteristic function of an α-stable random variable, $\alpha \in (0, 2)$, if and only if*

$$\psi(t) = itc - b|t|^\alpha \left(1 + i\kappa \operatorname{sgn}(t) w_\alpha(t)\right), \quad t \in \mathbb{R}$$

where $b > 0, c \in \mathbb{R}$, $\kappa \in [-1, 1]$ and

$$w_\alpha(t) = \begin{cases} \tan(\frac{\alpha\pi}{2}) & \alpha \neq 1 \\ \frac{2}{\pi} \log|t| & \alpha = 1. \end{cases}$$

It is easy to check that a random variable X is α-stable if and only if for every positive integer $n > 1$, there exists a constant b_n such that the characteristic function of X satisfies

$$\phi^n(t) = e^{b_n t} \phi(n^{1/\alpha} t), \quad t \in \mathbb{R}.$$

Definition 0.5 *A random variable X is said to be infinitely divisible if for every integer $n > 1$, there are n independent identically distributed random variables $X_1^{(n)}, \dots, X_n^{(n)}$ such that $X_1^{(n)} + \cdots + X_n^{(n)}$ has the same law as X.*

It is easy to see that a random variable X is infinitely divisible if and only if the characteristic function ϕ of X satisfies the following property: for every integer $n > 1$, $(\phi(t))^{1/n}$ is the characteristic function of a random variable. Obviously, stable random variables are infinitely divisible.

Theorem 0.6 *A random variable X is the weak limit of sums of the form*

$$X_1^{(n)} + \cdots + X_n^{(n)}$$

where $X_1^{(n)}, \dots, X_n^{(n)}$ are independent and identically distributed if and only if X is infinitely divisible.

Theorem 0.7 *A random variable X is infinitely divisible if and only if its characteristic function ϕ is given by*

$$\log \phi(t) = i\beta t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - itx 1_{\{|x|<1\}}\right) \nu(dx)$$

for some $\beta \in \mathbb{R}$, $\sigma \geq 0$ and a measure ν such that $\int (x^2 \wedge 1) \nu(dx) < \infty$.

The formula above is called the Lévy-Khintchine formula. The measure ν is called the Lévy measure of the random variable X.

Other examples of infinitely divisible random variables: Poisson random variables, exponential random variables and Gamma random variables.

References

[1] L. Breiman, *Probbaility*, Addison-Wesley, 1968.

[2] B. V. Gnedenco and A. V. Kolmogorov, *Limit distributions for sums of independent random variables*, Addison-Wesley, 1954.