Math 562, Final Homework

Due 12/09

1. Let B be a standard one-dimensional Brownian motion and let $\{\mathcal{F}_t\}$ be the Brownian filtration associated with B. Show that if T is a stopping time with respect to $\{\mathcal{F}_t\}$ with $E[T] < \infty$, then

$$E[B_T] = 0, \quad E[B_T^2] = E[T].$$

2. For any two continuous semimartingales

$$X_t = X_0 + M_t + A_t, \quad Y_t = Y_0 + N_t + C_t$$

where M and N are continuous local martinagles, and A and C are adapted continuous processes of finite variation with $A_0 = C_0 = 0$, the Stratonovich integral of Y with respect to X is defined as

$$\int_0^t Y_s \circ dX_s = \int_0^t Y_s dM_s + \int_0^t Y_s dA_s + \frac{1}{2} < M, N >_t, \quad t \geq 0.$$

Suppose that $X = (X^{(1)}, \ldots, X^{(d)})$ is a vector of continuous semimartingales with

$$X_t^{(i)} = X_0^{(i)} + M_t^{(i)} + A_t^{(i)}, \quad t \geq 0, i = 1, \ldots, d$$

where each $M^{(i)}$ is a continuous local martingale and each $A^{(i)}$ is an adapted continuous processes of finite variation vanishing at zero. Show that if $f : \mathbf{R}^d \to \mathbf{R}$ is of class C^3, then

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(X_s) \circ dX_s^{(i)}.$$

3. Let $d \geq 2$ be an integer and let $B = \{B_t : t \geq 0\}$, $\{P_x\}_{x \in \mathbf{R}^d}$ be a d-dimensional Brownian motion on $(\Omega, \mathcal{F})$. The process $R = \{R_t = \|B_t\| : t \geq 0\}$ together with the family of probability measures $\{\hat{P}_{(r)}\}_{r \geq 0} = \{P_{(r,0,\ldots,0)}\}_{r \geq 0}$ on $(\Omega, \mathcal{F})$ is called a Bessel process of dimension d, where $\| \cdot \|$ stands for the Euclidean norm on $\mathbf{R}^d$. Show that the Bessel process of dimension d starting at $r \geq 0$ satisfies the integral equation

$$R_t = r + \int_0^t \frac{d-1}{2R_s} ds + W_t, \quad t \geq 0$$

where $W = \{W_t, t \geq 0\}$ is the standard Brownian motion

$$W_t = \sum_{i=1}^d W_t^{(i)} \quad \text{with } W_t^{(i)} = \int_0^t \frac{B_s^{(i)}}{R_s} dB_s^{(i)}, 1 \leq i \leq d.$$