0.1 Gaussian processes

An $\mathbb{R}^d$-valued random variable $\xi = (\xi_1, \dots, \xi_d)$ is called a Gaussian random variable if, for any $y \in \mathbb{R}^d$, $\langle y, \xi \rangle$ is a real-valued Gaussian random variable.

This is equivalent to saying that ξ has characteristic function

$$\phi_\xi(y) = \mathbb{E}^{i\langle y,\xi\rangle} = \exp\left(i\mathbb{E}\langle y, \xi\rangle - \frac{\mathrm{Var}(\langle y, \xi\rangle)}{2}\right), \quad \forall y \in \mathbb{R}^d. \tag{0.1}$$

If ξ is an $\mathbb{R}^d$-valued Gaussian random variable, we define, for $j = 1, \dots, d$, $m_j = \mathbb{E}\xi_j$. Then $m = (m_1, \dots, m_d)$ is called the mean of ξ. The matrix $\Sigma = (\Sigma_{jk})$ defined by

$$\Sigma_{jk} = \mathbb{E}\left((\xi_j - m_j)(\xi_k - m_k)\right)$$

is called the covariance matric of ξ. The characteristic function of ξ can be written in the form

$$\phi_\xi(y) = \exp\left(i\langle y, m\rangle - \frac{1}{2}\langle y, \Sigma y^T\rangle\right), \quad \forall y \in \mathbb{R}^d.$$

It is easy to check that the covariance matric of any $\mathbb{R}^d$-valued Gaussian random variable ξ is symmetric and nonnegative definite.

0.1 Lemma. *Let ξ be an $\mathbb{R}^d$-valued Gaussian random variable with mean m and a diagonal covariance matrix. Then $\xi_1, \dots, \xi_d$ are independent.*

Proof. Omitted. ∎

0.2 Proposition. *For any $m \in \mathbb{R}^d$ and any symmetric nonnegative definite $d \times d$ matrix Σ, there is an $\mathbb{R}^d$-valued Gaussian random variable ξ with mean m and covariance matrix Σ.*

Proof. This is left as a homework problem. ∎

0.3 Lemma. *Let ξ be an $\mathbb{R}^d$-valued Gaussian random variable with mean m and a positive definite covariance matrix Σ. Then Ξ has a density function given by*

$$f(x) = \frac{1}{(2\pi)^{d/2}(\det\Sigma)^{1/2}} \exp\left(\frac{1}{2}\langle x - m, \Sigma^{-1}(x - m)^T\rangle\right), \quad \forall x \in \mathbb{R}^d.$$

Proof. Omitted. ∎

A function $\Sigma(s,t)$ on $\mathbb{R}_+ \times \mathbb{R}_+$ is said to be nonnegative definite if for any $n \geqslant 1$ and $t_1, \ldots, t_n$ in $\mathbb{R}_+$, the $n \times n$ matrix Σ defined by $\Sigma_{jk} = \Sigma(t_j, t_k)$ is symmetric and nonnegative definite.

A function ϕ on $\mathbb{R}_+$ is said to be nonnegative definite if the function $\Sigma(s,t) = \phi(s-t)$ is nonnegative definite.

A real-valued process $X = (X_t : t \geqslant 0)$ is called a Gaussian process if, for any $n \geqslant 1$ and $0 \leqslant t_1 \leqslant \ldots \leqslant t_n < \infty$, $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian random variable.

If X is a Gaussian process, we define

$$m(t) = \mathbb{E}X_t, \quad t \geqslant 0$$

and

$$\Sigma(s,t) = \mathbb{E}((X_s - m(s)(X_t - m(t))), \quad s,t \geqslant 0.$$

$m(t)$ is called the mean function of X and $\Sigma(s,t)$ is called the covariance kernel of X. If $m(t) = 0$ for all $t \geqslant 0$, then X is called a centered Gaussian process.

A Gaussian process is characterized by its mean function and covariance kernel.

The covariance kernel of any Gaussian process is nonnegative definite.

0.4 Proposition. *For any real-valued function $m(t)$ on $\mathbb{R}_+$ and any nonegative definite function Σ on $\mathbb{R}_+ \times \mathbb{R}_+$, there is a Gaussian process $X = (X_t : t \geqslant 0)$ with mean function m and covariance kernel Σ.*

Proof. This is a consequence of the Kolmogorov exttension theorem. We omit the details. ∎

0.5 Proposition. *A standard 1-dimensional Brownian motion is a centered Gaussian process with covariance kernel $\Sigma(s,t) = s \wedge t$.*

Proof. Easy to check. ∎