

Dirichlet Heat Kernel Estimates for $\Delta^{\alpha/2} + \Delta^{\beta/2}$

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Abstract

For $d \geq 1$ and $0 < \beta < \alpha < 2$, consider a family of pseudo differential operators $\{\Delta^\alpha + a^\beta \Delta^{\beta/2}; a \in [0, 1]\}$ on \mathbb{R}^d that evolves continuously from $\Delta^{\alpha/2}$ to $\Delta^{\alpha/2} + \Delta^{\beta/2}$. It gives rise to a family of Lévy processes $\{X^a, a \in [0, 1]\}$ on \mathbb{R}^d , where each X^a is the independent sum of a symmetric α -stable process and a symmetric β -stable process with weight a . For any $C^{1,1}$ open set $D \subset \mathbb{R}^d$, we establish explicit sharp two-sided estimates (uniform in $a \in [0, 1]$) for the transition density function of the subprocess $X^{a,D}$ of X^a killed upon leaving the open set D . The infinitesimal generator of $X^{a,D}$ is the non-local operator $\Delta^\alpha + a^\beta \Delta^{\beta/2}$ with zero exterior condition on D^c . As consequences of these sharp heat kernel estimates, we obtain uniform sharp Green function estimates for $X^{a,D}$ and uniform boundary Harnack principle for X^a in D with explicit decay rate.

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1 Introduction

It is well-known that, for a second order elliptic differential operator \mathcal{L} on \mathbb{R}^d satisfying some natural conditions, there is a diffusion process X on \mathbb{R}^d with \mathcal{L} as its infinitesimal generator. The fundamental solution $p(t, x, y)$ of $\partial_t u = \mathcal{L}u$ (also called the heat kernel of \mathcal{L}) is the transition density function of X . Thus obtaining sharp two-sided estimates for $p(t, x, y)$ is a fundamental problem in both analysis and probability theory. Such relationship is also true for a large class of Markov processes with discontinuous sample paths, which constitute an important family of stochastic processes in probability theory. They have been widely used in various applications.

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One of the most important and most widely used family of Markov processes is the family of (rotationally) symmetric α -stable processes on \mathbb{R}^d , $0 < \alpha \leq 2$. A symmetric α -stable process $X = \{X_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$ on \mathbb{R}^d is a Lévy process such that

$$\mathbb{E}_x \left[e^{i\xi \cdot (X_t - X_0)} \right] = e^{-t|\xi|^\alpha} \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

When $\alpha = 2$, X is a Brownian motion on \mathbb{R}^d whose infinitesimal generator is the Laplacian Δ . When $0 < \alpha < 2$, the infinitesimal generator of a symmetric α -stable process X in \mathbb{R}^d is the fractional Laplacian $\Delta^{\alpha/2}$, which is a prototype of nonlocal operators. The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2} u(x) = \mathcal{A}(d, -\alpha) \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |y-x| > \varepsilon\}} (u(y) - u(x)) \frac{dy}{|x-y|^{d+\alpha}} \quad (1.1)$$

for some constant $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$. Here and in the sequel, we use $:=$ as a way of definition. Here Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$ for every $\lambda > 0$.

Two-sided heat kernel estimates for diffusions in \mathbb{R}^d have a long history and many beautiful results have been established. See [11, 13] and the references therein. But, due to the complication near the boundary, two-sided estimates for the transition density functions of killed diffusions in a domain D (equivalently, the Dirichlet heat kernels) have been established only recently. See [12, 13, 14] for upper bound estimates and [29] for the lower bound estimates of the Dirichlet heat kernels in bounded $C^{1,1}$ domains. In a recent paper [3], we succeeded in establishing sharp two-sided estimates for the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$ with zero exterior condition on D^c (or equivalently, the transition density function of the killed α -stable process) in any $C^{1,1}$ open set.

The approach developed in [3] can be adapted to establish heat kernel estimates of other jump processes in open subsets of \mathbb{R}^d . In [4], the ideas of [3] were adapted to establish two-sided heat kernel estimates of censored stable processes in $C^{1,1}$ open subsets of \mathbb{R}^d . One of the main tools used in [4] is the boundary Harnack principle established in [2] and [17].

In [5] the ideas of [3] were adapted to establish two-sided heat kernel estimates of relativistic stable processes in $C^{1,1}$ open subsets of \mathbb{R}^d . One of main facts we used in [5] is that relativistic stable processes can be regarded as perturbations of symmetric stable processes in bounded open sets and therefore the Green functions of killed relativistic stable processes in bounded open sets are comparable to the Green functions of killed stable processes in the same open sets.

The goal of this paper is to prove sharp two-sided estimates for the independent sum of an α -stable process and a β -stable process, $0 < \beta < \alpha < 2$, in $C^{1,1}$ open subsets of \mathbb{R}^d . Note that these processes can not be obtained from symmetric stable processes through a combination of Girsanov transform and Feynman-Kac transform. So the method of [5] can not be used to establish the comparability of the Green functions of these processes and the Green functions of symmetric stable processes in bounded open sets. Since the differences of the Lévy measures of these processes and those of symmetric stable processes have infinite total mass, the method of [22] and [16] also could not be used to establish the comparability of the Green functions of these processes and the Green functions of symmetric stable processes in bounded open sets. The approach of this paper will be described in the second paragraph below after the statement of Corollary 1.2.

Let us first recall some basic facts about the independent sum of stable processes and state our main result.

Throughout the remainder of this paper, we assume that $d \geq 1$ and $0 < \beta < \alpha < 2$. The Euclidean distance between x and y will be denoted as $|x - y|$. We will use $B(x, r)$ to denote the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$.

Suppose X is a symmetric α -stable process and Y is a symmetric β -stable process on \mathbb{R}^d and that X and Y are independent. For any $a \geq 0$, we define X^a by $X_t^a := X_t + aY_t$. We will call the process X^a the independent sum of the symmetric α -stable process X and the symmetric β -stable process Y with weight a . The infinitesimal generator of X^a is $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$. Let $p^a(t, x, y)$ denote the transition density of X^a (or equivalently the heat kernel of $\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2}$) with respect to the Lebesgue measure on \mathbb{R}^d . We will use $p(t, x, y) = p^0(t, x, y)$ to denote the transition density of $X = X^0$. Recently it is proven in [8] that

$$p^1(t, x, y) \asymp \left(t^{-d/\alpha} \wedge t^{-d/\beta} \right) \wedge \left(\frac{t}{|x - y|^{d+\alpha}} + \frac{t}{|x - y|^{d+\beta}} \right) \quad \text{on } (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \quad (1.2)$$

Here and in the sequel, for $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$; for any two positive functions f and g , $f \asymp g$ means that there is a positive constant $c \geq 1$ so that $c^{-1}g \leq f \leq cg$ on their common domain of definition.

For every open subset $D \subset \mathbb{R}^d$, we denote by $X^{a,D}$ the subprocess of X^a killed upon leaving D . The infinitesimal generator of $X^{a,D}$ is $(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2})|_D$, the sum of two fractional Laplacians in D with zero exterior condition. It is known (see [8]) that $X^{a,D}$ has a Hölder continuous transition density $p_D^a(t, x, y)$ with respect to the Lebesgue measure.

Unlike the case of the symmetric α -stable process $X := X^0$, X^a does not have the stable scaling for $a > 0$. Instead, the following approximate scaling property is true and will be used several times in the rest of this paper: If $\{X_t^{a,D}, t \geq 0\}$ is the subprocess of X^a killed upon leaving D , then $\{\lambda^{-1}X_{\lambda^\alpha t}^{a,D}, t \geq 0\}$ is the subprocess of $\{X_t^{a\lambda^{(\alpha-\beta)/\beta}}, t \geq 0\}$ killed upon leaving $\lambda^{-1}D$. So for any $\lambda > 0$, we have

$$p_{\lambda^{-1}D}^{a\lambda^{(\alpha-\beta)/\beta}}(t, x, y) = \lambda^d p_D^a(\lambda^\alpha t, \lambda x, \lambda y) \quad \text{for } t > 0 \text{ and } x, y \in \lambda^{-1}D. \quad (1.3)$$

In particular, letting $a = 1$, $\lambda = a^{\beta/(\alpha-\beta)}$ and $D = \mathbb{R}^d$, we get

$$p^a(t, x, y) = a^{\frac{\beta d}{\alpha-\beta}} p^1(a^{\frac{\alpha\beta}{\alpha-\beta}} t, a^{\frac{\beta}{\alpha-\beta}} x, a^{\frac{\beta}{\alpha-\beta}} y) \quad \text{for } t > 0 \text{ and } x, y \in \mathbb{R}^d.$$

So we deduce from (1.2) that for any $M > 0$ there exists a constants $C > 1$ depending only on d, α, β and M such that for any $a \in (0, M]$ and $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$

$$C^{-1} f^a(t, x, y) \leq p^a(t, x, y) \leq C f^a(t, x, y), \quad (1.4)$$

where

$$f^a(t, x, y) := \left((a^\beta t)^{-d/\beta} \wedge t^{-d/\alpha} \right) \wedge \left(\frac{t}{|x - y|^{d+\alpha}} + \frac{a^\beta t}{|x - y|^{d+\beta}} \right).$$

The purpose of this paper is to establish the following two-sided sharp estimates on $p_D^a(t, x, y)$ in Theorem 1.1 for every $t > 0$. To state this theorem, we first recall that an open set D in \mathbb{R}^d (when $d \geq 2$) is said to be a (uniform) $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$

and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$ -function $\phi = \phi_z : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \leq \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(z)| \leq \Lambda_0|x - z|$, and an orthonormal coordinate system CS_z with its origin at z such that

$$B(z, R_0) \cap D = \{y = (\tilde{y}, y_d) \text{ in } CS_z : |y| < R_0, y_d > \phi(\tilde{y})\}.$$

The pair (R_0, Λ_0) is called the characteristics of the $C^{1,1}$ open set D . Note that a $C^{1,1}$ open set D with characteristics (R_0, Λ_0) can be unbounded and disconnected; the distance between two distinct components of D is at least R_0 . It is well known that any $C^{1,1}$ open set D satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition*: there exists $r_0 < R_0$ such that for every $x \in D$ with $\delta_{\partial D}(x) < r_0$ and $y \in \mathbb{R}^d \setminus \bar{D}$ with $\delta_{\partial D}(y) < r_0$, there are $z_x, z_y \in \partial D$ so that $|x - z_x| = \delta_{\partial D}(x)$, $|y - z_y| = \delta_{\partial D}(y)$ and that $B(x_0, r_0) \subset D$ and $B(y_0, r_0) \subset \mathbb{R}^d \setminus \bar{D}$ for $x_0 = z_x + r_0(x - z_x)/|x - z_x|$ and $y_0 = z_y + r_0(y - z_y)/|y - z_y|$. By a $C^{1,1}$ open set in \mathbb{R} we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is a positive constant R_0 .

Theorem 1.1 *Suppose $M > 0$. Let D be a $C^{1,1}$ open subset of \mathbb{R}^d and $\delta_D(x)$ the Euclidean distance between x and D^c .*

- (i) *For every $T > 0$, there is a positive constant $C_1 = C_1(D, M, \alpha, \beta, T) \geq 1$ such that for every $a \in (0, M]$,*

$$C_1^{-1} f_D^a(t, x, y) \leq p_D^a(t, x, y) \leq C_1 f_D^a(t, x, y),$$

where

$$f_D^a(t, x, y) := \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \left(\frac{t}{|x-y|^{d+\alpha}} + \frac{a^\beta t}{|x-y|^{d+\beta}}\right)\right).$$

- (ii) *Suppose in addition that D is bounded. For every $T > 0$, there is a constant $C_2 = C_2(D, M, \alpha, \beta, T) \geq 1$ so that for every $a \in (0, M]$ and $(t, x, y) \in [T, \infty) \times D \times D$,*

$$C_2^{-1} e^{-\lambda_1 t} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \leq p_D^a(t, x, y) \leq C_2 e^{-\lambda_1 t} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2},$$

where $\lambda_1 > 0$ is the smallest eigenvalue of $-(\Delta^{\alpha/2} + a^\beta \Delta^{\beta/2})|_D$.

Letting $a \rightarrow 0$, Theorem 1.1 recovers the heat kernel estimates for symmetric α -stable processes obtained in [3]. By integrating the two-sided heat kernel estimates in Theorem 1.1 with respect to t , we obtain the following estimates on the Green function $G_D^a(x, y) := \int_0^\infty p_D^a(t, x, y) dt$, which mean that, for bounded $C^{1,1}$ domains D , G_D^a and G_D^0 are comparable, see [9] and [20]. To the best of our knowledge, the Green function estimates in the corollary below are new.

Corollary 1.2 *Suppose $M > 0$. For any bounded $C^{1,1}$ -open set D in \mathbb{R}^d , there is a constant $C_3 = C_3(D, M, \alpha, \beta) \geq 1$ so that for every $a \in (0, M]$,*

$$C_3^{-1} g_D(x, y) \leq G_D^a(x, y) \leq C_3 g_D(x, y) \quad \text{for } x, y \in D,$$

where

$$g_D(x, y) := \begin{cases} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) |x-y|^{\alpha-d} & \text{when } d > \alpha, \\ \log \left(1 + \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d = 1 = \alpha, \\ (\delta_D(x) \delta_D(y))^{(\alpha-1)/2} \wedge \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{|x-y|} & \text{when } d = 1 < \alpha. \end{cases} \quad (1.5)$$

Theorem 1.1(i) will be established through Theorems 2.8 and 3.5, which give the upper bound and lower bound estimates, respectively. Theorem 1.1(ii) is a consequence of the intrinsic ultracontractivity of X^a in a bounded open set and the continuity of eigenvalues proved in [10]. In fact, the upper bound estimates in both Theorem 1.1 and Corollary 1.2 hold for any open set D with a weak version of the *uniform exterior ball condition* in place of the $C^{1,1}$ condition, while the lower bound estimates in both Theorem 1.1 and Corollary 1.2 hold for any open set D with the *uniform interior ball condition* in place of the $C^{1,1}$ condition (see Theorems 2.8 and 3.5, and the proofs for Theorem 1.1(ii) and Corollary 1.2).

Although we follow the main ideas we developed in [3], there are several new difficulties in obtaining two-sided Dirichlet heat kernel estimates for X^a : Even though the boundary Harnack principle has been extended in [19] to a large class of pure jump Lévy processes including X^a , the explicit decay rate of harmonic functions of X^a near the boundary of D was unknown. Instead, following the approach in [6], we establish necessary estimates using suitably chosen subharmonic and superharmonic functions of the process X^a . As in [6], we need to use finite range (or truncated) symmetric β -stable process \widehat{Y}^λ obtained from Y by suppressing all its jumps of size larger than λ . The infinitesimal generator of \widehat{Y}^λ is

$$\widehat{\Delta}_\lambda^{\beta/2} u(x) := \mathcal{A}(d, -\alpha) \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: \varepsilon < |y-x| \leq \lambda\}} (u(y) - u(x)) \frac{dy}{|x-y|^{d+\beta}}. \quad (1.6)$$

When $\lambda = 1$, we will simply denote $\widehat{\Delta}_\lambda^{\beta/2}$ by $\widehat{\Delta}^{\beta/2}$. We first establish the desired estimates for the Lévy process $\widehat{X}^a := X + a\widehat{Y}^{1/a}$. The infinitesimal generator of \widehat{X}^a is $\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}$. The desired estimates for $X^a = X + aY$ can then be obtained by adding back those jumps of Y of size larger than $1/a$. To obtain the lower bound of $p^a(t, x, y)$, we use the Dirichlet heat kernel estimate for the fractional Laplacian in [3] and a comparison of the killed subordinate stable process with the subordinate killed stable process where we will use some of the results obtained in [26].

We like to point out that unlike [3] the boundary Harnack principle for X^a is not used in this paper, which indicates that it might be possible to obtain sharp heat kernel estimate for processes for which the boundary Harnack principle fails.

As a consequence of Corollary 1.2, we have the following uniform boundary Harnack principle with explicit decay rate.

Theorem 1.3 *Suppose that $M > 0$. For any $C^{1,1}$ open set D in \mathbb{R}^d with the characteristics (R_0, Λ) , there exists a positive constant $C_4 = C_4(\alpha, \beta, d, \Lambda, R_0, M) \geq 1$ such that for $a \in [0, M]$, $r \in (0, R_0]$, $Q \in \partial D$ and any nonnegative function u in \mathbb{R}^d that is harmonic in $D \cap B(Q, r)$ with respect to X^a and vanishes continuously on $D^c \cap B(Q, r)$, we have*

$$\frac{u(x)}{u(y)} \leq C_4 \frac{\delta_D^{\alpha/2}(x)}{\delta_D^{\alpha/2}(y)} \quad \text{for every } x, y \in D \cap B(Q, r/2). \quad (1.7)$$

Throughout this paper, we will use capital letters C_1, C_2, \dots to denote constants in the statements of results, and their labeling will be fixed. The lower case constants c_1, c_2, \dots will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lower case constants starts anew in every proof. The dependence of the constants on dimension d may not be mentioned explicitly. For every function f , let $f^+ := f \vee 0$. We will use ∂ to denote a cemetery point and for every function f , we extend its definition to ∂ by setting $f(\partial) = 0$. We will use dx to denote the Lebesgue measure in \mathbb{R}^d . For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure.

2 Upper bound estimate

Throughout this section we assume that D is an open set satisfying the uniform exterior ball condition with radius $r_0 > 0$ in the following sense: for every $z \in \partial D$ and $r \in (0, r_0)$, there is a ball B^z of radius r such that $B^z \subset \mathbb{R}^d \setminus \overline{D}$ and $\partial B^z \cap \partial D = \{z\}$. The goal of this section is to establish the upper bound for the transition density (heat kernel) $p_D^\alpha(t, x, y)$. One of the main difficulties of getting the upper bound for $p_D^\alpha(t, x, y)$ is to obtaining the correct boundary decay rate.

Recall that $\Delta^{\alpha/2}$ and $\widehat{\Delta}_\lambda^{\beta/2}$ are defined by (1.1) and (1.6). The next two lemmas can be proved by direct computation, whose proofs can be found in [17] and [6], respectively.

For $p > 0$, let $w_p(x) := (x_1^+)^p$.

Lemma 2.1 *For any $x \in (0, \infty) \times \mathbb{R}^{d-1}$, we have*

$$\Delta^{\alpha/2} w_{\alpha/2}(x) = 0. \quad (2.1)$$

Moreover, for every $p \in (\alpha/2, \alpha)$, there is a positive constant $C_5 = C_5(d, \alpha, p)$ such that for every $x \in (0, \infty) \times \mathbb{R}^{d-1}$

$$\Delta^{\alpha/2} w_p(x) = C_5 x_1^{p-\alpha}. \quad (2.2)$$

Lemma 2.2 *There are constants $R_* \in (0, 1)$, $C_6 > C_7 > 0$ depending on p, d and α only such that for every $x \in (0, R_*) \times \mathbb{R}^{d-1}$*

$$C_7 x_1^{p-\alpha} \leq \widehat{\Delta}^{\alpha/2} w_p(x) \leq C_6 x_1^{p-\alpha} \quad \text{for } \alpha/2 < p < \alpha, \quad (2.3)$$

$$|\widehat{\Delta}^{\alpha/2} w_p(x)| \leq C_6 |\log x_1| \quad \text{for } p = \alpha \quad (2.4)$$

and

$$|\widehat{\Delta}^{\alpha/2} w_p(x)| \leq C_6 \quad \text{for } p > \alpha. \quad (2.5)$$

In the remainder of this paper, R_* will always stand for the constant in Lemma 2.2. The following result and its proof are similar to Lemma 3.2 of [6] and the proof there. For reader's convenience, we spell out the details of the proof here.

Lemma 2.3 *Assume that $r_1 \in (0, 1/2]$ and $p \geq \frac{\alpha}{2}$. Let $\delta_1 := R_* \wedge (r_1/4)$, $U := \{z \in \mathbb{R}^d : r_1 < |z| < 3r_1/2\}$ and*

$$h_p(y) := \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2} \right)^p \mathbf{1}_{U \cap \{z_d > 0, |\tilde{z}| < r_1/2\}}(y).$$

Then there exist $C_i = C_i(\alpha, p, r_1) > 0$, $i = 8, \dots, 12$, such that

(i) when $p \in (\alpha/2, \alpha)$, we have for all $y \in \{z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + \delta_1, |\tilde{z}| < r_1/4\}$,

$$C_8 \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2} \right)^{p-\alpha} \leq \widehat{\Delta}^{\alpha/2} h_p(y) \leq C_9 \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2} \right)^{p-\alpha} \quad (2.6)$$

and

$$C_8 \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2} \right)^{p-\alpha} \leq \Delta^{\alpha/2} h_p(y) \leq C_9 \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2} \right)^{p-\alpha}; \quad (2.7)$$

(ii) when $p > \alpha$, we have

$$|\widehat{\Delta}^{\alpha/2} h_p(y)| \leq C_{10} \quad \text{for all } y \in \{z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + \delta_1, |\tilde{z}| < r_1/4\}; \quad (2.8)$$

(iii) when $p = \alpha/2$, we have

$$|\Delta^{\alpha/2} h_{\alpha/2}(y)| \leq C_{11} \quad \text{for all } y \in \{z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + \delta, |\tilde{z}| < r_1/4\}; \quad (2.9)$$

(iv) when $p = \alpha$, we have for every $y \in \{z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + \delta, |\tilde{z}| < r_1/4\}$,

$$|\widehat{\Delta}^{\alpha/2} h_{\alpha/2}(y)| \leq C_{12} \left| \log \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2} \right) \right|. \quad (2.10)$$

Proof. Let

$$\Gamma(\tilde{y}) := \sqrt{r_1^2 - |\tilde{y}|^2} \quad \text{and} \quad \underline{h}(y) := y_d - \Gamma(\tilde{y}), \quad y \in U.$$

Fix $x \in \{z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + R_* \wedge (r_1/8), |\tilde{z}| < r_1/4\}$ and choose a point $x_0 \in \partial B_+(0, r_1) := \{z_d > 0, |z| = r_1\}$ satisfying $\tilde{x} = \tilde{x}_0$. Denote by $\vec{n}(x_0)$ the inward unit normal vector at x_0 for the exterior ball $B(0, r_1)^c$ and set $\Phi(y) = \langle y - x_0, \vec{n}(x_0) \rangle$ for $y \in \mathbb{R}^d$. $\Pi = \{y : \Phi(y) = 0\}$ is the plane tangent to $\partial B_+(0, r_1)$ at the point x_0 . Let $\Gamma^* : \tilde{x} \in \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be the function describing the hyperplane Π , that is, $\langle (\tilde{x}, \Gamma^*(\tilde{x})) - x_0, \vec{n}(x_0) \rangle = 0$. We also let

$$\begin{aligned} E &:= \{y = (\tilde{y}, y_d) : y \in U, |y - x| < r_1/4\}, \\ A &:= \{y : \Gamma^*(\tilde{y}) > y_d > \Gamma(\tilde{y}), |y - x| < r_1/4\} \end{aligned}$$

and $\bar{h}(y) := (y_d - \Gamma^*(\tilde{y})) \mathbf{1}_{\{y_d > \Gamma^*(\tilde{y})\}}(y)$ for $y \in \mathbb{R}^d$. Since $\nabla \Gamma(\tilde{x}) - \nabla \Gamma^*(\tilde{x}) = 0$, by the mean value theorem

$$|\bar{h}(y) - \underline{h}(y)| \leq |\Gamma(\tilde{y}) - \Gamma^*(\tilde{y})| \leq \Lambda |\tilde{y} - \tilde{x}|^2, \quad y \in E. \quad (2.11)$$

Let $\delta_\Pi(y) = \text{dist}(y, \Pi)$ for $y \in \mathbb{R}^d$ and $U_{\Gamma^*} = \{y \in \mathbb{R}^d : y_d > \Gamma^*(\tilde{y})\}$. Let $b_x := \sqrt{1 + |\nabla \Gamma(\tilde{x})|^2}$ and

$$h_{x,p}(y) := (\bar{h}(y))^p.$$

Note that $h_{x,p}(x) = h_p(x)$ and $B(x, r_1/4) \cap U \subset E$. Since $\bar{h}(y) = b_x \delta_\Pi(y)$ on D_{Γ^*} , by Lemma 2.1,

$$\Delta^{\alpha/2} h_{x,\alpha/2}(x) = 0 \quad (2.12)$$

and, if $\alpha/2 < p < \alpha$,

$$\Delta^{\alpha/2} h_{x,p}(x) = c_1 b_x^p \delta_{\Pi}^{p-\alpha}(x) = c_1 b_x^\alpha (\underline{h}(x))^{p-\alpha} \quad (2.13)$$

for some $c_1 > 0$. By Lemma 2.2, there are constants $c_i > 0$, $i = 2 \dots 6$, such that for $y \in D_{\Gamma^*}$ and $\delta_{\Pi}(y) < R_*$, when $\alpha/2 < p < \alpha$,

$$c_2 (\underline{h}(x))^{p-\alpha} \leq c_3 b_x^p \delta_{\Pi}^{p-\alpha}(x) \leq \widehat{\Delta}^{\alpha/2} h_{x,p}(x) = b_x^p \widehat{\Delta}^{\alpha/2} (\delta_{\Pi}(x))^p \leq c_4 b_x^p \delta_{\Pi}^{p-\alpha}(x) \leq c_5 (\underline{h}(x))^{p-\alpha}, \quad (2.14)$$

when $p > \alpha$,

$$|\widehat{\Delta}^{\alpha/2} h_{x,p}(x)| = b_x^p |\widehat{\Delta}^{\alpha/2} (\delta_{\Pi}(x))^p| \leq c_6 \quad (2.15)$$

and when $p = \alpha$,

$$|\widehat{\Delta}^{\alpha/2} h_{x,p}(x)| \leq c_6 |\log(\underline{h}(x))|. \quad (2.16)$$

Note that

$$\begin{aligned} |\widehat{\Delta}^{\alpha/2} (h_p - h_{x,p})(x)| &= \mathcal{A}(d, -\alpha) \left| \lim_{\varepsilon \downarrow 0} \int_{\{1 \geq |y-x| > \varepsilon\}} \frac{(h_p(y) - h_{p,x}(y))}{|x-y|^{d+\alpha}} dy \right| \\ &\leq \mathcal{A}(d, -\alpha) \left| \int_{\{1 \geq |y-x| > r_1/4\}} \frac{(h_p(y) - h_{p,x}(y))}{|x-y|^{d+\alpha}} dy \right| \end{aligned} \quad (2.17)$$

$$\begin{aligned} &+ \mathcal{A}(d, -\alpha) \lim_{\varepsilon \downarrow 0} \int_{\{r_1/4 \geq |y-x| > \varepsilon\}} \frac{|h_p(y) - h_{p,x}(y)|}{|x-y|^{d+\alpha}} dy \\ &\leq c_7 + \mathcal{A}(d, -\alpha) \int_A \frac{h_p(y) + h_{p,x}(y)}{|x-y|^{d+\alpha}} dy + \mathcal{A}(d, -\alpha) \int_E \frac{|h_p(y) - h_{p,x}(y)|}{|x-y|^{d+\alpha}} dy \\ &=: c_7 + I_1 + I_2 \end{aligned} \quad (2.18)$$

and, similarly,

$$\begin{aligned} &|\Delta^{\alpha/2} (h_p - h_{x,p})(x)| \\ &\leq \mathcal{A}(d, -\alpha) \left| \int_{\{|y-x| > r_1/4\}} \frac{(h_p(y) - h_{p,x}(y))}{|x-y|^{d+\alpha}} dy \right| \\ &+ \mathcal{A}(d, -\alpha) \int_A \frac{h_p(y) + h_{p,x}(y)}{|x-y|^{d+\alpha}} dy + \mathcal{A}(d, -\alpha) \int_E \frac{|h_p(y) - h_{p,x}(y)|}{|x-y|^{d+\alpha}} dy =: I_3 + I_1 + I_2. \end{aligned} \quad (2.19)$$

Since for $y \in B(x, r_1/4)^c$,

$$|h_{x,p}(y) - h_{x,p}(x)| \leq c_8 |y-x|^p \quad \text{and} \quad |h_p(y)| \leq c_8$$

and $h_p(y) = 0$ for $|\tilde{y}| > r_1/2$, for $\alpha/2 \leq p < \alpha$ we get

$$\begin{aligned} I_3 &\leq \mathcal{A}(d, -\alpha) \int_{B(x, r_1/4)^c} \frac{|h_{x,p}(y) - h_{x,p}(x)|}{|x-y|^{d+\alpha}} dy + \mathcal{A}(d, -\alpha) \int_{B(x, r_1/4)^c \cap \{|\tilde{y}| \leq r_1/2\}} \frac{|h_p(y) - h_p(x)|}{|x-y|^{d+\alpha}} dy \\ &+ \mathcal{A}(d, -\alpha) \left| \int_{B(x, r_1/4)^c \cap \{|\tilde{y}| > r_1/2\}} \frac{h_p(x)}{|x-y|^{d+\alpha}} dy \right| \\ &\leq c_9 \int_{B(x, r_1/4)^c} \frac{1}{|x-y|^{d+\alpha-p}} dy + c_9 \int_{B(x, r_1/4)^c} \frac{1}{|x-y|^{d+\alpha}} dy \leq c_{10} < \infty. \end{aligned} \quad (2.20)$$

We claim that, if $p \geq \alpha/2$,

$$I_1 + I_2 \leq c_{11} < \infty. \quad (2.21)$$

Note that for $y \in A$

$$\begin{aligned} |h_{x,p}(y)| + |h_p(y)| &\leq |y_d - \Gamma^*(\tilde{y})|^p + |y_d - \Gamma(\tilde{y})|^p \leq 2|\Gamma(\tilde{y}) - \Gamma^*(\tilde{y})|^p \\ &\leq 2|\Gamma(\tilde{y}) - \Gamma(\tilde{x}) - \nabla\Gamma(\tilde{x}) \cdot (\tilde{y} - \tilde{x})|^p \leq 2c_{12}^p |\tilde{y} - \tilde{x}|^{2p}. \end{aligned} \quad (2.22)$$

Furthermore, since $|\Gamma(\tilde{y}) - \Gamma^*(\tilde{y})| \leq c_{13}|\tilde{y} - \tilde{x}|^2 \leq c_{12}r^2$ on $|y - x| = r$, this together with (2.22) yields that

$$\begin{aligned} I_1 &\leq c_{14} \int_0^{r_1/4} r^{2p-\alpha-d} \int_{|y-x|=r} \mathbf{1}_A(y) m_{d-1}(dy) dr \\ &= c_{14} \int_0^{r_1/4} r^{2p-\alpha-d} m_{d-1}(\{y : |y-x|=r, \Gamma^*(\tilde{y}) > y_d > \Gamma(\tilde{y})\}) dr \\ &\leq c_{15} \int_0^{r_1/4} r^{2p-\alpha} dr < \infty. \end{aligned}$$

Note that for $y \in E$

$$|h_p(y) - h_{x,p}(y)| \leq c_{16}|(\bar{h}(y))^p - (\underline{h}(y))^p| \leq c_{17}(\bar{h}(y))^{(p-1)_-} |\bar{h}(y) - \underline{h}(y)|, \quad (2.23)$$

where $(p-1)_- := (p-1) \wedge 0$. In the last inequality above, we have used the inequalities

$$|b^p - a^p| \leq b^{p-1}|b-a| \quad \text{for } a, b > 0, 0 < p \leq 1$$

and

$$|b^p - a^p| \leq (p+1)|b-a| \quad \text{for } a, b \in (0, 1), p > 1.$$

For $y = (\tilde{y}, y_d) \in \mathbb{R}^d$, we use an affine coordinate system $z = (\tilde{z}, z_d)$ to represent it so that $z_d = y_d - \Gamma^*(\tilde{y})$ and \tilde{z} is the coordinates in an orthogonal coordinate system centered at x_0 for the $(d-1)$ -dimensional hyperplane Π for the point $(\tilde{y}, \Gamma^*(\tilde{y}))$. Denote such an affine transformation $y \mapsto z$ by $z = \Psi(y)$. It is clear that there is a constant $c_{18} > 1$ so that for every $y \in \mathbb{R}^d$,

$$c_{18}^{-1}|\tilde{y} - \tilde{x}| \leq |\tilde{z}| \leq c_{18}|\tilde{y} - \tilde{x}|, \quad c_{18}^{-1}|y - x| \leq |\Psi(y) - \Psi(x)| \leq c_{18}|y - x|$$

and that

$$\Psi(E) \subset \{z = (\tilde{z}, z_d) \in \mathbb{R}^d : |\tilde{z}| < c_{18}r_1 \text{ and } 0 < z_d \leq c_{18}r_1\}.$$

Denote $x_d - \Gamma^*(\tilde{x})$ by w ; that is, $\Psi(x) = (\tilde{0}, w)$. Hence by (2.11) and (2.23) and applying the transform Ψ , we have by using polar coordinates for \tilde{z} on the hyperplane Π ,

$$\begin{aligned} I_2 &\leq c_{19} \int_E \frac{\bar{h}(y)^{(p-1)_-} |\tilde{y} - \tilde{x}|^2}{|y-x|^{d+\alpha}} dy \leq c_{19} \int_{\Psi(E)} \frac{z_d^{(p-1)_-} |\tilde{z}|^2}{|z - (\tilde{0}, w)|^{d+\alpha}} dz \\ &\leq c_{20} \int_0^{c_{18}r_1} z_d^{(p-1)_-} \left(\int_0^{c_{18}r_1} \frac{r^{d-2}}{(r + |z_d - w|)^{d+\alpha-2}} dr \right) dz_d \\ &\leq c_{20} \int_0^{c_{18}r_1} z_d^{(p-1)_-} \left(\int_0^{c_{18}r_1} \frac{1}{(r + |z_d - w|)^\alpha} dr \right) dz_d \\ &\leq c_{21} \int_0^{c_{18}r_1} z_d^{(p-1)_-} \left(\frac{1}{|z_d - w|^{\alpha-1}} - \frac{1}{(c_{18}r_1 + |z_d - w|)^{\alpha-1}} \right) dz_d \\ &< c_{22} \int_0^{c_{18}r_1} \frac{1}{z_d^{(1-p)^+} |z_d - w|^{\alpha-1}} dz_d \leq c_{23} < \infty, \end{aligned}$$

where all the constants depend on α , p and r_1 . The last inequality is due to the fact that since $p > 0$, $0 < \alpha < 2$ and $(1-p)^+ + \alpha - 1 = \max\{\alpha - p, \alpha - 1\} < 1$, by the dominated convergence theorem, $\phi(w) := \int_0^{c_{18}r_1} \frac{1}{z_d^{(1-p)^+} |z_d - w|^{\alpha-1}} dz_d$ is a strictly positive continuous function in $x_d \in [0, c_{18}r_1]$ and hence is bounded.

Thus we have proved the claim (2.21). The desired estimates (2.6)-(2.10) now follow from (2.12)-(2.21). \square

It is well-known that X^1 has Lévy intensity

$$J^1(x, y) = j^1(|x - y|) = \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} + \frac{\mathcal{A}(d, -\beta)}{|x - y|^{d+\beta}}.$$

A scaling argument yields that

$$J^a(x, y) = j^a(|x - y|) = \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} + a^\beta \frac{\mathcal{A}(d, -\beta)}{|x - y|^{d+\beta}}.$$

Put

$$\psi^a(r) := 1 + a^\beta \frac{\mathcal{A}(d, -\beta)}{\mathcal{A}(d, -\alpha)} r^{\alpha-\beta}, \quad r \geq 0. \quad (2.24)$$

Clearly for $a \in (0, M]$ and $r > 0$, $1 \leq \psi^a(r) \leq 1 + cM^\beta r^{\alpha-\beta}$ and

$$J^a(x, y) = j^a(|x - y|) = \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} \psi^a(|x - y|).$$

The Lévy intensity gives rise to a Lévy system for X^a , which describes the jumps of the process X^a : for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, $x \in \mathbb{R}^d$ and stopping time T (with respect to the filtration of X^a),

$$\mathbb{E}_x \left[\sum_{s \leq T} f(s, X_{s-}^a, X_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X_s^a, y) J^a(X_s^a, y) dy \right) ds \right]. \quad (2.25)$$

(See, for example, [7, Proof of Lemma 4.7] and [8, Appendix A]).

For any open set $D \subset \mathbb{R}^d$, let $\tau_D^a = \tau^a(D) := \inf\{t > 0 : X_t^a \notin D\}$ denote the first exit time from D by X^a .

The next lemma follows immediately from a special case of [19, Proposition 2.10 and Lemma 3.6] and the scaling property of X^a .

Lemma 2.4 *For any $b, M \in (0, \infty)$, there exists $C_{13} = C_{13}(M, b, \alpha, \beta) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $a \in [0, M]$ and $r \in (0, b]$,*

$$\mathbb{E}_x \left[\tau_{B(x_0, r)}^a \right] \leq C_{13} r^{\alpha/2} (r - |x - x_0|)^{\alpha/2}, \quad \text{for } x \in B(x_0, r). \quad (2.26)$$

For $\lambda > 0$, $\widehat{Y}^\lambda = (\widehat{Y}_t^\lambda, \mathbb{P}_x)$ is a Lévy process in \mathbb{R}^d such that

$$\mathbb{E}_x \left[e^{i\xi \cdot (\widehat{Y}_t^\lambda - \widehat{Y}_0^\lambda)} \right] = e^{-t\psi(\xi)} \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d,$$

with

$$\psi(\xi) = \mathcal{A}(d, -\beta) \int_{\{|y| \leq \lambda\}} \frac{1 - \cos(\xi \cdot y)}{|y|^{d+\beta}} dy.$$

In other words, \widehat{Y}^λ is a pure jump symmetric Lévy process in \mathbb{R}^d with a Lévy density given by $\mathcal{A}(d, -\beta)|x|^{-d-\beta} \mathbf{1}_{\{|x| \leq \lambda\}}$. For $a > 0$, suppose $\widehat{Y}^{1/a}$ is independent of the symmetric α -stable process X on \mathbb{R}^d . Define

$$\widehat{X}_t^a := X_t + a\widehat{Y}_t^{1/a}, \quad t \geq 0.$$

We will call the process \widehat{X}^a the independent sum of the symmetric α -stable process X and the truncated symmetric β -stable process $\widehat{Y}^{1/a}$ with weight $a > 0$. The infinitesimal generator of \widehat{X}^a is $\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2}$.

For any open set $U \subset \mathbb{R}^d$, let $\widehat{\tau}_U^a = \inf\{t > 0 : \widehat{X}_t^a \notin U\}$ be the first exit time from U by \widehat{X}^a . The truncated process \widehat{X}^a will be used in the proof of next lemma.

Lemma 2.5 *Assume $r_1 \in (0, \frac{1}{4}]$ and $M > 0$. Let $U := \{z \in \mathbb{R}^d : r_1 < |z| < 3r_1/2\}$. There are constants $C_{14} = C_{14}(r_1, \alpha) > 0$ and $C_{15} = C_{15}(r_1, M, \alpha, \beta) > 0$ such that for every $a \in [0, M]$*

$$\mathbb{E}_x[\tau_U^a] \leq C_{14} \mathbb{P}_x\left(|X_{\tau_U^a}^a| \geq 3r_1/2\right) \leq C_{15} \delta_U(x)^{\alpha/2}, \quad \text{for } r_1 < |x| < 5r_1/4. \quad (2.27)$$

Proof. The first inequality in (2.27) is easy. In fact, by the Lévy system (2.25) with

$$f(s, x, y) = \mathbf{1}_U(x) \mathbf{1}_{\{5r_1 < |y| < 10r_1\}}(y)$$

and $T = \tau_U^a$, we have that for $x \in U$

$$\begin{aligned} & \mathbb{P}_x\left(|X_{\tau_U^a}^a| \geq 3r_1/2\right) \geq \mathbb{P}_x\left(10r_1 > |X_{\tau_U^a}^a| > 5r_1\right) \\ &= \mathbb{E}_x \left[\int_0^{\tau_U^a} \int_{\{5r_1 < |y| < 10r_1\}} J^a(X_s^a, y) dy ds \right] \\ &\geq \mathbb{E}_x \left[\int_0^{\tau_U^a} \int_{\{5r_1 < |y| < 10r_1\}} \frac{\mathcal{A}(d, -\alpha)}{|X_s^a - y|^{d+\alpha}} dy ds \right] \geq c_1 \mathbb{E}_x[\tau_U^a], \end{aligned}$$

where $c_1 = c(r_1, \alpha) > 0$.

It is enough to prove the second inequality in (2.27) for $r_1 < |x| < r_1 + \delta$ for some small $\delta > 0$. Without loss of generality, we assume $\tilde{x} = \tilde{0}$ and $x_d > 0$. Let $p > 0$ be such that $p \neq \beta$ and

$$\alpha - (\beta/2) < p < (\alpha - (\beta/2) + (\alpha - \beta)/3) \wedge \alpha.$$

Note that $\alpha/2 < p < 3\alpha/2 - \beta$. Define

$$\begin{aligned} h(y) &:= \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2}\right)^{\alpha/2} \mathbf{1}_{U \cap \{z_d > 0, |\tilde{z}| < r_1/2\}}(y), \\ g_p(y) &:= \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2}\right)^p \mathbf{1}_{U \cap \{z_d > 0, |\tilde{z}| < r_1/2\}}(y), \end{aligned}$$

and let ϕ be a smooth function on \mathbb{R}^d with bounded first and second partial derivatives such that $\phi(y) = 2^{4+p} |\tilde{y}|^2 / r_1^2$ for $y \in \{z_d > 0, r_1 < |y| < 4r_1/5, |\tilde{z}| < r_1/4\}$ and $2^p \leq \phi(y) \leq 4^p$ if $|\tilde{y}| \geq r_1/2$ or $|y| \geq 3r_1/2$.

Since $r_1 \leq 1/4$, it is easy to see that $\|g_p\|_\infty < 1$. Now we define

$$u(y) := h(y) + \phi(y) - g_p(y).$$

By Taylor's expansion with the remainder of order 2, we get that for any $a \in (0, M]$ and $y \in \mathbb{R}^d$,

$$|(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})\phi(y)| \leq \|\Delta^{\alpha/2}\phi\|_\infty + M^\beta \|\widehat{\Delta}^{\beta/2}\phi\|_\infty \leq c_2(\alpha, \beta, M) < \infty. \quad (2.28)$$

Moreover, by (2.6)–(2.8), there exist $c_3 = c_3(\alpha, \beta) > 0$ and $\delta_1 = \delta_1(\alpha, \beta) \in (0, r_1/8)$ such that

$$\Delta^{\alpha/2} g_p(y) \geq c_3 \delta_D(y)^{p-\alpha} \quad \text{for } y \in \left\{ z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + \delta_1, |\tilde{z}| < r_1/4 \right\}$$

and

$$\widehat{\Delta}^{\beta/2} g_p(y) \geq -c_3 \delta_D(y)^{(p-\beta) \wedge 0} \quad \text{for } y \in \left\{ z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + \delta_1, |\tilde{z}| < r_1/4 \right\}.$$

Thus there exist $c_4 = c_4(\alpha, \beta, M) > 0$ and $\delta_2 = \delta_2(\alpha, \beta, M) \in (0, \delta_1)$ such that for all $a \in (0, M]$ and $y \in \left\{ z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + \delta_2, |\tilde{z}| < r_1/4 \right\}$,

$$(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})g_p(y) \geq c_3 \delta_D(y)^{p-\alpha} - c_3 M^\beta \delta_D(y)^{(p-\beta) \wedge 0} \geq c_4 \delta_D(y)^{p-\alpha}. \quad (2.29)$$

Furthermore by (2.6) and (2.8)–(2.10), there exist $c_5 = c_5(\alpha, \beta, M) > 0$ and $\delta_3 = \delta_3(\alpha, \beta) \in (0, \delta_1)$ such that for all $a \in (0, M]$ and for every $y \in \left\{ z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + \delta_3, |\tilde{z}| < r_1/4 \right\}$,

$$\begin{aligned} \left| (\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})h(y) \right| &\leq |\Delta^{\alpha/2}h(y)| + M^\beta |\widehat{\Delta}^{\beta/2}h(y)| \\ &\leq \begin{cases} c_5 + c_5 \delta_D(y)^{(\alpha/2-\beta) \wedge 0} & \text{if } \beta \neq \alpha/2, \\ c_5 + c_5 |\log \delta_D(y)| & \text{if } \beta = \alpha/2. \end{cases} \end{aligned} \quad (2.30)$$

Since $p - \alpha < \alpha/2 - \beta$, by (2.28)–(2.29), there exists $\delta_4 = \delta_4(\alpha, \beta, M) \in (0, \delta_2 \wedge \delta_3)$ such that for all $a \in (0, M]$ and $y \in V := \left\{ z \in \mathbb{R}^d : z_d > 0, r_1 < |z| < r_1 + \delta_4, |\tilde{z}| < r_1/4 \right\}$

$$(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})u(y) \leq c_2 + c_5 + c_5 \left(\delta_D(y)^{(\alpha/2-\beta) \wedge 0} + |\log \delta_D(y)| \right) - c_4 \delta_D(y)^{p-\alpha} \leq 0. \quad (2.31)$$

Let η be a non-negative smooth radial function with compact support in \mathbb{R}^d such that $\eta(x) = 0$ for $|x| > 1$ and $\int_{\mathbb{R}^d} \eta(x) dx = 1$. For $k \geq 1$, define $\eta_k(x) = 2^{kd} \eta(2^k x)$. Set $u^{(k)}(z) := (\eta_k * u)(z)$. As $(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})u^{(k)} = \eta_k * (\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})u$, we have by (2.31) that

$$(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2})u^{(k)} \leq 0$$

on $V_k := \left\{ z \in \mathbb{R}^d : z_d > 0, r_1 + 2^{-k} < |z| < r_1 + \delta_4 - 2^{-k} \text{ and } |\tilde{z}| < r_1/4 - 2^{-k} \right\}$. Since $u^{(k)}$ is a bounded smooth function on \mathbb{R}^d with bounded first and second partial derivatives, by Ito's formula and the Lévy system (2.25),

$$M_t^k := u^{(k)}(\widehat{X}_t^a) - u^{(k)}(\widehat{X}_0^a) - \int_0^t \left(\Delta^{\alpha/2} + a^\beta \widehat{\Delta}^{\beta/2} \right) u^{(k)}(\widehat{X}_s^a) ds \quad (2.32)$$

is a martingale. Thus it follows from (2.32) that $t \mapsto u^{(k)}(\widehat{X}_{t \wedge \widehat{V}_k^a}^a)$ is a bounded supermartingale. Since V_k increases to V and u is bounded and continuous on \overline{V} , we conclude that

$$t \mapsto u\left(\widehat{X}_{t \wedge \widehat{V}^a}^a\right) \text{ is a bounded supermartingale.} \quad (2.33)$$

We observe that, since $\phi(x) = 0$,

$$u(x) \leq \delta_U(x)^{\alpha/2}. \quad (2.34)$$

We also observe that, since $\phi \geq 2g_p$ outside of $\{z \in U : z_d > 0, |\tilde{z}| < r_1/2\}$ and

$$u(y) \geq \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2}\right)^{\alpha/2} - \left(y_d - \sqrt{r_1^2 - |\tilde{y}|^2}\right)^p > c_6$$

on $\{z_d > 0, r_1 + \delta_4 \leq |z| < 3r_1/2, |\tilde{z}| < r_1/2\}$, we have

$$u(y) \geq c_7 > 0 \quad \text{for } y \in V^c \setminus \overline{B(0, r_1)}, \quad (2.35)$$

where c_7 depends on δ_4, α, β and r_1 . Therefore, by (2.33)-(2.35) we get

$$\delta_U(x)^{\alpha/2} \geq u(x) \geq \mathbb{E}_x \left[u(\widehat{X}_{\tau_V^a}^a) \right] \geq c_7 \mathbb{P}_x \left(\widehat{X}_{\tau_V^a}^a \in V^c \setminus \overline{B(0, r_1)} \right) \geq c_7 \mathbb{P}_x \left(|\widehat{X}_{\tau_V^a}^a| \geq 3r_1/2 \right). \quad (2.36)$$

Note that there exist $c_8 = c_8(\alpha, d, r_1) > 0$ and $c_9 = c_9(\beta, d, r_1) > 0$ such that for $z \in U$,

$$\int_{\{|y| \geq 2r_1\}} \frac{dy}{|z-y|^{d+\alpha}} \leq c_8 \int_{\{2r_1 \leq |y| < 3r_1\}} \frac{dy}{|z-y|^{d+\alpha}}$$

and

$$\int_{\{|y| \geq 2r_1\}} \frac{dy}{|z-y|^{d+\beta}} \leq c_9 \int_{\{2r_1 \leq |y| < 3r_1\}} \frac{dy}{|z-y|^{d+\beta}}.$$

Thus by (2.25), there exists a positive constant $c_{10} = c_{10}(d, \alpha, \beta, M)$ such that for any $a \in (0, M]$,

$$\begin{aligned} \mathbb{P}_x \left(|X_{\tau_V^a}^a| \geq 2r_1 \right) &= \mathbb{E}_x \left[\int_0^{\tau_V^a} \int_{\{|y| \geq 2r_1\}} J^a(X_s^a, y) dy ds \right] \\ &\leq c_{10} \mathbb{E}_x \left[\int_0^{\tau_V^a} \int_{\{2r_1 \leq |y| < 3r_1\}} J^a(X_s^a, y) dy ds \right] \\ &= c_{10} \mathbb{P}_x \left(3r_1 > |X_{\tau_V^a}^a| \geq 2r_1 \right). \end{aligned} \quad (2.37)$$

Since $r_1 \leq 1/4$ and the processes X and Y do not jump simultaneously, we have by (2.36) that there is a positive constant $c_{11} = c_{11}(d, \alpha, \beta, M, r_1)$ such that for all $a \in (0, M]$,

$$\begin{aligned} \mathbb{P}_x \left(|X_{\tau_V^a}^a| \geq 3r_1/2 \right) &\leq (c_{10} + 1) \mathbb{P}_x \left(3r_1 > |X_{\tau_V^a}^a| \geq 3r_1/2 \right) \\ &= (c_{10} + 1) \mathbb{P}_x \left(3r_1 > |\widehat{X}_{\tau_V^a}^a| \geq 3r_1/2 \right) \\ &\leq (c_{10} + 1) \mathbb{P}_x \left(|\widehat{X}_{\tau_V^a}^a| \geq 3r_1/2 \right) \leq c_{11} \delta_U(x)^{\alpha/2}. \end{aligned}$$

□

Lemma 2.6 *Assume $M > 0$ and $r_1 \in (0, \frac{1}{4}]$. Let $E = \{x \in \mathbb{R}^d : |x| > r_1\}$. Then for every $T > 0$, there is a constant $C_{16} = C_{16}(r_1, \alpha, \beta, T, M) > 0$ such that for every $a \in [0, M]$,*

$$p_E^a(t, x, y) \leq C_{16} \delta_E(x)^{\alpha/2} J^a(x, y) \quad \text{for } r_1 < |x| < 5r_1/4, |y| \geq 2r_1 \text{ and } t \leq T.$$

Proof. Define $U := \{z \in \mathbb{R}^d : r_1 < |z| < 3r_1/2\}$. Since X^a satisfies the hypothesis **H** in [28], by [28, Theorem 1], $X_{\tau_U^a}^a \notin \partial U$ with probability 1. For $r_1 < |x| < 5r_1/4$, $|y| \geq 2r_1$ and $t \in (0, T]$, it follows from the strong Markov property of X^a and (2.25) that

$$\begin{aligned}
p_E^a(t, x, y) &= \mathbb{E}_x \left[p_E^a(t - \tau_U^a, X_{\tau_U^a}^a, y) : \tau_U^a < t \right] \\
&= \int_0^t \int_U p_U(s, x, z) \left(\int_{\{w: |w| > 3r_1/2\}} J^a(z, w) p_E^a(t - s, w, y) dw \right) dz ds \\
&= \int_0^t \int_U p_U(s, x, z) \left(\int_{\{w: (3r_1/4) + (|y|/2) \geq |w| > 3r_1/2\}} J^a(z, w) p_E^a(t - s, w, y) dw \right) dz ds \\
&\quad + \int_0^t \int_U p_U(s, x, z) \left(\int_{\{w: |w| > (3r_1/4) + (|y|/2)\}} J^a(z, w) p_E^a(t - s, w, y) dw \right) dz ds \\
&=: I + II.
\end{aligned}$$

Note that for $|w| \leq (3r_1/4) + (|y|/2)$,

$$|w - y| \geq |y| - |w| \geq \frac{1}{2} \left(|y| - \frac{3r_1}{2} \right) \geq \frac{|y|}{8} \geq \frac{|x - y|}{16}. \quad (2.38)$$

Since $p_E^a(t - s, w, y) \leq p^a(t - s, w, y)$, by (1.4) and (2.38), there exist constants $c_1 = c_1(\alpha, \beta, M) > 0$ and $c_2 = c_2(\alpha, \beta, M) > 0$ such that for $a \in (0, M]$

$$\begin{aligned}
I &\leq \int_0^t \int_U p_U^a(s, x, z) \left(\int_{\{w: (3r_1/4) + (|y|/2) \geq |w| > 3r_1/2\}} J^a(z, w) c_1 T J^a(w, y) dw \right) dz ds \\
&\leq c_2 T J^a(x, y) \int_0^t \int_U p_U^a(s, x, z) \left(\int_{\{w: 3|x-y|/4 \geq |w| > 3r_1/2\}} J^a(z, w) dw \right) dz ds \\
&= c_2 T J^a(x, y) \mathbb{P}_x \left(3r_1/2 < |X_{\tau_U^a}^a| \leq 3|x - y|/4; \tau_U^a \leq t \right) \\
&\leq c_2 T J^a(x, y) \mathbb{P}_x \left(|X_{\tau_U^a}^a| > 3r_1/2 \right).
\end{aligned}$$

By Lemma 2.5, we have for $|x| \in (r_1, 5r_1/4)$,

$$\mathbb{P}_x \left(|X_{\tau_U^a}^a| > 3r_1/2 \right) \leq c_3 \delta_U(x)^{\alpha/2} = c_3 \delta_E(x)^{\alpha/2}$$

for some positive constant $c_3 = c_3(r_1, \alpha, \beta, M)$. Thus

$$I \leq c_4 (T \vee 1) \delta_E(x)^{\alpha/2} J^a(x, y) \quad (2.39)$$

for some positive constant $c_4 = c_4(r_1, \alpha, \beta, M)$. On the other hand, for $z \in U$ and $w \in \mathbb{R}^d$ with $|w| > (3r_1/4) + (|y|/2)$,

$$|z - w| \geq |w| - |z| \geq \frac{1}{2} \left(|y| - \frac{3r_1}{2} \right) \geq \frac{|y|}{8} \geq \frac{|x - y|}{16}.$$

Thus by the symmetry of $p_E^a(t-s, w, y)$ in (w, y) , we have

$$\begin{aligned} II &\leq c_5 J^a(x, y) \int_0^t \int_U p_U^a(s, x, z) \left(\int_{\{w: |w| > (3r_1/4) + (|y|/2)\}} p_E^a(t-s, y, w) dw \right) dz ds \\ &\leq c_5 J^a(x, y) \int_0^\infty \int_U p_U^a(s, x, z) dz ds \\ &= c_5 J^a(x, y) \mathbb{E}_x [\tau_U^a] \leq c_6 \delta_E(x)^{\alpha/2} J^a(x, y) \end{aligned}$$

for some positive constants $c_k = c_k(r_1, \alpha, \beta, M)$, $k = 5, 6$. In the last inequality, we used Lemma 2.5 to deduce that $\mathbb{E}_x [\tau_U^a] \leq c \delta_U(x)^{\alpha/2} = c \delta_E(x)^{\alpha/2}$ for some positive constant $c = c(r_1, \alpha, \beta, M)$. This together with (2.39) proves the lemma. \square

Theorem 2.7 *Assume that $M > 0$ and D is an open set that satisfies the uniform exterior ball condition with radius $r_0 > 0$. Then for every $T > 0$, there is a constant $C_{17} = C_{17}(r_0/T, \alpha, \beta, M) > 0$ such that for all $a \in (0, M]$, $\lambda \in (0, T]$ and $x, y \in \lambda^{-1}D$,*

$$p_{\lambda^{-1}D}^a(1, x, y) \leq C_{17} (1 \wedge J^a(x, y)) \delta_{\lambda^{-1}D}(x)^{\alpha/2}.$$

Proof. Note that for every $\lambda \in (0, T]$, $\lambda^{-1}D$ satisfies the uniform exterior ball condition with radius r_0/T . For $x, y \in \lambda^{-1}D$, let $z \in \partial(\lambda^{-1}D)$ be that $|x - z| = \delta_{\lambda^{-1}D}(x)$. Let $B_z \subset (\lambda^{-1}D)^c$ be the ball with radius $r_1 := 4^{-1} \wedge (r_0/T)$ so that $\partial B_z \cap \partial(\lambda^{-1}D) = \{z\}$. Since, by (1.4)

$$p_{\lambda^{-1}D}^a(1, x, y) \leq p^a(1, x, y) \leq c(1 \wedge J^a(x, y)),$$

it suffices to prove the theorem for $x \in \lambda^{-1}D$ with $\delta_{\lambda^{-1}D}(x) < r_1/4$. When $\delta_{\lambda^{-1}D}(x) < r_1/4$ and $|x - y| \geq 5r_1$, we have $\delta_{B_z^c}(y) > 2r_1$ and so by Lemma 2.6, there is a constant $c_1 > 0$ that depends only on $(r_0/T, d, \alpha, \beta, M)$ such that for $t \in (0, 1]$,

$$p_{\lambda^{-1}D}^a(t, x, y) \leq p_{(\overline{B_z})^c}^a(t, x, y) \leq c_1 \delta_{(\overline{B_z})^c}(x)^{\alpha/2} J^a(x, y) = c_1 \delta_{\lambda^{-1}D}(x)^{\alpha/2} J^a(x, y). \quad (2.40)$$

So it remains to show that when $\delta_{\lambda^{-1}D}(x) < r_1/4$ and $|x - y| < 5r_1$, there exists a positive constant $c_2 = c_2(r_0/T, d, \alpha, \beta, M)$ such that

$$p_{\lambda^{-1}D}^a(1, x, y) \leq c_2 \delta_{\lambda^{-1}D}(x)^{\alpha/2}. \quad (2.41)$$

Let $z_x \in \partial(\lambda^{-1}D)$ be such that $|x - z_x| = \delta_{\lambda^{-1}D}(x)$ and $z_0 \in \mathbb{R}^d$ so that

$$B(z_0, r_1) \subset (\lambda^{-1}D)^c \quad \text{and} \quad \partial B(z_0, r_1) \cap \partial(\lambda^{-1}D) = \{z_x\}.$$

Define $U := \{w \in \mathbb{R}^d : |w - z_0| \in (r_1, 8r_1)\}$. Note that

$$x, y \in U \cap \lambda^{-1}D \quad \text{and} \quad \delta_U(x) = \delta_{\lambda^{-1}D}(x).$$

By the strong Markov property and the symmetry of $p_{\lambda^{-1}D}^a(1, x, y)$ in x and y , we have

$$p_{\lambda^{-1}D}^a(1, x, y) = p_{U \cap \lambda^{-1}D}^a(1, x, y) + \mathbb{E}_y \left[p_{\lambda^{-1}D}^a(1 - \tau_{U \cap \lambda^{-1}D}^a, X_{\tau_{U \cap \lambda^{-1}D}^a}^a, x); \tau_{U \cap \lambda^{-1}D}^a < 1 \right].$$

By the semigroup property and (1.4),

$$\begin{aligned}
p_{U \cap \lambda^{-1}D}^a(1, x, y) &= \int_{U \cap \lambda^{-1}D} p_{U \cap \lambda^{-1}D}^a(1/2, x, z) p_{U \cap \lambda^{-1}D}^a(1/2, z, y) dz \\
&\leq \|p^a(1/2, \cdot, \cdot)\|_\infty \mathbb{P}_x(\tau_{U \cap \lambda^{-1}D}^a > 1/2) \\
&\leq c_3 \mathbb{E}_x[\tau_{U \cap \lambda^{-1}D}^a] \leq c_3 \mathbb{E}_x[\tau_U^a] \\
&\leq c_4 \delta_U(x)^{\alpha/2} = c_4 \delta_{\lambda^{-1}D}(x)^{\alpha/2}.
\end{aligned}$$

In the last inequality, we used Lemma 2.5.

On the other hand, we have $X_{\tau_{U \cap \lambda^{-1}D}^a}^a \in U^c \cap \lambda^{-1}D$ on $\{\tau_{U \cap \lambda^{-1}D}^a < 1\}$, and so

$$|X_{\tau_{U \cap \lambda^{-1}D}^a}^a - x| \geq 7r_1, \quad \text{on } \{\tau_{U \cap \lambda^{-1}D}^a < 1\}.$$

Consequently, by (2.40) for $p_{\lambda^{-1}D}^a(1 - \tau_{U \cap \lambda^{-1}D}^a, X_{\tau_{U \cap \lambda^{-1}D}^a}^a, x)$,

$$\begin{aligned}
&\mathbb{E}_y \left[p_{\lambda^{-1}D}^a(1 - \tau_{U \cap \lambda^{-1}D}^a, X_{\tau_{U \cap \lambda^{-1}D}^a}^a, x); \tau_{U \cap \lambda^{-1}D}^a < 1 \right] \\
&\leq \mathbb{E}_y \left[c_1 \delta_{\lambda^{-1}D}(x)^{\alpha/2} J^a(X_{\tau_{U \cap \lambda^{-1}D}^a}^a, x); \tau_{U \cap \lambda^{-1}D}^a < 1 \right] \\
&\leq c_1 ((7r_1)^{-d-\alpha} + M^\beta (7r_1)^{-d-\beta}) \delta_{\lambda^{-1}D}(x)^{\alpha/2} \mathbb{P}_y(\tau_{U \cap \lambda^{-1}D}^a < 1) \\
&\leq c_1 ((7r_1)^{-d-\alpha} + M^\beta (7r_1)^{-d-\beta}) \delta_{\lambda^{-1}D}(x)^{\alpha/2}.
\end{aligned}$$

This completes the proof for (2.41) and hence the theorem. \square

Theorem 2.8 *Assume that $M > 0$ and that D is an open set that satisfies the uniform exterior ball condition with radius $r_0 > 0$. For every $T > 0$, there exists a positive constant $C_{18} = C_{18}(T, r_0, \alpha, \beta, M)$ such that for every $a \in [0, M]$, $t \in (0, T]$ and $x, y \in D$,*

$$p_D^a(t, x, y) \leq C_{18} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) (t^{-d/\alpha} \wedge t J^a(x, y)). \quad (2.42)$$

Proof. Fix $T, M > 0$. By Theorem 2.7, there exists a positive constant $c_1 = c_1(T, r_0, \alpha, \beta, M)$ such that for every $t \in (0, T]$,

$$p_{t^{-1/\alpha}D}^{at^{(\alpha-\beta)/(\alpha\beta)}}(1, x, y) \leq c_1 \left(1 \wedge J^{at^{(\alpha-\beta)/(\alpha\beta)}}(x, y) \right) \delta_{t^{-1/\alpha}D}(x)^{\alpha/2}. \quad (2.43)$$

Thus by (1.3), (1.4) and (2.43), for every $t \leq T$,

$$\begin{aligned}
p_D^a(t, x, y) &= t^{-d/\alpha} p_{t^{-1/\alpha}D}^{at^{(\alpha-\beta)/(\alpha\beta)}}(1, t^{-1/\alpha}x, t^{-1/\alpha}y) \\
&\leq c_1 t^{-d/\alpha} \left(1 \wedge J^{at^{(\alpha-\beta)/(\alpha\beta)}}(t^{-1/\alpha}x, t^{-1/\alpha}y) \right) \delta_{t^{-1/\alpha}D}(t^{-1/\alpha}x)^{\alpha/2} \\
&= c_1 \left(t^{-d/\alpha} \wedge t J^a(x, y) \right) \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \\
&\leq c_2 p^a(t, x, y) \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}.
\end{aligned}$$

By symmetry, the above inequality holds with the roles of x and y interchanged. Using the semi-group property for $t \leq T$,

$$\begin{aligned} p_D^a(t, x, y) &= \int_D p_D^a(t/2, x, z) p_D^a(t/2, z, y) dz \\ &\leq c_3 \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{t} \int_D p^a(t/2, x, z) p^a(t/2, z, y) dz \\ &\leq c_3 \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{t} p^a(t, x, y). \end{aligned}$$

This proves the upper bound (2.42) by noting that

$$(1 \wedge a)(1 \wedge b) = \min\{1, a, b, ab\} \quad \text{for } a, b > 0.$$

□

3 Lower bound estimate

Lemma 3.1 *For any positive constants Λ , κ and b , there exists $C_{19} = C_{19}(\Lambda, \kappa, b, \alpha, \beta, M) > 0$ such that for every $z \in \mathbb{R}^d$, $\lambda \in (0, \Lambda]$ and $a \in (0, M]$,*

$$\inf_{\substack{y \in \mathbb{R}^d \\ |y-z| \leq \kappa \lambda^{1/\alpha}}} \mathbb{P}_y \left(\tau_{B(z, 2\kappa \lambda^{1/\alpha})}^a > b\lambda \right) \geq C_{19}.$$

Proof. By [8, Proposition 4.9], there exists $\varepsilon = \varepsilon(\Lambda, \kappa, \alpha, \beta) > 0$ such that for every $\lambda \in (0, \Lambda]$,

$$\inf_{y \in \mathbb{R}^d} \mathbb{P}_y \left(\tau_{B(y, \kappa \lambda^{1/\alpha}/2)}^1 > \varepsilon \lambda \right) \geq \frac{1}{2}.$$

Suppose $b > \varepsilon$ then by the parabolic Harnack principle in [8, Proposition 4.12]

$$c_1 p_{B(y, \kappa \lambda^{1/\alpha})}^1(\varepsilon \lambda, y, w) \leq p_{B(y, \kappa \lambda^{1/\alpha})}^1(b\lambda, y, w) \quad \text{for } w \in B(y, \kappa \lambda^{1/\alpha}/2),$$

where the constant $c_1 = c_1(\kappa, b, \alpha, \beta, \Lambda) > 0$ is independent of $y \in \mathbb{R}^d$, $\lambda \in (0, \Lambda]$. Thus

$$\begin{aligned} \mathbb{P}_y \left(\tau_{B(y, \kappa \lambda^{1/\alpha})}^1 > b\lambda \right) &= \int_{B(y, \kappa \lambda^{1/\alpha})} p_{B(y, \kappa \lambda^{1/\alpha})}^1(b\lambda, y, w) dw \\ &\geq \int_{B(y, \kappa \lambda^{1/\alpha}/2)} p_{B(y, \kappa \lambda^{1/\alpha})}^1(b\lambda, y, w) dw \\ &\geq c_1 \int_{B(y, \kappa \lambda^{1/\alpha}/2)} p_{B(y, \kappa \lambda^{1/\alpha}/2)}^1(\varepsilon \lambda, y, w) dw \geq c_1/2. \end{aligned} \tag{3.1}$$

For the general case, by (1.3) and (3.1),

$$\begin{aligned}
& \inf_{\substack{y \in \mathbb{R}^d \\ |y-z| \leq \kappa \lambda^{1/\alpha}}} \mathbb{P}_y \left(\tau_{B(z, 2\kappa \lambda^{1/\alpha})}^a > b\lambda \right) \\
& \geq \mathbb{P}_0 \left(\tau_{B(0, \kappa \lambda^{1/\alpha})}^a > b\lambda \right) \\
& = \int_{B(0, \kappa \lambda^{1/\alpha})} p_{B(0, \kappa \lambda^{1/\alpha})}^a(b\lambda, 0, w) dw \\
& = \int_{B(0, \kappa \lambda^{1/\alpha} a^{-\frac{\beta}{\alpha-\beta}})} p_{B(0, \kappa \lambda^{1/\alpha} a^{-\frac{\beta}{\alpha-\beta}})}^1(a^{\frac{\alpha\beta}{\alpha-\beta}} b\lambda, 0, z) dz \\
& = \mathbb{P}_0 \left(\tau^1(B(0, \kappa \lambda^{1/\alpha} a^{-\frac{\beta}{\alpha-\beta}})) > a^{\frac{\alpha\beta}{\alpha-\beta}} b\lambda \right) \\
& \geq \mathbb{P}_0 \left(\tau^1(B(0, \kappa \lambda^{1/\alpha} M^{-\frac{\beta}{\alpha-\beta}})) > M^{\frac{\alpha\beta}{\alpha-\beta}} b\lambda \right) \geq c_2(\Lambda, \kappa, b, \alpha, \beta, M) > 0.
\end{aligned}$$

This proves the lemma. \square

Recall that ψ^a is defined in (2.24).

Proposition 3.2 *Suppose that $M, T > 0$ and $(t, x, y) \in (0, T] \times D \times D$ with $\delta_D(x) \geq t^{1/\alpha} \geq 2|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$. Then there exists a positive constant $C_{20} = C_{20}(M, \alpha, \beta, T)$ such that for all $a \in (0, M]$*

$$p_D^a(t, x, y) \geq C_{20} t^{-d/\alpha}. \quad (3.2)$$

Proof. Let $t \in (0, T]$ and $x, y \in D$ with $\delta_D(x) \geq t^{1/\alpha} \geq 2|x-y|\psi^a(|x-y|)^{-1/(d+\alpha)}$. By the parabolic Harnack principle in [8, Proposition 4.12] and the scaling property, there exists $c_1 = c_1(M, \alpha, \beta, T) > 0$ such that for all $a \in (0, M]$,

$$p_D^a(t/2, x, w) \leq c_1 p_D^a(t, x, y) \quad \text{for } w \in B(x, 2t^{1/\alpha}/3).$$

This together with Lemma 3.1 yields that

$$\begin{aligned}
p_D^a(t, x, y) & \geq \frac{1}{c_1 |B(x, t^{1/\alpha}/2)|} \int_{B(x, t^{1/\alpha}/2)} p_D^a(t/2, x, w) dw \\
& \geq c_2 t^{-d/\alpha} \int_{B(x, t^{1/\alpha}/2)} p_{B(x, t^{1/\alpha}/2)}^a(t/2, x, w) dw \\
& = c_2 t^{-d/\alpha} \mathbb{P}_x \left(\tau_{B(x, t^{1/\alpha}/2)}^a > t/2 \right) \geq c_3 t^{-d/\alpha},
\end{aligned}$$

where $c_i = c_i(T, \alpha, \beta, M) > 0$ for $i = 2, 3$. \square

Lemma 3.3 *Suppose that $M, T > 0$, D is an open subset of \mathbb{R}^d and $(t, x, y) \in (0, T] \times D \times D$ with $\min\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$ and $|x-y|^\alpha \geq 2^{-\alpha} t \psi^a(|x-y|)^{\alpha/(d+\alpha)}$. Then there exists a constant $C_{21} = C_{21}(\alpha, \beta, T, M) > 0$ such that for $a \in (0, M]$*

$$\mathbb{P}_x \left(X_t^{a, D} \in B(y, 2^{-1} t^{1/\alpha}) \right) \geq C_{21} t^{d/\alpha+1} J^a(x, y).$$

Proof. For $t \in (0, T]$, it follows from Lemma 3.1 that, starting at $z \in B(y, 4^{-1}t^{1/\alpha})$, with probability at least $c_1 = c_1(\alpha, \beta, T, M) > 0$, for any $a \in (0, M]$, the process X^a does not move more than $6^{-1}t^{1/\alpha}$ by time t . Thus, it suffices to show that there exists a constant $c_2 = c_2(\alpha, \beta, T, M) > 0$ such that

$$\mathbb{P}_x \left(X^{a,D} \text{ hits the ball } B(y, 4^{-1}t^{1/\alpha}) \text{ by time } t \right) \geq c_2 t^{d/\alpha+1} J^a(x, y) \quad (3.3)$$

for all $a \in (0, M]$, $t \in (0, T]$ and $|x - y|^\alpha \geq 2^{-\alpha} t \psi^a(|x - y|)^{\alpha/(d+\alpha)}$.

Let $B_x := B(x, 6^{-1}t^{1/\alpha})$, $B_y := B(y, 6^{-1}t^{1/\alpha})$ and $\tau_x^a := \tau_{B_x}^a$. It follows from Lemma 3.1 that there exists $c_3 = c_3(\alpha, \beta, T, M) > 0$ such that for $a \in (0, M]$ and $t \in (0, T]$,

$$\mathbb{E}_x [t \wedge \tau_x^a] \geq t \mathbb{P}_x (\tau_x^a \geq t) \geq c_3 t. \quad (3.4)$$

By the Lévy system in (2.25),

$$\begin{aligned} & \mathbb{P}_x \left(X^{a,D} \text{ hits the ball } B(y, 4^{-1}t^{1/\alpha}) \text{ by time } t \right) \\ & \geq \mathbb{P}_x (X_{t \wedge \tau_x^a}^a \in B(y, 4^{-1}t^{1/\alpha}) \text{ and } t \wedge \tau_x^a \text{ is a jumping time}) \\ & \geq \mathbb{E}_x \left[\int_0^{t \wedge \tau_x^a} \int_{B_y} J^a(X_s^a, u) du ds \right]. \end{aligned} \quad (3.5)$$

Note that

$$|x - y| \geq 2^{-1}t^{1/\alpha} \psi^a(|x - y|)^{1/(d+\alpha)} \geq 2^{-1}t^{1/\alpha}.$$

Moreover, if $s < \tau_x^a$ and $u \in B_y$,

$$|X_s^a - u| \leq |x - y| + |x - X_s^a| + |y - u| \leq 2|x - y|.$$

Thus from (3.5) we get that for any $a \in (0, M]$ and $t \in (0, T]$,

$$\begin{aligned} & \mathbb{P}_x \left(X^{a,D} \text{ hits the ball } B(y, 4^{-1}t^{1/\alpha}) \text{ by time } t \right) \\ & \geq \mathbb{E}_x [t \wedge \tau_x^a] \int_{B_y} j^a(2|x - y|) du \\ & \geq c_4 t |B_y| j^a(2|x - y|) \geq c_5 t^{d/\alpha+1} j^a(2|x - y|) \geq c_5 2^{-d-\alpha} t^{d/\alpha+1} j^a(|x - y|) \end{aligned}$$

for some positive constants $c_i = c_i(\alpha, \beta, T, M)$, $i = 4, 5$. Here in the second inequality, (3.4) is used. \square

Proposition 3.4 *Suppose that $T > 0$, $M > 0$, D is an open subset of \mathbb{R}^d and $(t, x, y) \in (0, T] \times D \times D$ with $\min\{\delta_D(x), \delta_D(y)\} \geq (t/2)^{1/\alpha}$ and $|x - y|^\alpha \geq 2^{-\alpha-1} t \psi^a(|x - y|)^{\alpha/(d+\alpha)}$. Then there exists a constant $C_{22} = C_{22}(\alpha, \beta, T, M) > 0$ such that for any $a \in (0, M]$,*

$$p_D^a(t, x, y) \geq C_{22} t J^a(x, y). \quad (3.6)$$

Proof. By the semigroup property, Proposition 3.2 and Lemma 3.3, there exist positive constants $c_1 = c_1(\alpha, \beta, T, M)$ and $c_2 = c_2(\alpha, \beta, T, M)$ such that for any $t \in (0, T]$ and $a \in (0, M]$

$$\begin{aligned} p_D^a(t, x, y) &= \int_D p_D^a(t/2, x, z) p_D^a(t/2, z, y) dz \\ &\geq \int_{B(y, 2^{-1}(t/2)^{1/\alpha})} p_D^a(t/2, x, z) p_D^a(t/2, z, y) dz \\ &\geq c_1 t^{-d/\alpha} \mathbb{P}_x \left(X_{t/2}^{a, D} \in B(y, 2^{-1}(t/2)^{1/\alpha}) \right) \\ &\geq c_2 t J^a(x, y). \end{aligned}$$

□

In the rest of this section, we assume that D is an open set in \mathbb{R}^d satisfying the uniform interior ball condition with radius $r_0 > 0$ in the following sense: For every $x \in D$ with $\delta_D(x) < r_0$, there is $z_x \in \partial D$ so that $|x - z_x| = \delta_D(x)$ and $B(x_0, r_0) \subset D$ for $x_0 := z_x + r_0(x - z_x)/|x - z_x|$. Clearly, a (uniform) $C^{1,1}$ open set satisfies the uniform interior ball condition.

The goal of this section is to prove the following lower bound for the heat kernel $p_D^a(t, x, y)$.

Theorem 3.5 *For any $M > 0$ and $T > 0$, there exists positive constant $C_{23} = C_{23}(\alpha, \beta, T, M, r_0)$ such that for all $a \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D^a(t, x, y) \geq C_{23} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge t J^a(x, y) \right).$$

To prove this result, we will first prove a lower bound estimates on the Green function of $X^{a,U}$

$$G_U^a(x, y) := \int_0^\infty p_U^a(t, x, y) dt$$

when U is a bounded $C^{1,1}$ open set. The tool we use to establish the Green function lower bound is a subordinate killed α -stable process in U . We first introduce this subordinate killed process first.

Assume that U is a bounded $C^{1,1}$ open set in \mathbb{R}^d and R_1 the radius in the uniform interior and exterior ball conditions. Then it follows from [3, Theorem 1.1] that the killed α -stable process X^U on U has a density $p_U(t, x, y)$ satisfying the following condition: for any $T > 0$ there exist positive constants $c_2 > c_1$ depending only on α, T, R_1 and d such that for any $(t, x, y) \in (0, T] \times U \times U$,

$$p_U(t, x, y) \geq c_1 \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right), \quad (3.7)$$

$$p_U(t, x, y) \leq c_2 \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}} \right) \left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right). \quad (3.8)$$

Let $\{T_t^a : t \geq 0\}$ be a subordinator, independent of X^a , with Laplace exponent

$$\phi^a(\lambda) = \lambda + a^\beta \lambda^{\beta/\alpha}.$$

Then the process $\{Z_t^{a,U} : t \geq 0\}$ defined by $Z_t^{a,U} = X_{T_t^a}^U$ is called a subordinate killed stable process in U . Since ϕ^a is a complete Bernstein function, the subordinator T^a has a decreasing potential

density $u^a(x)$. In fact $u^a(x)$ is completely monotone. (See [21, 25] for the details.) Then it follows from [25] that the Green function $R_U^a(x, y)$ of $Z^{a,U}$ is given by

$$R_U^a(x, y) = \int_0^\infty p_U(t, x, y) u^a(t) dt. \quad (3.9)$$

It follows from [26] that the Green function G_U^a of $X^{a,U}$ and the Green function R_U^a of $Z^{a,U}$ satisfy the following relation:

$$R_U^a(x, y) \leq G_U^a(x, y) \quad (x, y) \in U \times U. \quad (3.10)$$

So we can get a lower bound on $G_U^a(x, y)$ by establishing a lower bound on $R_U^a(x, y)$. The following result gives sharp two-sided estimates on $R_U^a(x, y)$ and the idea of the proof is similar to that of [24].

Theorem 3.6 *Suppose that $M > 0$ and U is a bounded $C^{1,1}$ open set in U . There exist positive constant $C_{25} > C_{24}$ depending only on $(\alpha, \beta, d, R_1, M, \text{diam}(U))$ such that for all $a \in (0, M]$,*

$$R_U^a(x, y) \geq C_{24} \begin{cases} \left(1 \wedge \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right) |x-y|^{\alpha-d} & \text{when } d > \alpha, \\ \log \left(1 + \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d = 1 = \alpha, \\ (\delta_U(x) \delta_U(y))^{(\alpha-1)/2} \wedge \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x-y|} & \text{when } d = 1 < \alpha, \end{cases}$$

and

$$R_U^a(x, y) \leq C_{25} \begin{cases} \left(1 \wedge \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right) |x-y|^{\alpha-d} & \text{when } d > \alpha, \\ \log \left(1 + \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right) & \text{when } d = 1 = \alpha, \\ (\delta_U(x) \delta_U(y))^{(\alpha-1)/2} \wedge \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x-y|} & \text{when } d = 1 < \alpha. \end{cases}$$

Proof. Since the drift coefficient of T^a is 1, we know that $u^a(t) \leq 1$ for all $t > 0$. Now the upper bound on R_U^a follows immediately from (3.9) and [3, Corollary 1.2]. Thus we only need to prove the lower bound.

By using a scaling argument, one can easily check that

$$u^a(t) = u^1(a^{\frac{\alpha}{\alpha-\beta}} t) \quad t > 0. \quad (3.11)$$

Let $T = \text{diam}(U)$. Since $u^1(t)$ is a completely monotone function with $u^1(0+) = 1$, by (3.11),

$$u^a(t) \geq u^1(M^{\frac{\alpha}{\alpha-\beta}} T) \quad \text{for every } t \in (0, T] \text{ and } a \in (0, M]. \quad (3.12)$$

Using (3.12), (3.9) and [3, (4.2)] we get that

$$\begin{aligned} & \int_0^T \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}}\right) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) u^a(t) dt \\ & \geq \frac{u^1(M^{\frac{\alpha}{\alpha-\beta}} T)}{|x-y|^{d-\alpha}} \int_{\frac{|x-y|^\alpha}{T}}^\infty \left(u^{\frac{d}{\alpha}-2} \wedge u^{-3}\right) \left(1 \wedge \frac{\sqrt{u} \delta_U(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \left(1 \wedge \frac{\sqrt{u} \delta_U(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) du. \end{aligned} \quad (3.13)$$

Now we can follow the proof of [3, Corollary 1.2] to get the desired lower bound. In fact, when $d > \alpha$, the desired lower bound follows from (3.13) and [3, (4.3) and (4.7)]. Let

$$u_0 := \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x-y|^\alpha}.$$

When $d = \alpha = 1$, by (3.13) and [3, (4.3) and (4.9)],

$$\begin{aligned}
R_U^a(x, y) &\geq u^1(M^{\frac{\alpha}{\alpha-\beta}}T) \int_0^T p_U(t, x, y) dt \\
&\geq c_1 \left(1 \wedge \frac{\delta_U(x)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) \left(1 \wedge \frac{\delta_U(y)^{\alpha/2}}{|x-y|^{\alpha/2}}\right) + c_1 \log(u_0 \vee 1) + c_1 u_0 \left(\left(1/u_0\right) \wedge 1 - \frac{|x-y|^\alpha}{T}\right) \\
&\geq c_2(1 \wedge u_0) + c_2 \log(u_0 \vee 1) + c_2 u_0 \left(\left(1/u_0\right) \wedge 1 - \frac{|x-y|^\alpha}{T}\right) \\
&\geq c_3(1 \wedge u_0) + c_3 \log(u_0 \vee 1) \geq c_4 \log \left(1 + \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x-y|^\alpha}\right).
\end{aligned}$$

Lastly, in the case $d = 1 < \alpha < 2$. By (3.13), [3, (4.3) and (4.7)] and the first display in part (iii) of the proof of [3, Corollary1.2], we have

$$\begin{aligned}
R_U^a(x, y) &\geq u^1(M^{\frac{\alpha}{\alpha-\beta}}T) \int_T^\infty p_U(t, x, y) dt \\
&\geq c_5 \frac{1}{|x-y|^{1-\alpha}} (1 \wedge u_0) \\
&\quad + c_5 \frac{1}{|x-y|^{1-\alpha}} \left(\left((u_0 \vee 1)^{1-(1/\alpha)} - 1 \right) + c_5 u_0 \left((u_0 \vee 1)^{-1/\alpha} - \left(\frac{|x-y|^\alpha}{T} \right)^{1/\alpha} \right) \right) \\
&\geq c_6 \frac{1}{|x-y|^{1-\alpha}} \left(u_0 \wedge u_0^{1-(1/\alpha)} \right) \\
&= c_6 \left((\delta_U(x) \delta_U(y))^{(\alpha-1)/2} \wedge \frac{\delta_U(x)^{\alpha/2} \delta_U(y)^{\alpha/2}}{|x-y|} \right).
\end{aligned}$$

□

By integrating the lower bound in Theorem 3.6 with respect to y and applying (3.10), we obtain the following lower bound on $\mathbb{E}_x[\tau_U^a]$

Corollary 3.7 *Suppose that $M > 0$ and U is a bounded $C^{1,1}$ open set in U . Then there exists a constant $C_{26} = C_{26}(\alpha, \beta, d, M, R_1, \text{diam}(U)) > 0$ such that for every $a \in (0, M]$ and $x \in U$,*

$$\mathbb{E}_x[\tau_U^a] \geq C_{26} \delta_U(x)^{\alpha/2}.$$

We will first establish Theorem 3.5 for small T , that is, we will first assume that

$$t \leq T_0 := \left(\frac{r_0}{16}\right)^\alpha. \quad (3.14)$$

By integrating (1.3) with respect to t and y , we have that for every open set U , $\lambda > 0$ and $x \in U$,

$$\mathbb{E}_x[\tau_U^a] = \int_U G_U^a(x, z) dz = \lambda^\alpha \int_{\lambda^{-1}U} G_{\lambda^{-1}U}^{a\lambda^{(\alpha-\beta)/\beta}}(\lambda^{-1}x, y) dy = \lambda^\alpha \mathbb{E}_{\lambda^{-1}x} \left[\tau^{a\lambda^{(\alpha-\beta)/\beta}}(\lambda^{-1}U) \right]. \quad (3.15)$$

Lemma 3.8 *Suppose that $M > 0$, $\kappa \in (0, 1)$ and that $(t, x) \in (0, T_0] \times D$ with $\delta_D(x) \leq 3t^{1/\alpha} < r_0/4$. Let $z_x \in \partial D$ be such that $|z_x - x| = \delta_D(x)$ and define $\mathbf{n}(z_x) := (x - z_x)/|x - z_x|$. Put $x_1 = z_x + 3t^{1/\alpha}\mathbf{n}(z_x)$ and $B = B(x_1, 3t^{1/\alpha})$. Suppose that x_0 is a point on the line segment connecting z_x and $z_x + 6t^{1/\alpha}\mathbf{n}(z_x)$ such that $B(x_0, 2\kappa t^{1/\alpha}) \subset B \setminus \{x\}$. Then for any $b > 0$, there exists a constant $C_{27} = C_{27}(\kappa, \alpha, \beta, r_0, b, M) > 0$ such that for all $a \in (0, M]$*

$$\mathbb{P}_x \left(X_{bt}^{a,D} \in B(x_0, \kappa t^{1/\alpha}) \right) \geq C_{27} t^{-1/2} \delta_D(x)^{\alpha/2}. \quad (3.16)$$

Proof. Let $0 < \kappa_1 \leq \kappa$ and assume first that $2^{-4}\kappa_1 t^{1/\alpha} < \delta_D(x) \leq 3t^{1/\alpha}$. Repeating the proof of Lemma 3.3, we get that, in this case, there exists a constant $c_1 = c_1(\alpha, \beta, \kappa_1, M, r_0, b) > 0$ such that for all $a \in (0, M]$

$$\mathbb{P}_x \left(X_{bt}^{a,D} \in B(x_0, \kappa_1 t^{1/\alpha}) \right) \geq c_1 t^{d/\alpha+1} J^a(x, x_0) \geq c_1 \mathcal{A}(d, -\alpha) t^{d/\alpha+1} |x - x_0|^{-d-\alpha}$$

for all $t \leq T_0$. Using the fact that $|x - x_0| \in [2\kappa t^{1/\alpha}, 6t^{1/\alpha}]$ we get that for all $a \in (0, M]$,

$$\mathbb{P}_x \left(X_{bt}^{a,D} \in B(x_0, \kappa_1 t^{1/\alpha}) \right) \geq c_2 > 0 \quad (3.17)$$

for some constant $c_2 = c_2(\alpha, \beta, \kappa_1, M, r_0, b)$. By taking $\kappa_1 = \kappa$, this shows that (3.16) holds for all $b > 0$ in the case when $2^{-4}\kappa t^{1/\alpha} < \delta_D(x) \leq 3t^{1/\alpha}$.

So it suffices to consider the case that $\delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$. We now show that there is some $b_0 > 1$ so that (3.16) holds for every $b \geq b_0$ and $\delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$. For simplicity, we assume without loss of generality that $x_0 = 0$ and let $\widehat{B} := B(0, \kappa t^{1/\alpha})$. Let $x_2 = z_x + 4^{-1}\kappa \mathbf{n}(z_x) t^{1/\alpha}$ and $B_2 := B(x_2, 4^{-1}\kappa t^{1/\alpha})$. Observe that since $B(0, 2\kappa t^{1/\alpha}) \subset B \setminus \{x\}$,

$$\kappa/2 t^{1/\alpha} \leq |y - z| \leq 6t^{1/\alpha} \quad \text{for } y \in B_2 \text{ and } z \in B(0, \kappa t^{1/\alpha}). \quad (3.18)$$

By the strong Markov property of X^a at the first exit time $\tau_{B_2}^a$ from B_2 and Lemma 3.1,

$$\begin{aligned} & \mathbb{P}_x \left(X_{bt}^a \in B(0, \kappa t^{1/\alpha}) \right) \\ & \geq \mathbb{P}_x \left(\tau_{B_2}^a < bt, X_{\tau_{B_2}^a}^a \in B(0, 2^{-1}\kappa t^{1/\alpha}) \text{ and } |X_s^a - X_{\tau_{B_2}^a}^a| < \kappa/2 \text{ for } s \in [\tau_{B_2}^a, \tau_{B_2}^a + bt^{1/\alpha}] \right) \\ & \geq c_3 \mathbb{P}_x \left(\tau_{B_2}^a < bt \text{ and } X_{\tau_{B_2}^a}^a \in B(0, 2^{-1}\kappa t^{1/\alpha}) \right). \end{aligned} \quad (3.19)$$

It follows from (3.15) and Corollary 3.7 that

$$\begin{aligned} \mathbb{P}_x \left(X_{\tau_{B_2}^a}^a \in B(0, 2^{-1}\kappa t^{1/\alpha}) \right) &= \int_{B(0, 2^{-1}\kappa t^{1/\alpha})} \int_{B_2} G_{B_2}^a(x, y) J^a(y, z) dy dz \\ &\geq \mathcal{A}(d, -\alpha) \int_{B(0, 2^{-1}\kappa t^{1/\alpha})} \int_{B_2} G_{B_2}^a(x, y) \frac{dy dz}{|y - z|^{d+\alpha}} \\ &\geq \frac{c_4}{t} \mathbb{E}_x \left[\tau_{B_2}^a \right] \\ &= c_4 \mathbb{E}_{x/t^{1/\alpha}} \left[\tau^{at^{(\alpha-\beta)/\alpha\beta}}(B(x_2/t^{1/\alpha}, 4^{-1}\kappa)) \right] \\ &\geq c_5 \left(\frac{\delta_D(x)}{t^{1/\alpha}} \right)^{\alpha/2} \end{aligned} \quad (3.20)$$

for some positive constants c_4, c_5 depending only on $\alpha, \beta, r_0, \kappa$ and M . Note that, by (1.3)

$$\int_{B(x_2, 4^{-1}\kappa t^{1/\alpha})} p_{B(x_2, 4^{-1}\kappa t^{1/\alpha})}^a(bt, x, z) dz = \int_{B(t^{-1/\alpha}x_2, 4^{-1}\kappa)} p_{B(t^{-1/\alpha}x_2, 4^{-1}\kappa)}^{at^{(\alpha-\beta)/\alpha\beta}}(b, t^{-1/\alpha}x, w) dw.$$

Since $at^{(\alpha-\beta)/\alpha\beta} \leq MT_0^{(\alpha-\beta)/\alpha\beta}$, by applying Theorem 2.8 to the right hand side of the above display, we get

$$\begin{aligned} \mathbb{P}_x(\tau_{B_2}^a \geq bt) &\leq b^{-d/\alpha} \int_{B(t^{-1/\alpha}x_2, 4^{-1}\kappa)} \frac{\delta_{B(t^{-1/\alpha}x_2, 4^{-1}\kappa)}(t^{-1/\alpha}x)^{\alpha/2}}{\sqrt{b}} dw \\ &\leq c_6 b^{-d/\alpha-1/2} \delta_{t^{-1/\alpha}D}(t^{-1/\alpha}x)^{\alpha/2} = c_6 b^{-d/\alpha-1/2} \left(\frac{\delta_D(x)}{t^{1/\alpha}} \right)^{\alpha/2}, \end{aligned} \quad (3.21)$$

for some positive constant c_6 depending only on $\alpha, \beta, r_0, \kappa$ and M . Define

$$b_0 := \left(\frac{2c_6}{c_5} \right)^{\frac{2\alpha}{2d+\alpha}}.$$

We have by (3.19)–(3.21) that for $b \geq b_0$,

$$\begin{aligned} \mathbb{P}_x(X_{bt}^a \in \widehat{B}) &\geq c_3 \left(\mathbb{P}_x(X_{\tau_{B_2}^a}^a \in B(0, 2^{-1}\kappa t^{1/\alpha})) - \mathbb{P}_x(\tau_{B_2}^a \geq bt) \right) \\ &\geq c_3 (c_5/2) \left(\frac{\delta_D(x)}{t^{1/\alpha}} \right)^{\alpha/2}. \end{aligned} \quad (3.22)$$

(3.17) and (3.22) show that (3.16) holds for every $b \geq b_0$ and for every $x \in D$ with $\delta_D(x) \leq 3t^{1/\alpha}$.

Now we deal with the case $0 < b < b_0$ and $\delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$. If $\delta_D(x) \leq 3(bt/b_0)^{1/\alpha}$, we have from (3.16) for the case of $b = b_0$ that

$$\begin{aligned} \mathbb{P}_x \left(X_{bt}^a \in B(x_0, \kappa t^{1/\alpha}) \right) &\geq \mathbb{P}_x \left(X_{b_0(bt/b_0)}^a \in B(x_0, \kappa (bt/b_0)^{1/\alpha}) \right) \\ &\geq c_7 \left(\frac{\delta_D(x)}{(bt/b_0)^{1/\alpha}} \right)^{\alpha/2} = c_8 \left(\frac{\delta_D(x)}{t^{1/\alpha}} \right)^{\alpha/2}. \end{aligned}$$

If $3(bt/b_0)^{1/\alpha} < \delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$ (in this case $\kappa > 3 \cdot 2^4(b/b_0)^{1/\alpha}$), we get (3.16) from (3.17) by taking $\kappa_1 = (b/b_0)^{1/\alpha}$. The proof of the lemma is now complete. \square

Proposition 3.9 *Suppose that $M > 0$ and $(t, x, y) \in (0, T_0] \times D \times D$ with $|x - y| \leq t^{1/\alpha} \psi^a(|x - y|)^{1/(d+\alpha)}$, $\delta_D(x) \leq 2t^{1/\alpha}$ and $\delta_D(y) \leq r_0/5$. Then there exists a constant $C_{28} = C_{28}(\alpha, \beta, M, r_0) > 0$ such that for all $a \in (0, M]$,*

$$p_D^a(t, x, y) \geq C_{28} t^{-d/\alpha-1} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}. \quad (3.23)$$

Proof. Under the assumptions of the proposition, there are points $z_x, z_y \in \partial D$ and $x_0, y_0 \in D$ such that $\delta_D(x) = |x - z_x|$, $\delta_D(y) = |y - z_y|$, $\partial B(x_0, 4t^{1/\alpha}) \cap \partial D = \{z_x\}$ and $\partial B(y_0, 4t^{1/\alpha}) \cap \partial D = \{z_y\}$. Observe that

$$\delta_D(x_0) = \delta_D(y_0) = 4t^{1/\alpha} \quad \text{and} \quad |x - x_0|, |y - y_0| \in [t^{1/\alpha}, 4t^{1/\alpha}).$$

By the semigroup property, with $B := B(x_0, 4^{-1}t^{1/\alpha})$ and $\tilde{B} := B(y_0, 4^{-1}t^{1/\alpha})$

$$\begin{aligned} p_D^a(t, x, y) &= \int_D p_D^a(t/3, x, z) \int_D p_D^a(t/3, z, w) p_D^a(t/3, w, y) dw dz \\ &\geq \int_B p_D^a(t/3, x, z) \int_{\tilde{B}} p_D^a(t/3, z, w) p_D^a(t/3, w, y) dw dz \\ &\geq \inf_{(z, w) \in B \times \tilde{B}} p_D^a(t/3, z, w) \int_B p_D^a(t/3, x, z) dz \int_{\tilde{B}} p_D^a(t/3, w, y) dw. \end{aligned}$$

Since for $z \in B$ and $w \in \tilde{B}$,

$$\delta_D(z) \geq \delta_D(x_0) - |x_0 - z| \geq t^{1/\alpha}, \quad \delta_D(w) \geq \delta_D(y_0) - |y_0 - w| \geq t^{1/\alpha},$$

$$|z - w| \leq |z - x_0| + |x_0 - x| + |x - y| + |y - y_0| + |y_0 - w| < 10t^{1/\alpha} \psi^a(|x - y|)^{1/(d+\alpha)},$$

by combining Proposition 3.2 and Proposition 3.4, we have that there exists $c_1 = c_1(\alpha, \beta, r_0, M) > 0$ such that for all $a \in (0, M]$,

$$\inf_{(z, w) \in B \times \tilde{B}} p_D^a(t/3, z, w) \geq c_1 t^{-d/\alpha}.$$

Since $\delta_D(x) \leq 2t^{1/\alpha} < r_0/8$ and $\delta_D(y) \leq 3t^{1/\alpha}$, we have by Lemma 3.8

$$p_D^a(t, x, y) \geq c_2 t^{-d/\alpha-1} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}$$

for some positive constant $c_2 = c_2(\alpha, \beta, M, r_0)$. □

Proposition 3.10 *Suppose that $M > 0$ and $(t, x, y) \in (0, T_0] \times D \times D$ with $\delta_D(x) \leq (t/2)^{1/\alpha} \leq \delta_D(y)$ and $|x - y|^\alpha \geq t\psi^a(|x - y|)^{\alpha/(d+\alpha)}$. Then there exists a constant $C_{29} = C_{29}(\alpha, \beta, M, r_0) > 0$ such that for all $a \in (0, M]$,*

$$p_D^a(t, x, y) \geq C_{29} t^{1/2} \delta_D(x)^{\alpha/2} J^a(x, y). \quad (3.24)$$

Proof. Since $\delta_D(x) \leq (t/2)^{1/\alpha} \leq r_0/16$, there are $z_x \in \partial D$ and $z_0 \in D$ such that $\delta_D(x) = |x - z_x|$ and $\partial B(z_0, 2t^{1/\alpha}) \cap \partial D = \{z_x\}$. Choose x_0 in $B(z_0, 2t^{1/\alpha})$ and $\kappa = \kappa(\alpha) \in (0, 1)$ such that

$$B(x_0, 2\kappa t^{1/\alpha}) \subset B(z_0, (2 - 2^{-2/\alpha})t^{1/\alpha}) \cap B(x, (1 - 2^{-1-2/\alpha})t^{1/\alpha}).$$

Such a ball $B(x_0, 2\kappa t^{1/\alpha})$ always exists because

$$2 < (2 - 2^{-1}) + (1 - 2^{-2}) < (2 - 2^{-2/\alpha}) + (1 - 2^{-1-2/\alpha}).$$

Since $|x - y| \geq t^{1/\alpha} \psi^a(|x - y|)^{1/(d+\alpha)}$, we get that for every $z \in B(x_0, \kappa t^{1/\alpha})$, $\delta_D(z) \geq (t/4)^{1/\alpha}$ and

$$|y - z| \geq |y - x| - |z - x| \geq 2^{-1}(t/4)^{1/\alpha} \psi^a(|x - y|)^{1/(d+\alpha)}.$$

On the other hand, for every $z \in B(x_0, \kappa t^{1/\alpha})$,

$$|z - y| \leq |z - x| + |x - y| \leq (1 - 2^{-1-2/\alpha})t^{1/\alpha} + |x - y| < 2|x - y|.$$

Thus by the semigroup property and Propositions 3.2 and 3.4, there exist positive constants c_1, c_2 and c_3 depending only on (α, β, r_0, M) such that for all $a \in (0, M]$,

$$\begin{aligned}
p_D^a(t, x, y) &= \int_D p_D^a(t/2, x, z) p_D^a(t/2, z, y) dz \\
&\geq \int_{B(x_0, \kappa t^{1/\alpha})} p_D^a(t/2, x, z) p_D^a(t/2, z, y) dz \\
&\geq c_1 t \int_{B(x_0, \kappa t^{1/\alpha})} p_D^a(t/2, x, z) J^a(z, y) dz \\
&\geq c_2 t j^a(2|x-y|) \int_{B(x_0, \kappa t^{1/\alpha})} p_D^a(t/2, x, z) dz \\
&\geq c_3 t j^a(|x-y|) \mathbb{P}_x \left(X_{t/2}^{a,D} \in B(x_0, \kappa t^{1/\alpha}) \right).
\end{aligned}$$

Applying Lemma 3.8, we arrive at the conclusion of the proposition. \square

Proposition 3.11 *Suppose that $M > 0$ and $(t, x, y) \in (0, T_0] \times D \times D$ with*

$$\max\{\delta_D(x), \delta_D(y)\} \leq (t/2)^{1/\alpha} \leq |x-y| \psi^a(|x-y|)^{-1/(d+\alpha)}.$$

Then there exists a constant $C_{30} = C_{30}(\alpha, \beta, M, r_0) > 0$ such that for all $a \in (0, M]$,

$$p_D^a(t, x, y) \geq C_{30} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} J^a(x, y). \quad (3.25)$$

Proof. As in the first paragraph of the proof of Proposition 3.9, set $z_x \in \partial D$ and $x_0 \in D$ so that $|x - z_x| = \delta_D(x)$ and $\partial B(x_0, 3t^{1/\alpha}) \cap \partial D = \{z_x\}$. Let $\kappa := 1 - 2^{-1/\alpha}$. Note that for every $z \in B(x_0, \kappa t^{1/\alpha})$, we have

$$4t^{1/\alpha} \geq \delta_D(z) \geq 2(t/2)^{1/\alpha}.$$

If $|y - z| \leq t^{1/\alpha} \psi^a(|y - z|)^{1/(d+\alpha)}$, we can apply Proposition 3.9 and the assumption

$$(t/2)^{1/\alpha} \leq |x - y| \psi^a(|x - y|)^{-1/(d+\alpha)}$$

to get that

$$p^a(t/2, z, y) \geq c_1 t^{1/2} \delta_D(y)^{\alpha/2} J^a(x, y).$$

If $|y - z| \geq t^{1/\alpha} \psi^a(|y - z|)^{1/(d+\alpha)}$, we can apply Proposition 3.10 to get that

$$p^a(t/2, z, y) \geq c_2 t^{1/2} \delta_D(y)^{\alpha/2} J^a(y, z).$$

For $z \in B(x_0, \kappa t^{1/\alpha})$, we have

$$\begin{aligned}
|z - y| &\leq |x - y| + |x_0 - x| + |x_0 - z| \leq |x - y| + 4t^{1/\alpha} \\
&\leq |x - y| + 2^{2+1/\alpha} (t/2)^{1/\alpha} \psi^a(|x - y|)^{1/(d+\alpha)} \\
&\leq (1 + 2^{2+1/\alpha}) |x - y|.
\end{aligned}$$

Thus if $|y - z| \geq t^{1/\alpha} \psi^a(|y - z|)^{1/(d+\alpha)}$, we have

$$p^a(t/2, z, y) \geq c_3 t^{1/2} \delta_D(y)^{\alpha/2} J^a(x, y).$$

Consequently we have for all $z \in B(x_0, \kappa t^{1/\alpha})$

$$p^\alpha(t/2, z, y) \geq c_4 t^{1/2} \delta_D(y)^{\alpha/2} J^\alpha(x, y).$$

Hence by the semigroup property we get

$$\begin{aligned} p_D^\alpha(t, x, y) &= \int_D p_D^\alpha(t/2, x, z) p_D^\alpha(t/2, z, y) dz \\ &\geq \int_{B(x_0, \kappa t^{1/\alpha})} p_D^\alpha(t/2, x, z) p_D^\alpha(t/2, z, y) dz \\ &\geq c_4 \int_{B(x_0, \kappa t^{1/\alpha})} p_D^\alpha(t/2, x, z) t^{1/2} \delta_D(y)^{\alpha/2} J^\alpha(x, y) dz \\ &= c_4 t^{1/2} \delta_D(y)^{\alpha/2} J^\alpha(x, y) \int_{B(x_0, \kappa t^{1/\alpha})} p_D^\alpha(t/2, x, z) dz \\ &= c_4 t^{1/2} \delta_D(y)^{\alpha/2} J^\alpha(x, y) \mathbb{P}_x \left(X_{t/2}^{a, D} \in B(x_0, \kappa t^{1/\alpha}) \right) \\ &= c_5 \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} J^\alpha(x, y). \end{aligned}$$

We arrive at the conclusion of the proposition. \square

Proof of Theorem 3.5. In this proof, for two non-negative functions f and g , the notation $f \asymp g$ means that there are positive constants c_1 and c_2 depending only on M , d , α and β so that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ in the common domain of definition for f and g .

We first assume that $t \leq T_0$.

1. We first consider the case $|x - y| \psi^\alpha(|x - y|)^{-1/(d+\alpha)} \leq t^{1/\alpha}$. We claim that in this case

$$p_D(t, x, y) \geq ct^{-d/\alpha} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right). \quad (3.26)$$

This will be proved by considering the following two possibilities.

- (a) $\max\{\delta_D(x), \delta_D(y), |x - y| \psi^\alpha(|x - y|)^{-1/(d+\alpha)}\} \leq t^{1/\alpha}$: Proposition 3.9 and symmetry yield (3.26)
- (b) $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha} \geq |x - y| \psi^\alpha(|x - y|)^{-1/(d+\alpha)}$:
If $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha} \geq 2|x - y| \psi^\alpha(|x - y|)^{-1/(d+\alpha)}$, (3.26) follows from Proposition 3.2.

If $\min\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$ and $|x - y| \leq t^{1/\alpha} < 2|x - y| \psi^\alpha(|x - y|)^{-1/(d+\alpha)}$,

$$\frac{t \psi^\alpha(|x - y|)}{|x - y|^{d+\alpha}} \asymp t^{-d/\alpha} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right).$$

If $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$, $\min\{\delta_D(x), \delta_D(y)\} < t^{1/\alpha}$ and $|x - y| \psi^\alpha(|x - y|)^{-1/(d+\alpha)} \leq t^{1/\alpha} < 2|x - y| \psi^\alpha(|x - y|)^{-1/(d+\alpha)}$,

$$\left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right) \asymp \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}} \right)$$

Thus by combining Proposition 3.4 and Proposition 3.9, we get (3.26) for the case of $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$ and $|x - y| \leq t^{1/\alpha} < 2|x - y| \psi^\alpha(|x - y|)^{-1/(d+\alpha)}$.

2. Now we consider the case $|x - y|\psi^\alpha(|x - y|)^{-1/(d+\alpha)} \geq t^{1/\alpha}$ and claim that

$$p_D(t, x, y) \geq c \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) t j^\alpha(|x - y|). \quad (3.27)$$

- (a) $\min\{\delta_D(x), \delta_D(y)\} \leq (t/2)^{1/\alpha}$ and $|x - y|\psi^\alpha(|x - y|)^{-1/(d+\alpha)} \geq t^{1/\alpha}$: By symmetry we can assume $\delta_D(x) \leq (t/2)^{1/\alpha}$. Thus combining Propositions 3.10 and 3.11, we have (3.27) for this case.
- (b) $\min\{\delta_D(x), \delta_D(y)\} \geq (t/2)^{1/\alpha}$ and $|x - y|\psi^\alpha(|x - y|)^{-1/(d+\alpha)} \geq t^{1/\alpha}$. In this case, clearly

$$\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \asymp \left(\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right).$$

Thus Proposition 3.4 yields (3.27).

We have arrived at the conclusion of Theorem 3.5 for $t \leq T_0$.

Assume $T = 2T_0$. Recall that $T_0 = (r_0/16)^\alpha$. For $(t, x, y) \in (T_0, 2T_0] \times D \times D$, let $x_0, y_0 \in D$ be such that $\max\{|x - x_0|, |y - y_0|\} < r_0$ and $\min\{\delta_D(x_0), \delta_D(y_0)\} \geq r_0/2$. Note that, since

$$j^\alpha(r) \leq c_1 j^\alpha(2r), \quad \text{for all } r > 0, \quad (3.28)$$

if $|x - y| \geq 4r_0$, then $\frac{1}{2}|x - y| \leq |x - y| - 2r_0 \leq |x_0 - y_0| \leq |x - y| + 2r_0 \leq \frac{3}{2}|x - y|$, and so $c_2^{-1} J^\alpha(x_0, y_0) \leq J^\alpha(x, y) \leq c_2 J^\alpha(x_0, y_0)$ for some constant $c_2 = c_2(M) > 1$. Thus by considering the cases $|x - y| \geq 4r_0$ and $|x - y| < 4r_0$, we have

$$(t/2)^{-d/\alpha} \wedge \frac{t J^\alpha(x_0, y_0)}{2} \geq c_3 \left(t^{-d/\alpha} \wedge (t J^\alpha(x, y))\right). \quad (3.29)$$

Similarly, there is a positive constant c_2 such that

$$\begin{aligned} (t/3)^{-d/\alpha} \wedge \frac{t J^\alpha(x, z)}{3} &\geq c_4 \left((t/(12))^{-d/\alpha} \wedge \frac{t J^\alpha(x_0, z)}{12}\right), \quad z \in D, \\ (t/3)^{-d/\alpha} \wedge \frac{t J^\alpha(w, y)}{3} &\geq c_4 \left((t/(12))^{-d/\alpha} \wedge \frac{t J^\alpha(w, y_0)}{12}\right), \quad w \in D. \end{aligned} \quad (3.30)$$

By (3.30) and the lower bound estimate in Theorem 3.5 for p_D^α on $(0, T_0] \times D \times D$, we have

$$\begin{aligned} p_D^\alpha(t, x, y) &= \int_{D \times D} p_D^\alpha(t/3, x, z) p_D^\alpha(t/3, z, w) p_D^\alpha(t/3, w, y) dz dw \\ &\geq c_5 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t/3}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t/3}}\right) \int_{D \times D} \left((t/3)^{-d/\alpha} \wedge \frac{t J^\alpha(x, z)}{3}\right) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right) \\ &\quad \cdot p_D^\alpha(t/3, z, w) \left((t/3)^{-d/\alpha} \wedge \frac{t J^\alpha(w, y)}{3}\right) \left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right) dz dw \\ &\geq c_6 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \int_{D \times D} \left(\left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{t J^\alpha(x_0, z)}{12}\right) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right) \\ &\quad \cdot p_D^\alpha(t/3, z, w) \left(\left(\frac{t}{12}\right)^{-d/\alpha} \wedge \frac{t J^\alpha(w, y_0)}{12}\right) \left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right) dz dw \end{aligned}$$

for some positive constants $c_i, i = 3, 4$. Let $D_1 := \{z \in D : \delta_D(z) > r_0/4\}$. Clearly, $x_0, y_0 \in D_1$ and

$$\min\{\delta_{D_1}(x_0), \delta_{D_1}(y_0)\} \geq r_0/4 = 4(T_0)^{1/\alpha} \geq 4(t/2)^{1/\alpha}. \quad (3.31)$$

By (1.4) and (3.29), we have

$$\begin{aligned} & \int_{D \times D} \left(\left(\frac{t}{12} \right)^{-d/\alpha} \wedge \frac{tJ^a(x_0, z)}{12} \right) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}} \right) \\ & \quad \cdot p_D^a(t/3, z, w) \left(\left(\frac{t}{12} \right)^{-d/\alpha} \wedge \frac{tJ^a(w, y_0)}{12} \right) \left(1 \wedge \frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}} \right) dzdw \\ & \geq c_7 \int_{D_1 \times D_1} \left(\left(\frac{t}{12} \right)^{-d/\alpha} \wedge \frac{tJ^a(x_0, z)}{12} \right) p_D^a(t/3, z, w) \left(\left(\frac{t}{12} \right)^{-d/\alpha} \wedge \frac{tJ^a(w, y_0)}{12} \right) dzdw \\ & \geq c_8 \int_{D_1 \times D_1} p^a(t/(12), x_0, z) p_{D_1}^a(t/3, z, w) p^a(t/(12), w, y_0) dzdw \\ & \geq c_8 \int_{D_1 \times D_1} p_{D_1}^a(t/(12), x_0, z) p_{D_1}^a(t/3, z, w) p_{D_1}^a(t/(12), w, y_0) dzdw \\ & = c_8 p_{D_1}^a(t/2, x_0, y_0) \geq c_9 \left((t/2)^{-d/\alpha} \wedge \frac{tJ^a(x_0, y_0)}{2} \right) \geq c_{10} \left(t^{-d/\alpha} \wedge (tJ^a(x, y)) \right) \end{aligned}$$

for some positive constants $c_i, i = 7, \dots, 10$. Here both Propositions 3.2 and 3.4 are used in the third inequality in view of (3.31). By repeating the argument above, we have proved Theorem 3.5. \square

Proof of Theorem 1.1. Theorems 2.8 and 3.5 give Theorem 1.1(i). By [15, 18], for any bounded open set D in \mathbb{R}^d , $X^{a,D}$ is intrinsically ultracontractive. Since the function $\psi^a(|x-y|)$ is bounded above and below by a positive constant if D is bounded, using the intrinsic ultracontractivity of X^a on bounded open set and the continuity of eigenvalues proved in [10], the proof of Theorem 1.1 (ii) is almost identical to the one of [5, Theorem 1.1(ii)]. We omit the details. \square

Proof of Corollary 1.2. The lower bound estimate in (1.5) follows from (3.10) and Theorem 3.6.

Since the function $\psi^a(|x-y|)$ is bounded above and below by a positive constant if D is bounded, by integrating the two-sided heat kernel estimates in Theorem 1.1 with respect to t , the proof of the upper bound estimate in (1.5) is identical to the one of [3, Corollary 1.2] so we omit its details here. \square

Theorem 3.12 (Uniform boundary Harnack principle) *Suppose $M, R \in (0, \infty)$ and that D is an open set in \mathbb{R}^d , $z \in \partial D$, $r \in (0, R)$ and that $B(A, \kappa r) \subset D \cap B(z, r)$. There exists $C_{31} = C_{31}(d, \alpha, \beta, \kappa, M, R) > 1$ such that for every $a \in (0, M]$, and any functions $u, v \geq 0$ on \mathbb{R}^d , positive regular harmonic for X^a in $D \cap B(z, 2r)$ and vanishing on $D^c \cap B(z, 2r)$, we have*

$$C_{31}^{-1} \frac{u(A)}{v(A)} \leq \frac{u(x)}{v(x)} \leq C_{31} \frac{u(A)}{v(A)}, \quad x \in D \cap B(z, r).$$

Proof. Note that by the approximate scaling property in (1.3), we have for every $r > 0$.

$$G_{B(0,r)}^a(x, y) = r^{\alpha-d} G_{B(0,1)}^{ar^{(\alpha-\beta)/\beta}}(x/r, y/r). \quad (3.32)$$

Thus applying [3, Corollary 1.2] and our Corollary 1.2 to (3.32), we have that for every $R, M > 0$, there exists $c = c(\alpha, \beta, R, M) > 0$ such that, for every $a \in (0, M]$ and $0 < r \leq R$

$$c^{-1} G_{B(x_0,r)}^a(x, y) \leq G_{B(x_0,r)}^a(x, y) \leq c G_{B(x_0,r)}^a(x, y), \quad \forall x, y \in B(x_0, r). \quad (3.33)$$

Using (3.33), we can get uniform estimates on the Poisson kernel

$$K_{B(x_0,r)}^a(x, z) := \int_{B(x_0,r)} G_{B(x_0,r)}^a(x, y) J^a(y, z) dy$$

of $B(x_0, r)$ with respect to X^a for $r \in (0, R]$. In particular, for $r < |z - x_0| < 2R$, $K_{B(x_0,r)}^a(x, z)$ is comparable to $K_{B(x_0,r)}(x, z)$, the Poisson kernel of $B(x_0, r)$ with respect to X for $r \in (0, R]$. Then using the uniform estimates on $K_{B(x_0,r)}^a(x, z)$ and (3.33) we can easily see that [27, Lemma 3.3] can be proved in the same way. Using the uniform estimates on the Poisson kernel of $B(x_0, r)$, (3.28) and (3.33) we can adapt the argument in [1, 19, 27] to get our uniform boundary Harnack principle. We omit the details. \square

Proof of Theorem 1.3 First we observe that Harnack inequality holds for the process $X := X^1$ by [21]. That is, there exists a constant $c_1 = c_1(\alpha, \beta, M) > 0$ such that for any $r \in (0, M^{\beta/(\alpha-\beta)}]$, $x_0 \in \mathbb{R}^d$ and any function $v \geq 0$ harmonic in $B(x_0, r)$ with respect to X , we have

$$v(x) \leq c_1 v(y) \quad \text{for all } x, y \in B(x_0, r/2). \quad (3.34)$$

Note that for any $a \in (0, M]$, X^a has the same distribution as $\{\lambda X_{\lambda^{-\alpha}t}, t \geq 0\}$, where $\lambda = a^{\beta/(\beta-\alpha)} \geq M^{\beta/(\beta-\alpha)}$. Consequently, if u is harmonic in $B(x_0, r)$ with respect to X^a , where $r \in (0, 1]$, then $v(x) := u(\lambda x)$ is harmonic in $B(\lambda^{-1}x_0, \lambda^{-1}r)$ with respect to X and $\lambda^{-1}r \leq M^{\beta/(\beta-\alpha)}$. So by (3.34)

$$u(\lambda x) = v(x) \leq c_1 v(y) = c_1 u(\lambda y) \quad \text{for all } x, y \in B(\lambda^{-1}x_0, \lambda^{-1}r/2).$$

That is,

$$u(x) \leq c_1 u(y) \quad \text{for all } x, y \in B(x_0, r/2). \quad (3.35)$$

In other words, uniform Harnack inequality holds (for every $r \leq 1$) for the family of processes $\{X^a, a \in (0, M]\}$.

Since D is $C^{1,1}$ open set, there exists $r_0 \leq R_0$ such that the following holds: for every $Q \in \partial D$ and $r \leq r_0$ there is a ball $B = B(z_Q^r, r)$ of radius r such that $B \subset \mathbb{R}^d \setminus \bar{D}$ and $\partial B \cap \partial D = \{Q\}$. In addition, it follows [23, Lemma 2.2] that, for each $Q \in \partial D$, we can choose a constant $c_2 = c_2(d, \Lambda) \in (0, 1/8]$ and a bounded $C^{1,1}$ open set U_Q with uniform characteristics (R_*, Λ_*) depending on (R_0, Λ) such that $B(Q, c_2 r_0) \cap D \subset U_Q \subset B(Q, r_0) \cap D$ and

$$\delta_D(y) = \delta_{U_Q}(y) \quad \text{for every } y \in B(Q, c_2 r_0) \cap D. \quad (3.36)$$

Assume $a \in [0, M]$, $r \in (0, c_2 r_0]$, $Q \in \partial D$ and u is nonnegative function in \mathbb{R}^d harmonic in $D \cap B(Q, r)$ with respect to X^a and vanishes continuously on $D^c \cap B(Q, r)$. Let $z_Q := z_Q^{c_2 r_0}$. By

the boundary Harnack principle (Theorem 3.12), there exists a constant $c_3 = c_3(\alpha, \beta, a, R_0, \Lambda, M)$ such that

$$\frac{u(x)}{u(y)} \leq c_3 \frac{G_{U_Q}^a(x, z_Q)}{G_{U_Q}^a(y, z_Q)} \quad \text{for every } x, y \in B(Q, r/8) \cap D.$$

Now applying Corollary 1.2 to $G_{U_Q}^a(x, z_Q)$ and $G_{U_Q}^a(y, z_Q)$, then using (3.36), we conclude that

$$\frac{u(x)}{u(y)} \leq c_4 \frac{\delta_{U_Q}^{\alpha/2}(x)}{\delta_{U_Q}^{\alpha/2}(y)} = c_4 \frac{\delta_D^{\alpha/2}(x)}{\delta_D^{\alpha/2}(y)} \quad \text{for every } x, y \in B(Q, c_2 r) \cap D \quad (3.37)$$

for some $c_4 = c_4(\alpha, \beta, a, R_0, \Lambda, M) > 0$.

Now Theorem 1.3 follows from the uniform Harnack principle in (3.35), (3.37) and a standard chain argument. \square

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