

# Boundary Harnack principle for $\Delta + \Delta^{\alpha/2}$

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## Abstract

For  $d \geq 1$  and  $\alpha \in (0, 2)$ , consider the family of pseudo differential operators  $\{\Delta + b\Delta^{\alpha/2}; b \in [0, 1]\}$  on  $\mathbb{R}^d$  that evolves continuously from  $\Delta$  to  $\Delta + \Delta^{\alpha/2}$ . In this paper, we establish a uniform boundary Harnack principle (BHP) with explicit boundary decay rate for nonnegative functions which are harmonic with respect to  $\Delta + b\Delta^{\alpha/2}$  (or equivalently, the sum of a Brownian motion and an independent symmetric  $\alpha$ -stable process with constant multiple  $b^{1/\alpha}$ ) in  $C^{1,1}$  open sets. Here a “uniform” BHP means that the comparing constant in the BHP is independent of  $b \in [0, 1]$ . Along the way, a uniform Carleson type estimate is established for nonnegative functions which are harmonic with respect to  $\Delta + b\Delta^{\alpha/2}$  in Lipschitz open sets. Our method employs a combination of probabilistic and analytic techniques.

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## 1 Introduction

Throughout this paper, we assume that  $d \geq 1$  is an integer and  $\alpha \in (0, 2)$ . Let  $X^0 = (X_t^0, t \geq 0)$  be a Brownian motion in  $\mathbb{R}^d$  and  $Y = (Y_t, t \geq 0)$  an independent symmetric  $\alpha$ -stable process in  $\mathbb{R}^d$ . The process  $X = (X_t, t \geq 0)$  defined by  $X_t = X_t^0 + Y_t$ , called *the independent sum of a Brownian motion and a symmetric  $\alpha$ -stable process*, is a prototype of a strong Markov process (actually a Lévy process) with both continuous and jump components. Such kind of processes take up an important place both in theory and in applications. The fact that  $X$  has both continuous and jump components is the source of many difficulties in investigating the potential theory of the process  $X$ . The main difficulty in studying  $X$  stems from the fact that it runs on two different scales: on the small scale the continuous part dominates, while on the large scale the jumps take over. Another

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difficulty is encountered when looking at the exit of  $X$  from an open set: for diffusions, the exit is through the boundary, while for the pure jump processes, typically the exit happens by jumping out from the set. For the process  $X$ , both cases will occur which makes the process  $X$  much more difficult to study.

Despite these difficulties, in the last few years significant progress has been made in understanding the potential theory of such processes. Green function estimates (for the whole space) and the Harnack inequality for some processes with both continuous and jump components were established in [15] and [16]. The parabolic Harnack inequality and heat kernel estimates were studied in [17] for the process  $X$ , and in [9] for much more general diffusions with jumps. Moreover, an a priori Hölder estimate is established in [9] for bounded parabolic functions. Very recently, the boundary Harnack principle for some one-dimensional Lévy processes with both continuous and jump components was studied in [14]. The principal goal of this paper is to establish the boundary Harnack principle for nonnegative functions which are harmonic with respect to the independent sum of a Brownian motion and a symmetric stable process in  $C^{1,1}$  open sets in  $\mathbb{R}^d$  for every  $d \geq 1$ . The process  $X$  studied in this paper, although quite specific, serves as a test case for more general processes with both continuous and jump parts. The study of this test case will hopefully shed new light on the understanding of the boundary behavior of nonnegative harmonic functions of general Markov processes.

Let us recall the classical boundary Harnack principle for the Brownian motion (or, equivalently, for the Laplacian): for any Lipschitz domain  $D$  in  $\mathbb{R}^d$ , there exist positive constants  $R_0$  and  $C$  with the property that for any  $Q \in \partial D$ ,  $r \in (0, R_0]$ , and any positive harmonic functions  $u$  and  $v$  in  $D \cap B(Q, r)$  that vanish continuously on  $\partial D \cap B(Q, r)$ , we have

$$(1.1) \quad \frac{u(x)}{u(y)} \leq C \frac{v(x)}{v(y)} \quad \text{for all } x, y \in D \cap B(Q, r/2).$$

Here, and in the sequel,  $B(Q, r)$  denotes the open ball in  $\mathbb{R}^d$  centered at  $Q$  with radius  $r > 0$ . This is a very deep result in classical potential theory and it was independently established in the late 1970's by Ancona, Dahlberg and Wu ([1, 11, 20]). Later, Bass and Burdzy developed a probabilistic method in [3] to prove the boundary Harnack principle and extended the boundary Harnack principle to more general domains (see also [2]). The boundary Harnack principle for symmetric stable processes was first established for bounded Lipschitz domains in [5] and then extended to more general open sets in [19]. Recently, the boundary Harnack principle has been extended in [13] to a large class of pure jump Lévy processes. The fact that a pure jump process may (and typically does) exit an open set by jumping out of it requires that, in the boundary Harnack principle for such processes, the nonnegative harmonic functions vanish continuously on  $D^c \cap B(Q, r)$ .

Let us now describe the result of this paper in more detail and at the same time fix the notations. A (rotationally) symmetric  $\alpha$ -stable process  $Y = (Y_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d)$  in  $\mathbb{R}^d$  is a Lévy process such that

$$\mathbb{E}_x \left[ e^{i\xi \cdot (Y_t - Y_0)} \right] = e^{-t|\xi|^\alpha} \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

The infinitesimal generator of a symmetric  $\alpha$ -stable process  $Y$  in  $\mathbb{R}^d$  is the fractional Laplacian  $\Delta^{\alpha/2}$ , which is a prototype of nonlocal operators. The fractional Laplacian can be written in the

form

$$(1.2) \quad \Delta^{\alpha/2}u(x) = \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: |y-x| > \varepsilon\}} (u(y) - u(x)) \frac{\mathcal{A}(d, \alpha)}{|x-y|^{d+\alpha}} dy,$$

where  $\mathcal{A}(d, \alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$ . Here  $\Gamma$  is the Gamma function defined by  $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$  for every  $\lambda > 0$ .

We now give a precise definition of the independent sum of two processes and state the main result of this paper. Suppose  $X^0$  is a Brownian motion in  $\mathbb{R}^d$  with generator  $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ , and  $Y$  is a symmetric  $\alpha$ -stable process in  $\mathbb{R}^d$ . Both  $X^0$  and  $Y$  satisfy a self-similarity, which will be used several times in this paper. That is, for every  $\lambda > 0$ ,  $\{\lambda^{-1/2}(X_{\lambda t}^0 - X_0^0), t \geq 0\}$  and  $\{\lambda^{-1/\alpha}(Y_{\lambda t} - Y_0), t \geq 0\}$  have the same distributions as that of  $\{X_t^0 - X_0^0, t \geq 0\}$  and  $\{Y_t - Y_0, t \geq 0\}$ , respectively. Assume that  $X^0$  and  $Y$  are independent. For any  $a > 0$ , we define  $X^a$  by  $X_t^a := X_t^0 + aY_t$ . We will call the process  $X^a$  the independent sum of the Brownian motion  $X^0$  and the symmetric  $\alpha$ -stable process  $Y$  with weight  $a > 0$ . The infinitesimal generator of  $X^a$  is  $\Delta + a^\alpha \Delta^{\alpha/2}$ . For every open subset  $D \subset \mathbb{R}^d$ , we denote by  $X^{a,D}$  the subprocess of  $X^a$  killed upon leaving  $D$ . The infinitesimal generator of  $X^{a,D}$  is  $(\Delta + a^\alpha \Delta^{\alpha/2})|_D$ . It is known (see [17]) that  $X^{a,D}$  has a continuous transition density  $p_D^a(t, x, y)$  with respect to the Lebesgue measure. We will use  $p^a(t, x, y)$  to denote the transition density of  $X^a$  (or equivalently, the heat kernel of  $\Delta + a^\alpha \Delta^{\alpha/2}$ ). The quadratic form  $(\mathcal{E}, \mathcal{F})$  associated with the generator  $\Delta + a^\alpha \Delta^{\alpha/2}$  of  $X^a$  is given by

$$\mathcal{F} = W^{1,2}(\mathbb{R}^d) := \left\{ u \in L^2(\mathbb{R}^d; dx) : \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^d; dx) \text{ for every } 1 \leq i \leq d \right\}$$

and for  $u, v \in \mathcal{F}$ ,

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla v(x) dx + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)) \frac{\mathcal{A}(d, \alpha) a^\alpha}{|x-y|^{d+\alpha}} dx dy.$$

In probability theory, the quadratic form  $(\mathcal{E}, W^{1,2}(\mathbb{R}^d))$  is called the Dirichlet form of  $X^a$ . A statement is said to hold quasi-everywhere (q.e. in abbreviation) if there is a set  $N$  having zero capacity with respect to  $(\mathcal{E}_1, W^{1,2}(\mathbb{R}^d))$  such that the statement holds everywhere outside  $N$ . Here  $\mathcal{E}_1(u, u) := \mathcal{E}(u, u) + \int_{\mathbb{R}^d} u(x)^2 dx$ . The function  $J^a(x, y) := a^\alpha \mathcal{A}(d, \alpha) |x-y|^{-(d+\alpha)}$  is the Lévy intensity of  $X^a$ . It determines the Lévy system for  $X^a$ , which describes the jumps of the process  $X^a$ : for any non-negative measurable function  $f$  on  $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ ,  $x \in \mathbb{R}^d$  and stopping time  $T$  (with respect to the filtration of  $X^a$ ),

$$(1.3) \quad \mathbb{E}_x \left[ \sum_{s \leq T} f(s, X_{s-}^a, X_s^a) \right] = \mathbb{E}_x \left[ \int_0^T \left( \int_{\mathbb{R}^d} f(s, X_s^a, y) J^a(X_s^a, y) dy \right) ds \right].$$

(See, for example [8], Appendix A.)

The purpose of this paper is to establish the following scale invariant version of the boundary Harnack principle in Theorem 1.4. To state this theorem, we first recall that an open set  $D$  in  $\mathbb{R}^d$  (when  $d \geq 2$ ) is said to be  $C^{1,1}$  if there exist a localization radius  $R > 0$  and a constant  $\Lambda > 0$  such that for every  $Q \in \partial D$ , there exist a  $C^{1,1}$ -function  $\phi = \phi_Q : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  satisfying  $\phi(0) = \nabla \phi(0) = 0$ ,  $\|\nabla \phi\|_\infty \leq \Lambda$ ,  $|\nabla \phi(x) - \nabla \phi(y)| \leq \Lambda|x-y|$ , and an orthonormal coordinate system  $CS_Q: y = (y_1, \dots, y_{d-1}, y_d) =: (\tilde{y}, y_d)$  with its origin at  $Q$  such that

$$B(Q, R) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \phi(\tilde{y})\}.$$

The pair  $(R, \Lambda)$  is called the characteristics of the  $C^{1,1}$  open set  $D$ . By a  $C^{1,1}$  open set in  $\mathbb{R}^d$  we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a  $C^{1,1}$  open set can be unbounded and disconnected.

For any  $x \in D$ , let  $\delta_D(x)$  denote the distance between  $x$  and  $\partial D$ . It is well known that any  $C^{1,1}$  open set  $D$  satisfies the uniform interior ball condition: there exists  $\tilde{R} \leq R$  such that for every  $x \in D$  with  $\delta_D(x) < \tilde{R}$ , there is  $Q_x \in \partial D$  so that  $|x - Q_x| = \delta_D(x)$  and that  $B(\tilde{x}, \tilde{R}) \subset D$  for  $\tilde{x} = Q_x + \tilde{R}(x - Q_x)/|x - Q_x|$ . Without loss of generality, throughout this paper, we assume that the characteristics  $(R, \Lambda)$  of a  $C^{1,1}$  open set satisfies  $R = \tilde{R} \leq 1$  and  $\Lambda \geq 1$ .

For any open set  $D \subset \mathbb{R}^d$ ,  $\tau_D^a := \inf\{t > 0 : X_t^a \notin D\}$  denotes the first exit time from  $D$  by  $X^a$ .

**Definition 1.1** A real-valued function  $u$  defined on  $\mathbb{R}^d$  is said to be harmonic in  $D \subset \mathbb{R}^d$  with respect to  $X^a$  if for every open set  $B$  whose closure is a compact subset of  $D$ ,

$$(1.4) \quad \mathbb{E}_x \left[ |u(X_{\tau_B^a}^a)| \right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x \left[ u(X_{\tau_B^a}^a) \right] \quad \text{for q.e. } x \in B.$$

Note that by using the Lévy system of  $X^a$ , we have

$$\begin{aligned} \mathbb{E}_x \left[ |u(X_{\tau_B^a}^a)| \right] &\geq \mathbb{E}_x \left[ |u(X_{\tau_B^a}^a)|; X_{\tau_B^a}^a \in \mathbb{R}^d \setminus \bar{B} \right] \\ &= \mathbb{E}_x \left[ \int_0^{\tau_B} \left( \int_{\mathbb{R}^d \setminus \bar{B}} |u(y)| \frac{\mathcal{A}(d, \alpha) a^\alpha}{|X_s^a - y|^{d+\alpha}} dy \right) ds \right]. \end{aligned}$$

Hence if  $u$  is a harmonic function in  $D$  with respect to  $X^a$ , then  $u(y)(1 \wedge |y|^{-(d+\alpha)})$  is integrable on  $B^c$  for any relatively compact open subset  $B$  with  $\bar{B} \subset D$ . It follows from Theorems 1.2 and 1.3 of [9] that all harmonic functions in  $D$  with respect to  $X^a$  are continuous on  $D$ , since every harmonic function in  $D$  with respect to  $X^a$  can be approximated locally uniformly in  $D$  by functions that are bounded on  $\mathbb{R}^d$  and harmonic with respect to  $X^a$  in relatively compact open subsets of  $D$ . Therefore, for any harmonic function  $u$  in  $D$ , (1.4) holds for every point  $x \in D$ . The above also implies that any harmonic function  $u$  in  $D$  with respect to  $X^a$  is locally bounded in  $D$  with  $\int_{\mathbb{R}^d} |u(y)|(1 \wedge |y|^{-(d+\alpha)}) dy < \infty$ . A function  $u$  is said to be in  $W_{\text{loc}}^{1,2}(D)$  if for every relatively compact subset  $B$  with  $\bar{B} \subset D$ , there is a function  $f \in W^{1,2}(\mathbb{R}^d)$  such that  $u = f$  a.e. on  $B$ . The following analytic characterization of a function  $u$  being harmonic in  $D$  with respect to  $X^a$  follows immediately from Example 2.14 in [7].

**Proposition 1.2** *Let  $D$  be an open subset of  $\mathbb{R}^d$ . Then the following are equivalent.*

- (i)  $u$  is harmonic in  $D$  with respect to  $X^a$ ;
- (ii)  $u$  is locally bounded in  $D$ ,  $\int_{\mathbb{R}^d} |u(y)|(1 \wedge |y|^{-(d+\alpha)}) dy < \infty$ ,  $u \in W_{\text{loc}}^{1,2}(D)$  and  $(\Delta + a^\alpha \Delta^{\alpha/2})u = 0$  in  $D$  in the distributional sense: for every  $\phi \in C_c^\infty(D)$

$$\int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla \phi(x) dx + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))(\phi(x) - \phi(y)) \frac{\mathcal{A}(d, \alpha) a^\alpha}{|x - y|^{d+\alpha}} dx dy = 0.$$

The following uniform Harnack principle will be used to prove the main result of this paper.

**Proposition 1.3 (Harnack principle)** *Suppose that  $M > 0$ . There exists a constant  $C_0 = C_0(\alpha, M) > 0$  such that for any  $r \in (0, 1]$ ,  $a \in [0, M]$ ,  $x_0 \in \mathbb{R}^d$  and any function  $u$  which is nonnegative in  $\mathbb{R}^d$  and harmonic in  $B(x_0, r)$  with respect to  $X^a$  we have*

$$u(x) \leq C_0 u(y) \quad \text{for all } x, y \in B(x_0, r/2).$$

Let  $Q \in \partial D$ . We will say that a function  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  vanishes continuously on  $D^c \cap B(Q, r)$  if  $u = 0$  on  $D^c \cap B(Q, r)$  and  $u$  is continuous at every point of  $\partial D \cap B(Q, r)$ .

**Theorem 1.4** *Suppose that  $M > 0$ . For any  $C^{1,1}$  open set  $D$  in  $\mathbb{R}^d$  with the characteristics  $(R, \Lambda)$ , there exists a positive constant  $C = C(\alpha, d, \Lambda, R, M)$  such that for  $a \in [0, M]$ ,  $r \in (0, R]$ ,  $Q \in \partial D$  and any nonnegative function  $u$  in  $\mathbb{R}^d$  that is harmonic in  $D \cap B(Q, r)$  with respect to  $X^a$  and vanishes continuously on  $D^c \cap B(Q, r)$ , we have*

$$(1.5) \quad \frac{u(x)}{u(y)} \leq C \frac{\delta_D(x)}{\delta_D(y)} \quad \text{for every } x, y \in D \cap B(Q, r/2).$$

When  $a$  changes from 0 to  $M$ ,  $\Delta + a^\alpha \Delta^{\alpha/2}$  changes continuously from  $\Delta$  to  $\Delta + M^\alpha \Delta^{\alpha/2}$ . So Theorem 1.4 says that the boundary Harnack principle (BHP) holds uniformly for the family  $\{\Delta + a^\alpha \Delta^{\alpha/2}, a \in [0, M]\}$  of pseudo differential operators in the sense that the constant  $C$  in (1.5) can be chosen to be independent of  $a \in [0, M]$ . Note that  $a = 0$  corresponds to the classical case of the boundary Harnack principle for the Laplacian. We will therefore in the rest of the paper assume that  $a \in (0, M]$ .

Unlike (1.1) and the paragraph above it, in this paper we are concerned with the above BHP for  $C^{1,1}$  open sets only. The main focus and goal of this paper is to get the explicit decay rate of harmonic functions near the boundary of  $D$  as in (1.5) and to show that the BHP is *uniform* in  $a \in [0, M]$ . We emphasize that (1.5) is not true in Lipschitz domains even in the classical case of BHP for the Laplacian. However, a uniform Carleson type estimate is shown to hold for Lipschitz open sets in Theorem 4.3. The BHP of above type is very useful in studying other fine properties of the process. For example, we will use it to derive sharp Green function estimates of  $X^a$  in  $C^{1,1}$  open sets in a forthcoming paper.

For  $a > 0$ ,  $X^a$  and  $X := X^1$  are in fact related by a scaling. More precisely, for  $a \in (0, M]$ ,  $X^a$  has the same distribution as  $\lambda X_{\lambda^{-2}t}$ , where  $\lambda = a^{\alpha/(\alpha-2)} \geq M^{\alpha/(\alpha-2)}$ . Consequently, if  $u$  is harmonic in an open set  $U$  with respect to  $X^a$ , then  $v(x) := u(\lambda x)$  is harmonic in  $\lambda^{-1}U$  with respect to  $X$ . Hence the uniform Harnack inequality of Proposition 1.3 follows from the Harnack inequality for  $X$ . The latter is known, see Theorem 6.7 of [9] or Theorem 4.5 of [17]. However the uniform BHP of Theorem 1.4 can not be obtained by such a scaling argument from the BHP of  $X$ . This is because for a  $C^{1,1}$  open set  $D$  with the characteristics  $(R, \lambda)$ ,  $\lambda^{-1}D$  is, in general, a  $C^{1,1}$  open set with  $C^{1,1}$  characteristics  $(R/\lambda, (\lambda \vee 1)\Lambda)$ , which tends to  $(0, \infty)$  as  $\lambda \rightarrow \infty$ .

For each fixed  $\alpha_0 \in (0, 2)$ , when  $\alpha$  changes from  $\alpha_0$  to 2, the operator  $\Delta + a^\alpha \Delta^{\alpha/2}$  evolves continuously from  $\Delta + a^{\alpha_0} \Delta^{\alpha_0/2}$  to  $(1 + a^2)\Delta$ . So in view of Theorem 1.4, it is reasonable to expect that one can get the BHP for  $\Delta + a^\alpha \Delta^{\alpha/2}$  uniformly both in  $a \in (0, M]$  and in  $\alpha \in [\alpha_0, 2)$ . We believe this is the case and that it can be achieved by carefully keeping track of all the comparison constants in the arguments of this paper. However in order to keep our exposition as transparent

as possible, we are content with establishing the results stated in Theorem 1.4 and leave the details of the proof for the last claim to interested readers.

Our method of establishing the above BHP is different from those in [5, 19] for symmetric stable processes and in [13] for more general subordinate Brownian motions. Instead, our approach adopts the ideas from [6] for the BHP of censored stable processes, which is further refined in [12]. That is, we use suitably chosen subharmonic and superharmonic functions of the process  $X^a$  (or equivalently, of  $\Delta + a^\alpha \Delta^{\alpha/2}$ ) to derive some exit distribution estimates that are needed to establish the BHP. However, had we done it in this way directly, we would only get the BHP for  $\Delta + a^\alpha \Delta^{\alpha/2}$  with  $\alpha \in (1, 2)$ . The reason is that, when  $D = \mathbb{H}_+^d := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_1 > 0\}$ , we need to consider testing function  $w_p(x) = (x_1 \vee 0)^p$  for  $p > 1$ . But for  $w_p$  to be  $\Delta^{\alpha/2}$ -differentiable in  $\mathbb{H}_+^d$  (see (1.2)), one requires  $p < \alpha$ , which would be impossible when  $\alpha \in (0, 1]$ . To circumvent this difficulty, for each  $\lambda > 0$ , we consider the finite range (or truncated) symmetric  $\alpha$ -stable process  $\widehat{Y}^\lambda$  obtained from  $Y$  by suppressing all its jumps of size larger than  $\lambda$ . The infinitesimal generator of  $\widehat{Y}^\lambda$  is

$$(1.6) \quad \widehat{\Delta}_{d,\lambda}^{\alpha/2} u(x) := \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d : \varepsilon < |y-x| < \lambda\}} (u(y) - u(x)) \frac{\mathcal{A}(d, \alpha)}{|x-y|^{d+\alpha}} dy.$$

When  $\lambda = 1$ , we will simply denote  $\widehat{\Delta}_{d,1}^{\alpha/2}$  by  $\widehat{\Delta}_d^{\alpha/2}$ . Then  $w_p$  is  $\widehat{\Delta}_d^{\alpha/2}$ -differentiable in  $\mathbb{H}_+^d$  for every  $p > 0$ . Observe that  $\widehat{X}^a := X^0 + a\widehat{Y}^{1/a}$  is a Lévy process obtained from  $X^a = X^0 + aY$  by suppressing all its jumps of size larger than 1 and that the infinitesimal generator of  $\widehat{X}^a$  is  $\Delta + a^\alpha \widehat{\Delta}_d^{\alpha/2}$ . From this, we can obtain suitable exit distribution estimates for the Lévy process  $\widehat{X}^a$ . The desired estimates for  $X^a$  can then be obtained from that for  $\widehat{X}^a$  by adding back those jumps of  $X^a$  of size larger than 1. Such an idea has already been used in [10] to study Schramm-Löwner evolutions driven by one-dimensional symmetric stable processes. We remark that the BHP in Theorem 1.4 for the case of  $a = 1$  has also been mentioned in Remark 5.2 of Guan [12]. However, no precise statement (such as the range of  $\alpha$ ) nor a proof is given in that paper.

The rest of the paper is organized as follows. In Section 2, we derive estimates on  $\widehat{\Delta}_d^{\alpha/2} w_p$ . These estimates are then used in Section 3 to obtain exit distribution (or harmonic measure) estimates for the finite range process  $\widehat{X}^a$  and then for the desired process  $X^a$ . In Section 4, we first give the proof of Proposition 1.3, and then establish a Carleson estimate for non-negative harmonic functions of  $\Delta + a^\alpha \Delta^{\alpha/2}$  in Lipschitz open sets. Then using these results, proof of Theorem 1.4 is presented.

Throughout this paper, we use the capital letters  $C_1, C_2, \dots$  to denote constants in the statement of the results, and their labeling will be fixed. The lowercase constants  $c_1, c_2, \dots$  will denote generic constants used in the proofs, whose exact values are not important and can change from one appearance to another. The labeling of the constants  $c_1, c_2, \dots$  starts anew in every proof. The dependence of the constant  $c$  on the dimension  $d \geq 1$  and  $\alpha \in (0, 2)$  may not be mentioned explicitly. The constant  $M > 0$  will be fixed throughout this paper. We will use “:=” to denote a definition, which is read as “is defined to be”. For  $a, b \in \mathbb{R}$ ,  $a \wedge b := \min\{a, b\}$  and  $a \vee b := \max\{a, b\}$ . For every function  $f$ , let  $f^+ := f \vee 0$ . We will use  $\partial$  to denote a cemetery point and for every function  $f$ , we extend its definition to  $\partial$  by setting  $f(\partial) = 0$ . We will use  $dx$  or  $m_d(dx)$  to denote the Lebesgue measure in  $\mathbb{R}^d$ . For a Borel set  $A \subset \mathbb{R}^d$ , we also use  $|A|$  to denote its Lebesgue measure and  $\text{diam}(A)$  to denote the diameter of the set  $A$ .

## 2 Truncated fractional Laplacian estimates for power functions

In this section, we give some estimates which will be used later. Recall that the fractional Laplacian  $\Delta^{\alpha/2}$  and the truncated fractional Laplacian  $\widehat{\Delta}_d^{\alpha/2} := \widehat{\Delta}_{d,1}^{\alpha/2}$  are defined in (1.2) and (1.6), respectively.

**Lemma 2.1** *For  $x \in \mathbb{R}^d$  and  $p > 0$ , set  $w_p(x) := (x_1^+)^p$ . Then there are constants  $R_* \in (0, 1)$ ,  $C_1 > C_2 > 0$  depending only on  $p, d$  and  $\alpha$  such that for every  $x \in \mathbb{R}^d$  with  $x_1 \in (0, R_*]$*

$$(2.1) \quad |\widehat{\Delta}_d^{\alpha/2} w_p(x)| \leq C_1 \quad \text{for } p > \alpha,$$

$$(2.2) \quad C_2 x_1^{p-\alpha} \leq \widehat{\Delta}_d^{\alpha/2} w_p(x) \leq C_1 x_1^{p-\alpha} \quad \text{for } \alpha/2 < p < \alpha,$$

and

$$(2.3) \quad -C_1 x_1^{p-\alpha} \leq \widehat{\Delta}_d^{\alpha/2} w_p(x) \leq -C_2 x_1^{p-\alpha} \quad \text{for } 0 < p < \alpha/2.$$

**Proof.** First note that using integration by parts and a change of variable, we get that for  $p, x > 0$  and  $\varepsilon \in (0, 1/(x+1))$ ,

$$(2.4) \quad \begin{aligned} \int_0^{1-\varepsilon} \frac{z^p - 1}{(1-z)^{\alpha+1}} dz &= \frac{1}{\alpha} \int_0^{1-\varepsilon} (z^p - 1) d(1-z)^{-\alpha} \\ &= \frac{1}{\alpha} (z^p - 1)(1-z)^{-\alpha} \Big|_0^{1-\varepsilon} - \frac{p}{\alpha} \int_0^{1-\varepsilon} z^{p-1} (1-z)^{-\alpha} dz \\ &= \frac{(1-\varepsilon)^p - 1}{\alpha \varepsilon^\alpha} + \frac{1}{\alpha} - \frac{p}{\alpha} \int_0^{1-\varepsilon} \frac{z^{p-1}}{(1-z)^\alpha} dz \end{aligned}$$

and

$$(2.5) \quad \begin{aligned} \int_{1+\varepsilon}^{\frac{x+1}{x}} \frac{z^p - 1}{(z-1)^{\alpha+1}} dz &= -\frac{1}{\alpha} \int_{1+\varepsilon}^{\frac{x+1}{x}} (z^p - 1) d(z-1)^{-\alpha} \\ &= -\frac{1}{\alpha} (z^p - 1)(z-1)^{-\alpha} \Big|_{1+\varepsilon}^{\frac{x+1}{x}} + \frac{p}{\alpha} \int_{1+\varepsilon}^{\frac{x+1}{x}} z^{p-1} (z-1)^{-\alpha} dz \\ &= \frac{(1+\varepsilon)^p - 1}{\alpha \varepsilon^\alpha} + \frac{1}{\alpha} x^\alpha - \frac{1}{\alpha} (x+1)^p x^{\alpha-p} + \frac{p}{\alpha} \int_{\frac{x}{x+1}}^{\frac{1}{\varepsilon+1}} z^{\alpha-p-1} (1-z)^{-\alpha} dz. \end{aligned}$$

For  $p > 0$  and  $x \in (0, 1)$ , by a change of variable

$$\begin{aligned} \widehat{\Delta}_1^{\alpha/2} w_p(x) &= \mathcal{A}(1, \alpha) \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \frac{w_p(y) - w_p(x)}{|x-y|^{1+\alpha}} \mathbf{1}_{\{\varepsilon < |y-x| \leq 1\}} dy \\ &= \mathcal{A}(1, \alpha) \lim_{\varepsilon \downarrow 0} \int_0^{x+1} \frac{y^p - x^p}{|x-y|^{1+\alpha}} \mathbf{1}_{\{|y-x| > \varepsilon\}} dy - \mathcal{A}(1, \alpha) x^p \int_{x-1}^0 \frac{dy}{|x-y|^{1+\alpha}} \\ &= \mathcal{A}(1, \alpha) x^{p-\alpha} \lim_{\varepsilon \downarrow 0} \int_0^{\frac{x+1}{x}} \frac{z^p - 1}{|z-1|^{1+\alpha}} \mathbf{1}_{\{|z-1| > \varepsilon/x\}} dz - \mathcal{A}(1, \alpha) x^p \int_{x-1}^0 (x-y)^{-1-\alpha} dy \\ &= \mathcal{A}(1, \alpha) x^{p-\alpha} \lim_{\varepsilon \downarrow 0} \left( \int_0^{1-\varepsilon} \frac{z^p - 1}{(1-z)^{1+\alpha}} dz + \int_{1+\varepsilon}^{\frac{x+1}{x}} \frac{z^p - 1}{(z-1)^{1+\alpha}} dz \right) - \frac{\mathcal{A}(1, \alpha)}{\alpha} (x^{p-\alpha} - x^p). \end{aligned}$$

So we have by (2.4)-(2.5) that for  $p > 0$  and  $x \in (0, 1)$ ,

$$\begin{aligned}
(2.6) \quad & \widehat{\Delta}_1^{\alpha/2} w_p(x) \\
&= \frac{\mathcal{A}(1, \alpha)}{\alpha} x^{p-\alpha} \lim_{\varepsilon \downarrow 0} \left( 1 + \frac{(1-\varepsilon)^p + (1+\varepsilon)^p - 2}{\varepsilon^\alpha} \right) - \frac{\mathcal{A}(1, \alpha)}{\alpha} (x^{p-\alpha} - x^p) + \frac{\mathcal{A}(1, \alpha)}{\alpha} x^p \\
&\quad - \frac{\mathcal{A}(1, \alpha)}{\alpha} (x+1)^p + \frac{\mathcal{A}(1, \alpha) p}{\alpha} x^{p-\alpha} \left( \int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1} - z^{p-1}}{(1-z)^\alpha} dz - \int_0^{\frac{x}{x+1}} \frac{z^{p-1}}{(1-z)^\alpha} dy \right) \\
&= \frac{\mathcal{A}(1, \alpha)}{\alpha} \left( 2x^p - (x+1)^p + px^{p-\alpha} \left( \int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1} - z^{p-1}}{(1-z)^\alpha} dz - \int_0^{\frac{x}{x+1}} \frac{z^{p-1}}{(1-z)^\alpha} dz \right) \right).
\end{aligned}$$

Note that

$$\sup_{x \in (0,1]} x^{p-\alpha} \left| \int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1} - z^{p-1}}{(1-z)^\alpha} dz - \int_0^{\frac{x}{x+1}} \frac{z^{p-1}}{(1-z)^\alpha} dz \right| < \infty.$$

So when  $p > \alpha$ ,

$$(2.7) \quad \sup_{x \in (0,1)} |\widehat{\Delta}_1^{\alpha/2} w_p(x)| < \infty.$$

On the other hand, when  $p \in (0, \alpha)$ ,

$$(2.8) \quad \sup_{x \in (0,1]} \left| \int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1} - z^{p-1}}{(1-z)^\alpha} dz - \int_0^{\frac{x}{x+1}} \frac{z^{p-1}}{(1-z)^\alpha} dz \right| < \infty.$$

As

$$\lim_{x \rightarrow 0^+} \left( \int_{\frac{x}{x+1}}^1 \frac{z^{\alpha-p-1} - z^{p-1}}{(1-z)^\alpha} dz - \int_0^{\frac{x}{x+1}} \frac{z^{p-1}}{(1-z)^\alpha} dz \right) \begin{cases} > 0 & \text{if } p \in (\alpha/2, \alpha) \\ < 0 & \text{if } p \in (0, \alpha/2) \end{cases},$$

we conclude from (2.6)-(2.8) that there are constants  $r_1 \in (0, 1)$  and  $C_1 > C_2 > 0$  depending on  $p$  and  $\alpha$  so that when  $p \in (\alpha/2, \alpha)$ ,

$$(2.9) \quad C_2 x^{p-\alpha} < \widehat{\Delta}_1^{\alpha/2} w_p(x) < C_1 x^{p-\alpha} \text{ for } x \in (0, r_1] \quad \text{and} \quad \sup_{x \in (r_1, 1)} |\widehat{\Delta}_1^{\alpha/2} w_p(x)| < \infty,$$

and for  $p \in (0, \alpha/2)$ ,

$$(2.10) \quad -C_1 x^{p-\alpha} < \widehat{\Delta}_1^{\alpha/2} w_p(x) < -C_2 x^{p-\alpha} \text{ for } x \in (0, r_1] \quad \text{and} \quad \sup_{x \in (r_1, 1)} |\widehat{\Delta}_1^{\alpha/2} w_p(x)| < \infty.$$

On the other hand, for  $x \geq 1$ ,

$$\begin{aligned}
\widehat{\Delta}_1^{\alpha/2} w_p(x) &= \mathcal{A}(1, \alpha) \lim_{\varepsilon \downarrow 0} \int_{x-1}^{x+1} \frac{w_p(y) - w_p(x)}{|x-y|^{1+\alpha}} \mathbf{1}_{\{|x-y|>\varepsilon\}} dy \\
&= \mathcal{A}(1, \alpha) x^{p-\alpha} \lim_{\varepsilon \downarrow 0} \int_{\frac{x-1}{x}}^{\frac{x+1}{x}} \frac{y^p - 1}{|y-1|^{1+\alpha}} \mathbf{1}_{\{|y-1|>\varepsilon\}} dy \\
&= \mathcal{A}(1, \alpha) x^{p-\alpha} \lim_{\varepsilon \downarrow 0} \left( \int_{\frac{x-1}{x}}^{1-\varepsilon} \frac{y^p - 1}{(1-y)^{1+\alpha}} dy + \int_{1+\varepsilon}^{\frac{x+1}{x}} \frac{y^p - 1}{(y-1)^{1+\alpha}} dy \right) \\
&= \mathcal{A}(1, \alpha) x^{p-\alpha} \int_0^{1/x} \frac{(1+u)^p + (1-u)^p - 2}{u^{1+\alpha}} du.
\end{aligned}$$

Note the above integrand

$$\frac{(1+u)^p + (1-u)^p - 2}{u^{1+\alpha}}$$

is of the order  $u^{1-\alpha}$  near zero. So for  $p > 0$  and  $\alpha \in (0, 2)$ , there is a constant  $c_0 = c_0(p, \alpha) > 0$  so that

$$(2.11) \quad |\widehat{\Delta}_1^{\alpha/2} w_p(x)| \leq c_0 x^{p-2} \quad \text{for } x \geq 1.$$

With  $r_1 \in (0, 1)$  as in (2.9)-(2.10), the above inequality in fact holds for  $x > r_1$ .

The estimates (2.7)-(2.10) prove the Lemma in dimension  $d = 1$ . Now we consider the case  $d \geq 2$ . For each fixed  $x \in \mathbb{R}^d$ , we use the spherical coordinates

$$(y_1, \dots, y_d) := x + (r \cos \theta_1, r \sin \theta_1 \cos \theta_2, \dots, r \sin \theta_1 \dots \cos \theta_{d-1}, r \sin \theta_1 \dots \sin \theta_{d-1})$$

where  $r \geq 0$ ,  $0 \leq \theta_1, \dots, \theta_{d-2} < \pi$  and  $0 \leq \theta_{d-1} < 2\pi$ . Let

$$\phi(\widehat{\theta}) := \phi(\theta_1, \dots, \theta_{d-2}) := \sin^{d-2} \theta_1 \sin^{d-3} \theta_2 \dots \sin \theta_{d-2}.$$

Then for  $x \in \mathbb{R}^d$  with  $x_1 > 0$  we have

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: 1 > |y-x| > \varepsilon\}} ((y_1^+)^p - x_1^p) \frac{dy}{|x-y|^{d+\alpha}} \\ &= \lim_{\varepsilon \downarrow 0} \int_0^\pi d\theta_1 \dots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} \int_\varepsilon^1 \frac{((r \cos \theta_1 + x_1)^+)^p - x_1^p}{r^{d+\alpha}} r^{d-1} dr \\ &= \lim_{\varepsilon \downarrow 0} \int_0^{\pi/2} d\theta_1 \int_0^\pi d\theta_2 \dots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (\cos \theta_1)^p \int_\varepsilon^1 \frac{((r + \frac{x_1}{\cos \theta_1})^+)^p - (\frac{x_1}{\cos \theta_1})^p}{r^{1+\alpha}} dr \\ & \quad + \lim_{\varepsilon \downarrow 0} \int_{\pi/2}^\pi d\theta_1 \int_0^\pi d\theta_2 \dots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (-\cos \theta_1)^p \int_\varepsilon^1 \frac{((-r - \frac{x_1}{\cos \theta_1})^+)^p - (-\frac{x_1}{\cos \theta_1})^p}{r^{1+\alpha}} dr. \end{aligned}$$

By the change of variable  $r = t - x_1 / \cos \theta_1$  for  $\theta \in [0, \pi/2)$  and  $r = -t - x_1 / \cos \theta_1 = -t + x_1 / \cos(\pi - \theta_1)$  for  $\theta \in (\pi/2, \pi]$ , we get

$$\begin{aligned} & \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d: 1 > |y-x| > \varepsilon\}} ((y_1^+)^p - x_1^p) \frac{dy}{|x-y|^{d+\alpha}} \\ &= \lim_{\varepsilon \downarrow 0} \int_0^{\pi/2} d\theta_1 \int_0^\pi d\theta_2 \dots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (\cos \theta_1)^p \int_{\varepsilon + \frac{x_1}{\cos \theta_1}}^{1 + \frac{x_1}{\cos \theta_1}} \frac{(t^+)^p - (\frac{x_1}{\cos \theta_1})^p}{|t - x_1 / \cos \theta_1|^{1+\alpha}} dt \\ & \quad + \lim_{\varepsilon \downarrow 0} \int_{\pi/2}^\pi d\theta_1 \int_0^\pi d\theta_2 \dots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (\cos(\pi - \theta_1))^p \int_{-1 + \frac{x_1}{\cos(\pi - \theta_1)}}^{-\varepsilon + \frac{x_1}{\cos(\pi - \theta_1)}} \frac{(t^+)^p - (\frac{x_1}{\cos(\pi - \theta_1)})^p}{|t - x_1 / \cos(\pi - \theta_1)|^{1+\alpha}} dt \\ &= \lim_{\varepsilon \downarrow 0} \int_0^{\pi/2} d\theta_1 \int_0^\pi d\theta_2 \dots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (\cos \theta_1)^p \int_{\varepsilon + \frac{x_1}{\cos \theta_1}}^{1 + \frac{x_1}{\cos \theta_1}} \frac{(t^+)^p - (\frac{x_1}{\cos \theta_1})^p}{|t - x_1 / \cos \theta_1|^{1+\alpha}} dt \\ & \quad + \lim_{\varepsilon \downarrow 0} \int_0^{\pi/2} d\theta_1 \int_0^\pi d\theta_2 \dots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (\cos \theta_1)^p \int_{-1 + \frac{x_1}{\cos \theta_1}}^{-\varepsilon + \frac{x_1}{\cos \theta_1}} \frac{(t^+)^p - (\frac{x_1}{\cos \theta_1})^p}{|t - x_1 / \cos \theta_1|^{1+\alpha}} dt \\ &= \int_0^{\pi/2} d\theta_1 \int_0^\pi d\theta_2 \dots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (\cos \theta_1)^p \left( \lim_{\varepsilon \downarrow 0} \int_{\{t \in \mathbb{R}: 1 > |t - \frac{x_1}{\cos \theta_1}| > \varepsilon\}} \frac{(t^+)^p - (\frac{x_1}{\cos \theta_1})^p}{|t - x_1 / \cos \theta_1|^{1+\alpha}} dt \right). \end{aligned}$$

Therefore we have

$$\begin{aligned}
& \widehat{\Delta}_d^{\alpha/2} w_p(x) \\
&= \frac{\mathcal{A}(d, \alpha)}{\mathcal{A}(1, \alpha)} \int_0^{\pi/2} d\theta_1 \int_0^\pi d\theta_2 \cdots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (\cos \theta_1)^p \widehat{\Delta}_1^{\alpha/2} w_p \left( \frac{x_1}{\cos \theta_1} \right) \\
&= \frac{\mathcal{A}(d, \alpha)}{\mathcal{A}(1, \alpha)} \int_0^{\arccos(x_1/r_1)} d\theta_1 \int_0^\pi d\theta_2 \cdots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (\cos \theta_1)^p \widehat{\Delta}_1^{\alpha/2} w_p \left( \frac{x_1}{\cos \theta_1} \right) \\
&\quad + \frac{\mathcal{A}(d, \alpha)}{\mathcal{A}(1, \alpha)} \int_{\arccos(x_1/r_1)}^{\pi/2} d\theta_1 \int_0^\pi d\theta_2 \cdots \int_0^\pi d\theta_{d-2} \int_0^{2\pi} \phi(\widehat{\theta}) d\theta_{d-1} (\cos \theta_1)^p \widehat{\Delta}_1^{\alpha/2} w_p \left( \frac{x_1}{\cos \theta_1} \right).
\end{aligned}$$

The conclusion (2.1)-(2.3) now follow immediately from the above equality and the estimates (2.7)-(2.11), where we use (2.11) to bound the second integral above by  $c_1 x_1^3 / r_1^3$  for some positive constant  $c_1$ .  $\square$

**Remark 2.2** A careful evaluation of (2.6) in fact shows that  $\lim_{x \rightarrow 0^+} \widehat{\Delta}_1^{\alpha/2} w_p(x) = c \neq 0$  when  $p > \alpha$ . At the first glance, this may look surprising, as in the Brownian motion case (which corresponds to  $\alpha = 2$ ),  $\Delta x_1^p = p(p-1)x_1^{p-2}$ . The bound in (2.1) is due to the non-local nature of the operator  $\widehat{\Delta}_d^{\alpha/2}$  for  $\alpha \in (0, 2)$ . However a more careful analysis of (2.6) reveals that for  $p > 0$ ,

$$\widehat{\Delta}_1^{\alpha/2} w_p(x) \asymp (2 - \alpha) ((p - \alpha)^{-1} - 1) + p(p - 1)x^{p-\alpha} \quad \text{for } x \in (0, r_1)$$

as  $\alpha \uparrow 2$ . It is not difficult to see that as  $\alpha \uparrow 2$ ,  $\widehat{\Delta}_1^{\alpha/2} w_p(x)$  converges to  $\Delta w_p(x)$ .  $\square$

Recall that for  $\lambda > 0$ , the operator  $\widehat{\Delta}_{d,\lambda}^{\alpha/2}$  is defined by (1.6). Note that

$$(2.12) \quad \widehat{\Delta}_{d,\lambda}^{\alpha/2} u(x) = \lambda^{-\alpha} (\widehat{\Delta}_d^{\alpha/2} u(\lambda \cdot)) (\lambda^{-1} x).$$

Thus, from Lemma 2.1 and (2.12), we get the following corollary.

**Corollary 2.3** For  $x \in \mathbb{R}^d$  and  $p > 0$ , set  $w_p(x) := (x_1^+)^p$ . Then there are constants  $R_* \in (0, 1/2)$ ,  $C_1 > C_2 > 0$  depending only on  $p, d$  and  $\alpha$  such that for every  $\lambda > 0$  and  $x \in \mathbb{R}^d$  with  $x_1 \in (0, \lambda R_*)$ ,

$$(2.13) \quad |\widehat{\Delta}_{d,\lambda}^{\alpha/2} w_p(x)| \leq C_1 \lambda^{p-\alpha} \quad \text{for } p > \alpha,$$

$$(2.14) \quad C_2 x_1^{p-\alpha} \leq \widehat{\Delta}_{d,\lambda}^{\alpha/2} w_p(x) \leq C_1 x_1^{p-\alpha} \quad \text{for } \alpha/2 < p < \alpha,$$

and

$$(2.15) \quad -C_1 x_1^{p-\alpha} \leq \widehat{\Delta}_{d,\lambda}^{\alpha/2} w_p(x) \leq -C_2 x_1^{p-\alpha} \quad \text{for } 0 < p < \alpha/2.$$

### 3 Estimates on harmonic measures

Recall that for any open set  $U \subset \mathbb{R}^d$ ,  $\tau_U^a = \inf\{t > 0 : X_t^a \notin U\}$  is the first exit time from  $U$  by  $X^a$ .

**Lemma 3.1** *For every  $b \in (0, \infty)$ , there exist  $C_3 = C_3(M, b) > 0$  and  $C_4 = C_4(M, b) > 0$  such that for every  $x_0 \in \mathbb{R}^d$ ,  $a \in (0, M]$  and  $r \in (0, b]$ ,*

$$(3.1) \quad C_3 r^2 \leq \mathbb{E}_{x_0} \left[ \tau_{B(x_0, r)}^a \right] \leq C_4 r^2.$$

**Proof.** See Lemmas 2.2 and 2.3 in [16] or Lemmas 2.3 and 2.4 in [9] for a proof.  $\square$

In the remainder of this section, we assume  $D$  is a  $C^{1,1}$  open set with characteristics  $(R, \Lambda)$ . Recall that we are always assuming that  $R \leq 1$  and  $\Lambda \geq 1$ . For notational convenience, throughout the rest of this section, we put

$$r_0 = \frac{R}{4\sqrt{1 + \Lambda^2}}.$$

Define

$$\rho_Q(x) := x_d - \phi_Q(\tilde{x}),$$

where  $(\tilde{x}, x_d)$  is the coordinates of  $x$  in  $CS_Q$ . Note that for every  $Q \in \partial D$  and  $x \in B(Q, R) \cap D$  we have

$$(3.2) \quad (1 + \Lambda^2)^{-1/2} \rho_Q(x) \leq \delta_D(x) \leq \rho_Q(x).$$

Recall that  $R_*$  is the constant in Lemma 2.1.

**Lemma 3.2** *Fix  $Q \in \partial D$  and the coordinate system  $CS_Q$  so that*

$$B(Q, R) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \phi(\tilde{y})\}.$$

For  $p > \alpha/2$ , let

$$h_p(y) := (\rho_Q(y))^p \mathbf{1}_{D \cap B(Q, 4r_0)}(y).$$

Then there exist  $C_i = C_i(\alpha, p, \Lambda, R) > 0$ ,  $i = 5, 6, 7$ , independent of the choice of the point  $Q \in \partial D$  such that

(i) *in the case  $\frac{\alpha}{2} < p < \alpha$ , for all  $x \in D$  such that  $\rho_Q(x) < r_0 \wedge R_*$  and  $|\tilde{x}| < r_0$ , we have*

$$(3.3) \quad C_6 (\rho_Q(x))^{p-\alpha} \leq \widehat{\Delta}_d^{\alpha/2} h_p(x) \leq C_5 (\rho_Q(x))^{p-\alpha};$$

(ii) *in the case  $p > \alpha$ , for all  $x \in D$  such that  $\rho_Q(x) < r_0 \wedge R_*$  and  $|\tilde{x}| < r_0$ , we have*

$$(3.4) \quad |\widehat{\Delta}_d^{\alpha/2} h_p(x)| \leq C_7.$$

**Proof.** In this proof our coordinate system is always  $CS_Q$ . Fix  $x = (\tilde{x}, x_d) \in D$  such that  $\rho_Q(x) < r_0 \wedge R_*$  and  $|\tilde{x}| < r_0$ , and choose a point  $x_0 \in \partial D$  satisfying  $\tilde{x} = \tilde{x}_0$ . Denote by  $\vec{n}(x_0)$  the inward unit normal vector at  $x_0$  for  $\partial D$  and set  $\Phi(y) = \langle y - x_0, \vec{n}(x_0) \rangle$  for  $y \in \mathbb{R}^d$ . Then  $\Pi := \{y : \Phi(y) = 0\}$  is the hyperplane tangent to  $\partial D$  at the point  $x_0$ . The function  $\Gamma^* : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  describing the plane  $\Pi$  is given by  $\Gamma^*(\tilde{y}) = \phi_Q(\tilde{x}_0) + \nabla \phi_Q(\tilde{x}_0)(\tilde{y} - \tilde{x}_0)$ , and it holds that  $\langle (\tilde{y}, \Gamma^*(\tilde{y})) - x_0, \vec{n}(x_0) \rangle = 0$ . We also let

$$\begin{aligned} A &:= \{y : \Gamma^*(\tilde{y}) < y_d < \phi_Q(\tilde{y}) \text{ and } |\tilde{y} - \tilde{x}| < r_0\} \cup \{y : \Gamma^*(\tilde{y}) > y_d > \phi_Q(\tilde{y}) \text{ and } |\tilde{y} - \tilde{x}| < r_0\}, \\ E &:= \{y \in D \setminus A : |\tilde{y} - \tilde{x}| < r_0 \text{ and } \rho_Q(y) < r_0(2 + \Lambda)\}. \end{aligned}$$

Note that, if  $|x - y| < r_0$  and  $y \in D$ ,

$$\rho_Q(y) \leq |y_d - x_d| + |x_d - \phi_Q(\tilde{x})| + |\phi_Q(\tilde{y}) - \phi_Q(\tilde{x})| < r_0(2 + \Lambda).$$

On the other hand if  $|\tilde{y} - \tilde{x}| < r_0$  and  $\rho_Q(y) < r_0(2 + \Lambda)$ , then

$$|y|^2 = |\tilde{y}|^2 + |y_d|^2 \leq (2r_0)^2 + (r_0(2 + \Lambda) + |\phi_Q(\tilde{y})|)^2 \leq \frac{4 + (2 + 3\Lambda)^2}{64(1 + \Lambda^2)} R^2 < R^2.$$

Consequently, we have

$$(3.5) \quad D \cap B(x, r_0) \subset D \cap \{y : |\tilde{y} - \tilde{x}| < r_0 \text{ and } \rho_Q(y) < r_0(2 + \Lambda)\} \subset D \cap B(0, R).$$

Let  $\bar{h}(y) := \bar{h}_x(y) := (y_d - \Gamma^*(\tilde{y}))^+$  for  $y \in \mathbb{R}^d$ . Since  $\nabla \phi_Q(\tilde{x}) = \nabla \Gamma^*(\tilde{x})$ , by the mean value theorem and the  $C^{1,1}$  condition on  $\phi_Q$ ,

$$(3.6) \quad \begin{aligned} |\bar{h}(y) - \rho_Q(y)| &= |\phi_Q(\tilde{y}) - \Gamma^*(\tilde{y})| \\ &= |\phi_Q(\tilde{y}) - \phi_Q(\tilde{x}) - \nabla \phi_Q(\tilde{x}) \cdot (\tilde{y} - \tilde{x})| \leq \Lambda |\tilde{y} - \tilde{x}|^2, \quad y \in E. \end{aligned}$$

For  $y \in \mathbb{R}^d$ , define  $\delta_\Pi(y) := \text{dist}(y, \Pi)$  and  $D_{\Gamma^*} = \{y \in \mathbb{R}^d : y_d > \Gamma^*(\tilde{y})\}$ . Let

$$b_x := (1 + |\nabla \phi_Q(\tilde{x})|^2)^{1/2} \quad \text{and} \quad h_{x,p}(y) := (\bar{h}(y))^p \quad \text{for } p > \alpha/2.$$

Note that  $1 \leq b_x \leq \sqrt{1 + \Lambda^2}$  and  $h_{x,p}(x) = h_p(x)$ .

Recall that  $R_*$  and  $C_1 > C_2 > 0$  are constants in Lemma 2.1. Since  $\bar{h}(y) = b_x \delta_\Pi(y)$  on  $D_{\Gamma^*}$ , by Lemma 2.1, it holds that for  $y \in D_{\Gamma^*}$  and  $\delta_\Pi(y) < R_*$ ,

$$(3.7) \quad C_2 b_x^p (\delta_\Pi(y))^{p-\alpha} \leq \widehat{\Delta}_d^{\alpha/2} h_{x,p}(y) = b_x^p \widehat{\Delta}_d^{\alpha/2} (\delta_\Pi(y))^p \leq C_1 b_x^p (\delta_\Pi(y))^{p-\alpha} \quad \text{when } \alpha/2 < p < \alpha$$

and

$$(3.8) \quad |\widehat{\Delta}_d^{\alpha/2} h_{x,p}(y)| = b_x^p |\widehat{\Delta}_d^{\alpha/2} (\delta_\Pi(y))^p| \leq C_1 b_x^p \leq C_1 (1 + \Lambda^2)^{p/2} \quad \text{when } p > \alpha.$$

Note that  $b_x \delta_\Pi(x) = \rho_Q(x)$ . Applying (3.2) and (3.7) to the point  $x$  gives that, for  $\alpha/2 < p < \alpha$

$$(3.9) \quad C_2 \rho_Q(x)^{p-\alpha} \leq C_2 b_x^\alpha \rho_Q(x)^{p-\alpha} \leq \widehat{\Delta}_d^{\alpha/2} h_{x,p}(x) \leq C_1 b_x^\alpha \rho_Q(x)^{p-\alpha} \leq C_1 (1 + \Lambda^2)^{\alpha/2} \rho_Q(x)^{p-\alpha}.$$

Note that by (3.5),

$$\begin{aligned}
(3.10) \quad & |\widehat{\Delta}_d^{\alpha/2}(h_p - h_{x,p})(x)| \\
&= \mathcal{A}(d, \alpha) \left| \lim_{\varepsilon \downarrow 0} \int_{\{1 \geq |y-x| > \varepsilon\}} \frac{(h_p(y) - h_{p,x}(y))}{|x-y|^{d+\alpha}} dy \right| \\
&\leq \mathcal{A}(d, \alpha) \left| \int_{\{1 \geq |y-x| > r_0\}} \frac{(h_p(y) - h_{p,x}(y))}{|x-y|^{d+\alpha}} dy \right| \\
&\quad + \mathcal{A}(d, \alpha) \lim_{\varepsilon \downarrow 0} \int_{\{r_0 \geq |y-x| > \varepsilon\}} \frac{|h_p(y) - h_{p,x}(y)|}{|x-y|^{d+\alpha}} dy \\
&\leq \mathcal{A}(d, \alpha) \left| \int_{\{1 \geq |y-x| > r_0\}} \frac{(h_p(y) - h_{p,x}(y))}{|x-y|^{d+\alpha}} dy \right| \\
&\quad + \mathcal{A}(d, \alpha) \int_A \frac{h_p(y) + h_{p,x}(y)}{|x-y|^{d+\alpha}} dy + \mathcal{A}(d, \alpha) \int_E \frac{|h_p(y) - h_{p,x}(y)|}{|x-y|^{d+\alpha}} dy \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

We claim that, if  $p > \alpha/2$ , then

$$(3.11) \quad I_1 + I_2 + I_3 \leq c_0$$

for some constant  $c_0 = c_0(\alpha, p, \Lambda, R)$ . Together with (3.8), (3.9) and (3.10) this will establish the desired estimates (3.3) and (3.4) with constants depending on  $\alpha, p, \Lambda$  and  $R$ .

Clearly  $I_1$  is bounded by some positive constant.

For  $y \in A$ , we have

$$\begin{aligned}
(3.12) \quad & |h_{x,p}(y)| + |h_p(y)| \leq |y_d - \Gamma^*(\tilde{y})|^p + |y_d - \phi_Q(\tilde{y})|^p \leq 2|\phi_Q(\tilde{y}) - \Gamma^*(\tilde{y})|^p \\
&\leq 2|\phi_Q(\tilde{y}) - \phi_Q(\tilde{x}) - \nabla \phi_Q(\tilde{x}) \cdot (\tilde{y} - \tilde{x})|^p \leq 2\Lambda^p |\tilde{y} - \tilde{x}|^{2p}.
\end{aligned}$$

Furthermore, since, on  $\{|\tilde{y} - \tilde{x}| = r \leq r_0\}$ ,  $|\phi_Q(\tilde{y}) - \Gamma^*(\tilde{y})| \leq \Lambda|\tilde{y} - \tilde{x}|^2 = \Lambda r^2$ ,

$$m_{d-1}(\{y : |\tilde{y} - \tilde{x}| = r, \Gamma^*(\tilde{y}) < y_d < \phi_Q(\tilde{y}) \text{ or } \Gamma^*(\tilde{y}) > y_d > \phi_Q(\tilde{y})\}) \leq c_1 r^d$$

for some constant  $c_1 > 0$  if  $r \leq r_0$ . This together with (3.12) yields that

$$\begin{aligned}
I_2 &\leq \mathcal{A}(d, \alpha) \int_0^{r_0} \int_{|\tilde{y}-\tilde{x}|=r} \mathbf{1}_A(y) \frac{|h_{x,p}(y)| + |h_p(y)|}{|\tilde{y} - \tilde{x}|^{d+\alpha}} m_{d-1}(dy) dr \\
&\leq c_2 \int_0^{r_0} r^{-d+2p-\alpha} m_{d-1}(\{y \in A : |\tilde{y} - \tilde{x}| = r\}) dr \\
&\leq c_1 c_2 \int_0^{r_0} r^{2p-\alpha} dr \leq c_3.
\end{aligned}$$

Note that for  $y \in E$

$$(3.13) \quad |h_p(y) - h_{x,p}(y)| = |(\bar{h}(y))^p - (\rho_Q(y))^p| \leq c_4 (\bar{h}(y))^{(p-1)_-} |\bar{h}(y) - \rho_Q(y)|,$$

where  $(p-1)_- := (p-1) \wedge 0$ . In the last inequality above, we have used the inequalities

$$|b^p - a^p| \leq b^{p-1} |b - a| \quad \text{for } a, b > 0, \quad 0 < p \leq 1$$

and

$$|b^p - a^p| \leq (p+1)|b-a| \quad \text{for } a, b \in (0, 1), \quad p > 1.$$

For  $y = (\tilde{y}, y_d) \in \mathbb{R}^d$ , we use an affine coordinate system  $z = (\tilde{z}, z_d)$  to represent it so that  $z_d = y_d - \Gamma^*(\tilde{y})$  and  $\tilde{z}$  are the coordinates in an orthogonal coordinate system centered at  $x_0$  for the  $(d-1)$ -dimensional hyperplane  $\Pi$  for the point  $(\tilde{y}, \Gamma^*(\tilde{y}))$ . Denote such an affine transformation  $y \mapsto z$  by  $z = \Psi(y)$ . It is clear that there is a constant  $c_5 = c_5(\Lambda, R) > 1$  so that for every  $y \in \mathbb{R}^d$ ,

$$c_5^{-1}|\tilde{y} - \tilde{x}| \leq |\tilde{z}| \leq c_5|\tilde{y} - \tilde{x}|, \quad c_5^{-1}|y - x| \leq |\Psi(y) - \Psi(x)| \leq c_5|y - x|$$

and that

$$\Psi(E) \subset \{z = (\tilde{z}, z_d) \in \mathbb{R}^d : |\tilde{z}| < c_5 r_0 \text{ and } 0 < z_d \leq c_5 r_0\}.$$

Denote  $x_d - \Gamma^*(\tilde{x})$  by  $w$ ; that is,  $\Psi(x) = (\tilde{0}, w)$ . Hence by (3.6) and (3.13) and applying the transform  $\Psi$ , we have by using polar coordinates for  $\tilde{z}$  on the hyperplane  $\Pi$ ,

$$\begin{aligned} I_3 &\leq c_6 \int_E \frac{\bar{h}(y)^{(p-1)-} |\tilde{y} - \tilde{x}|^2}{|y - x|^{d+\alpha}} dy \leq c_7 \int_{\Psi(E)} \frac{z_d^{(p-1)-} |\tilde{z}|^2}{|z - (\tilde{0}, w)|^{d+\alpha}} dz \\ &\leq c_8 \int_0^{c_5 r_0} z_d^{(p-1)-} \left( \int_0^{c_5 r_0} \frac{r^{d-2}}{(r + |z_d - w|)^{d+\alpha-2}} dr \right) dz_d \\ &\leq c_8 \int_0^{c_5 r_0} z_d^{(p-1)-} \left( \int_0^{c_5 r_0} \frac{1}{(r + |z_d - w|)^\alpha} dr \right) dz_d \\ &\leq c_9 \int_0^{c_5 r_0} z_d^{(p-1)-} \left( \frac{1}{|z_d - w|^{\alpha-1}} - \frac{1}{(c_5 r_0 + |z_d - w|)^{\alpha-1}} \right) dz_d \\ &< c_{10} \int_0^{c_5 r_0} \frac{1}{z_d^{(1-p)^+} |z_d - w|^{\alpha-1}} dz_d \leq c_{11} < \infty, \end{aligned}$$

where all constants depend on  $\alpha, p, \Lambda$  and  $R$ . The last inequality is due to the fact that since  $p > 0$ ,  $0 < \alpha < 2$  and  $(1-p)^+ + \alpha - 1 = \max\{\alpha - p, \alpha - 1\} < 1$ , by the dominated convergence theorem,  $\phi(w) := \int_0^{c_5 r_0} z_d^{-(1-p)^+} |z_d - w|^{1-\alpha} dz_d$  is a strictly positive continuous function in  $x_d \in [0, c_5 r_0]$  and hence is bounded. Thus we have proved the claim (3.11), hence completing the proof of the lemma.  $\square$

Since  $D$  is a  $C^{1,1}$  open set with characteristics  $(R, \Lambda)$ , for every  $\lambda \geq 1$ ,  $\lambda D$  is a  $C^{1,1}$  open set with uniform characteristics  $(R, \Lambda)$ . Thus, by the previous lemma and (2.12), we get the following as a corollary.

**Corollary 3.3** *Fix  $Q \in \partial D$  and the coordinate system  $CS_Q$  so that*

$$B(Q, R) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \phi(\tilde{y})\}.$$

Let

$$h_p(y) := (\rho_Q(y))^p \mathbf{1}_{D \cap B(Q, 4r_0)}(y).$$

Then there exist  $C_i = C_i(\alpha, p, \Lambda, R) > 0$ ,  $i = 5, 6, 7$ , independent of the choice of the point  $Q \in \partial D$  and  $\lambda \geq 1$  such that

(i) in the case  $\frac{\alpha}{2} < p < \alpha$ , for all  $x \in D$  such that  $\rho_Q(x) < r_0 \wedge R_*$  and  $|\tilde{x}| < r_0$ , we have

$$(3.14) \quad C_6 (\rho_Q(x))^{p-\alpha} \leq \widehat{\Delta}_{d,\lambda}^{\alpha/2} h_p(x) \leq C_5 (\rho_Q(x))^{p-\alpha};$$

(ii) in the case  $p > \alpha$ , for all  $x \in D$  such that  $\rho_Q(x) < r_0 \wedge R_*$  and  $|\tilde{x}| < r_0$ , we have

$$(3.15) \quad |\widehat{\Delta}_{d,\lambda}^{\alpha/2} h_p(x)| \leq C_7 \lambda^{p-\alpha}.$$

If  $(X_t^{a,D}, t \geq 0)$  is the subprocess in  $D$  of the independent sum of a Brownian motion and a symmetric  $\alpha$ -stable process on  $\mathbb{R}^d$  with weight  $a$ , then  $(\lambda X_{\lambda^{-2}t}^{a,D}, t \geq 0)$  is the subprocess in  $\lambda D$  of the independent sum of a Brownian motion and a symmetric  $\alpha$ -stable process on  $\mathbb{R}^d$  with weight  $a\lambda^{(\alpha-2)/\alpha}$ . So for any  $\lambda > 0$ , we have

$$(3.16) \quad p_{\lambda D}^{a\lambda^{(\alpha-2)/\alpha}}(t, x, y) = \lambda^{-d} p_D^a(\lambda^{-2}t, \lambda^{-1}x, \lambda^{-1}y) \quad \text{for } t > 0 \text{ and } x, y \in \lambda D.$$

By integrating the above equation with respect to  $t$ , we get

$$(3.17) \quad G_{\lambda D}^{a\lambda^{(\alpha-2)/\alpha}}(x, y) = \lambda^{2-d} G_D^a(\lambda^{-1}x, \lambda^{-1}y) \quad \text{for } x, y \in \lambda D$$

where

$$G_D^a(x, y) := \int_0^\infty p_D^a(t, x, y) dt$$

is the Green function of  $X^a$  in  $D$ . It is well known that the Lévy measure of  $X^1$  has the intensity

$$J^1(x, y) = j^1(|x - y|) = \mathcal{A}(d, \alpha)|x - y|^{-(d+\alpha)}.$$

Thus by a scaling argument, we get that the Lévy intensity of  $X^a$  is

$$J^a(x, y) = j^a(|x - y|) = a^\alpha \mathcal{A}(d, \alpha)|x - y|^{-(d+\alpha)},$$

which gives the Lévy system (1.3) of  $X^a$ .

By a  $\lambda$ -truncated symmetric  $\alpha$ -stable process in  $\mathbb{R}^d$  we mean a pure jump symmetric Lévy process  $\widehat{Y}^\lambda = (\widehat{Y}_t^\lambda, t \geq 0, \mathbb{P}_x)$  in  $\mathbb{R}^d$  with Lévy density  $\mathcal{A}(d, \alpha)|x|^{-d-\alpha} \mathbf{1}_{\{|x| < \lambda\}}$ . Note that the Lévy exponent  $\psi^\lambda$  of  $\widehat{Y}^\lambda$  defined by

$$\mathbb{E}_x \left[ e^{i\xi \cdot (\widehat{Y}_t^\lambda - \widehat{Y}_0^\lambda)} \right] = e^{-t\psi^\lambda(\xi)} \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d,$$

is given by

$$(3.18) \quad \psi^\lambda(\xi) = \mathcal{A}(d, \alpha) \int_{\{|y| < \lambda\}} \frac{1 - \cos(\xi \cdot y)}{|y|^{d+\alpha}} dy.$$

Suppose that  $\widehat{Y}^{\lambda/a}$  is a  $(\lambda/a)$ -truncated symmetric  $\alpha$ -stable process in  $\mathbb{R}^d$  which is independent of the Brownian motion  $X^0$ . For any  $a > 0$ , we define

$$\widehat{X}_t^{a,\lambda} := X_t^0 + a\widehat{Y}_t^{\lambda/a}, \quad t \geq 0.$$

Note that from (3.18) we can easily check that for any  $b > 0$ ,

$$(3.19) \quad \psi^\lambda(b\xi) = b^\alpha \psi^\lambda(\xi) \quad \text{for every } \xi \in \mathbb{R}^d.$$

Thus for any  $a > 0$  and  $\xi, x \in \mathbb{R}^d$ ,

$$\mathbb{E}_x \left[ e^{i\xi \cdot (\widehat{X}_t^{a,\lambda} - \widehat{X}_0^{a,\lambda})} \right] = e^{-t|\xi|^2} \mathbb{E}_x \left[ e^{i(a\xi) \cdot (\widehat{Y}_t^{\lambda/a} - \widehat{Y}_0^{\lambda/a})} \right] = e^{-t(|\xi|^2 + \psi^{\lambda/a}(a\xi))} = e^{-t(|\xi|^2 + a^\alpha \psi^\lambda(\xi))}$$

Therefore  $\widehat{X}^{a,\lambda}$  has the same distribution as the Lévy process obtained from  $X^a$  by removing jumps of size larger than  $\lambda$ . The above observation also gives us that the infinitesimal generator of  $\widehat{X}^{a,\lambda}$  is  $\Delta + a^\alpha \widehat{\Delta}_{d,\lambda}^{\alpha/2}$ , and the Lévy intensity for  $\widehat{X}^{a,\lambda}$  is

$$J^{a,\lambda}(x, y) := a^\alpha \mathcal{A}(d, \alpha) |x - y|^{-(d+\alpha)} \mathbf{1}_{\{|x-y| < \lambda\}}.$$

The Lévy intensity describes the jumps of the process  $\widehat{X}^{a,\lambda}$  through the Lévy system: for any non-negative measurable function  $f$  on  $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ ,  $x \in \mathbb{R}^d$  and stopping time  $T$  (with respect to the filtration of  $\widehat{X}^{a,\lambda}$ ),

$$(3.20) \quad \mathbb{E}_x \left[ \sum_{s \leq T} f(s, \widehat{X}_{s-}^{a,\lambda}, \widehat{X}_s^{a,\lambda}) \right] = \mathbb{E}_x \left[ \int_0^T \left( \int_{\mathbb{R}^d} f(s, \widehat{X}_s^{a,\lambda}, y) J^{a,\lambda}(\widehat{X}_s^{a,\lambda}, y) dy \right) ds \right].$$

For any open set  $U \subset \mathbb{R}^d$ , let  $\widehat{\tau}_U^{a,\lambda} = \inf\{t > 0 : \widehat{X}_t^{a,\lambda} \notin U\}$  be the first exit time from  $U$  by  $\widehat{X}^{a,\lambda}$ , and denote by  $\widehat{X}^{a,\lambda,U}$  the subprocess of  $\widehat{X}^{a,\lambda}$  killed upon leaving upon  $U$ . When  $\lambda = 1$ , we simply write  $\widehat{X}^a$  for  $\widehat{X}^{a,1}$  and  $\widehat{\tau}_U^a$  for  $\widehat{\tau}_U^{a,1}$ . The following scaling property will be used in the next lemma: by (3.19), we see that for every  $\lambda, a, b > 0$  and  $\xi, x \in \mathbb{R}^d$ ,

$$\begin{aligned} \mathbb{E}_x \left[ e^{i\xi \cdot (b(\widehat{X}_{b^{-2}t}^{a,\lambda} - \widehat{X}_0^{a,\lambda}))} \right] &= e^{-t|\xi|^2} \mathbb{E}_x \left[ e^{i(ab\xi) \cdot (\widehat{Y}_{b^{-2}t}^{\lambda/a} - \widehat{Y}_0^{\lambda/a})} \right] \\ &= e^{-t|\xi|^2} e^{-b^{-2}t\psi^{\lambda/a}(ab\xi)} = e^{-t(|\xi|^2 + a^\alpha b^{\alpha-2}\psi^{b\lambda}(\xi))}. \end{aligned}$$

Thus, if  $\{\widehat{X}_t^{a,\lambda,D}, t \geq 0\}$  is the subprocess of  $\{\widehat{X}_t^{a,\lambda}, t \geq 0\}$  in  $D$ , then  $\{b\widehat{X}_{b^{-2}t}^{a,\lambda,D}, t \geq 0\}$  is the subprocess of  $\{\widehat{X}_t^{ab(\alpha-2)/\alpha, b\lambda}, t \geq 0\}$  in  $bD$ . In particular, if  $\{\widehat{X}_t^{a,D}, t \geq 0\}$  is the subprocess of  $\{\widehat{X}_t^a, t \geq 0\}$  in  $D$ , then  $\{\lambda\widehat{X}_{\lambda^{-2}t}^{a,D}, t \geq 0\}$  is the subprocess of  $\{\widehat{X}_t^{a\lambda(\alpha-2)/\alpha, \lambda}, t \geq 0\}$  in  $\lambda D$ . So for any  $\lambda > 0$ , we have

$$(3.21) \quad \widehat{p}_{\lambda D}^{a\lambda(\alpha-2)/\alpha, \lambda}(t, x, y) = \lambda^{-d} \widehat{p}_D^{a,1}(\lambda^{-2}t, \lambda^{-1}x, \lambda^{-1}y) \quad \text{for } t > 0 \text{ and } x, y \in \lambda D$$

where  $\widehat{p}_D^{a,\lambda}(t, x, y)$  is the transition density of  $\widehat{X}^{a,\lambda,D}$ . By integrating the above equation with respect to  $t$ , we get

$$(3.22) \quad \widehat{G}_{\lambda D}^{a\lambda(\alpha-2)/\alpha, \lambda}(x, y) = \lambda^{2-d} \widehat{G}_D^{a,1}(\lambda^{-1}x, \lambda^{-1}y) \quad \text{for } x, y \in \lambda D$$

where

$$\widehat{G}_D^{a,\lambda}(x, y) := \int_0^\infty \widehat{p}_D^{a,\lambda}(t, x, y) dt$$

is the Green function of  $\widehat{X}^{a,\lambda}$  in  $D$ .

For our reader's convenience, we summarize some notations below.

Process	Generator	Lévy (jumping) kernel
$X^0$	$\Delta$	0
$Y$	$\Delta^{\alpha/2}$	$\mathcal{A}(d, \alpha)  z ^{-d-\alpha}$
$aY$	$a^\alpha \Delta^{\alpha/2}$	$a^\alpha \mathcal{A}(d, \alpha)   z ^{-d-\alpha}$
$\widehat{Y}^\lambda$	$\widehat{\Delta}_{d,\lambda}^{\alpha/2}$	$\mathcal{A}(d, \alpha)  z ^{-d-\alpha} \mathbf{1}_{\{ z <\lambda\}}$
$X^a := X^0 + aY$	$\Delta + a^\alpha \Delta^{\alpha/2}$	$a^\alpha \mathcal{A}(d, \alpha)  z ^{-d-\alpha}$
$\widehat{X}^{a,\lambda} := X^0 + a\widehat{Y}^{\lambda/a}$	$\Delta + a^\alpha \widehat{\Delta}_{d,\lambda}^{\alpha/2}$	$a^\alpha \mathcal{A}(d, \alpha)  z ^{-d-\alpha} \mathbf{1}_{\{ z <\lambda\}}$
$\widehat{X}^a := \widehat{X}^{a,1}$	$\Delta + a^\alpha \widehat{\Delta}_d^{\alpha/2}$	$a^\alpha \mathcal{A}(d, \alpha)  z ^{-d-\alpha} \mathbf{1}_{\{ z <1\}}$ .

Recall that  $\rho_Q(x) := x_d - \phi_Q(\tilde{x})$  for every  $Q \in \partial D$  and  $x \in \{y = (\tilde{y}, y_d) \in B(Q, R) : y_d > \phi_Q(\tilde{y})\}$ . We define for  $r_1, r_2 > 0$

$$D_Q(r_1, r_2) := \{y \in D : r_1 > \rho_Q(y) > 0, |\tilde{y}| < r_2\}.$$

**Lemma 3.4** *There are constants  $\delta_0 = \delta_0(R, M, \Lambda, \alpha) \in (0, r_0)$ ,  $C_8 = C_8(R, M, \Lambda, \alpha) > 0$  and  $C_9 = C_9(R, M, \Lambda, \alpha) > 0$  such that for every  $a \in (0, M]$ ,  $\lambda \geq 1$ ,  $Q \in \partial D$  and  $x \in D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)$  with  $\tilde{x} = 0$ ,*

$$(3.23) \quad \mathbb{P}_x \left( \widehat{X}_{\widehat{\tau}_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a}^a \in D_Q(2\lambda^{-1}\delta_0, \lambda^{-1}r_0) \right) \geq C_8 \lambda \delta_D(x),$$

$$(3.24) \quad \mathbb{P}_x \left( \widehat{X}_{\widehat{\tau}_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a}^a \in D \right) \leq C_9 \lambda \delta_D(x)$$

and

$$(3.25) \quad \mathbb{E}_x \left[ \widehat{\tau}_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a \right] \leq C_9 \lambda^{-1} \delta_D(x).$$

**Proof.** To derive the estimates in the lemma, it will be convenient to consider the scaled process  $\lambda \widehat{X}_{\lambda^{-2}t}^a$ , which has the same distribution as  $\widehat{X}^{a\lambda^{(\alpha-2)/\alpha}, \lambda}$ . The latter has infinitesimal generator  $\Delta + a^\alpha \lambda^{\alpha-2} \widetilde{\Delta}_{d,\lambda}^{\alpha/2}$ .

Without loss of generality, we assume  $Q = 0$  and let  $\phi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  be the  $C^{1,1}$ -function satisfying  $\phi(\tilde{0}) = \nabla \phi(\tilde{0}) = 0$ ,  $\|\nabla \phi\|_\infty \leq \Lambda$ ,  $|\nabla \phi(\tilde{y}) - \nabla \phi(\tilde{z})| \leq \Lambda |\tilde{y} - \tilde{z}|$  and  $CS_Q$  be the corresponding coordinate system such that

$$B(Q, R) \cap D = \{(\tilde{y}, y_d) \in B(0, R) \text{ in } CS_Q : y_d > \phi(\tilde{y})\}.$$

Note that, since  $D$  is a  $C^{1,1}$  open set with characteristics  $(R, \Lambda)$ , for every  $\lambda \geq 1$ ,  $\lambda D$  is a  $C^{1,1}$  open set with the same characteristics  $(R, \Lambda)$ . Let  $\phi_\lambda(\tilde{y}) := \phi(\lambda^{-1}\tilde{y}) : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ . Then  $\phi_\lambda$  satisfies  $\phi_\lambda(\tilde{0}) = \nabla \phi_\lambda(\tilde{0}) = 0$ ,  $\|\nabla \phi_\lambda\|_\infty \leq \Lambda$ ,  $|\nabla \phi_\lambda(\tilde{y}) - \nabla \phi_\lambda(\tilde{z})| \leq \Lambda |\tilde{y} - \tilde{z}|$  and

$$B(Q, R) \cap \lambda D = \{y \in B(0, R) \text{ in } CS_Q : y_d > \phi_\lambda(\tilde{y})\} \quad \text{for all } \lambda \geq 1.$$

We let

$$p := \frac{1}{2} \left( 1 + \left( 2 \wedge (3 - \alpha) \wedge \left( \frac{2}{\alpha} + \alpha - 1 \right) \right) \right)$$

so  $p \in (1, 3/2]$ , and define

$$\begin{aligned} \rho_\lambda(y) &:= y_d - \phi_\lambda(\tilde{y}), \\ h_\lambda(y) &:= \rho_\lambda(y) \mathbf{1}_{B(0, 4r_0) \cap \lambda D}(y), \\ h_{\lambda, p}(y) &:= h_\lambda(y)^p = (\rho_\lambda(y))^p \mathbf{1}_{B(0, 4r_0) \cap \lambda D}(y), \\ D(\lambda, r_1, r_2) &:= \{y \in \lambda D : 0 < \rho_\lambda(y) < r_1 \text{ and } |\tilde{y}| < r_2\}. \end{aligned}$$

Since  $\rho_\lambda(y) \leq \sqrt{1 + \Lambda^2} \delta_{\lambda D}(y)$  in view of (3.2), we have  $0 \leq h_\lambda \leq R \leq 1$ . It is easy to see that  $D(\lambda, r_1, r_2)$  is contained in  $D \cap B(0, R/4)$  for every  $r_1, r_2 \leq r_0$ . Note that the (vector-valued) Lipschitz function  $\nabla \phi_\lambda$  is differentiable almost everywhere. So for a.e.  $y \in B(0, 4r_0) \cap \lambda D$ ,

$$(3.26) \quad \Delta h_\lambda(y) = \Delta(y_d - \phi_\lambda(\tilde{y})) = -\Delta \phi_\lambda(\tilde{y})$$

and

$$\begin{aligned} \Delta h_{\lambda, p}(y) &= \Delta(y_d - \phi_\lambda(\tilde{y}))^p \\ &= p(p-1)(1 + |\nabla \phi_\lambda(\tilde{y})|^2)(\rho_\lambda(y))^{p-2} - p(\rho_\lambda(y))^{p-1} \Delta \phi_\lambda(\tilde{y}) \\ &\geq p(p-1)(1 + |\nabla \phi_\lambda(\tilde{y})|^2)(\rho_\lambda(y))^{p-2} - p(\rho_\lambda(y))^{p-1} \|\Delta \phi_\lambda\|_\infty. \end{aligned}$$

Thus, since  $p \in (1, 3/2]$ , we can choose a positive constant  $\delta_1 = \delta_1(R, M, \Lambda, \alpha) \in (0, r_0)$ , independent of  $\lambda$ , so that there is  $c_1 > 0$  such that

$$(3.27) \quad \Delta h_{\lambda, p}(y) \geq c_1(\rho_\lambda(y))^{p-2} > 0 \quad \text{for a.e. } y \in D(\lambda, \delta_1, r_0).$$

We divide the rest of the proof into three steps.

*Step 1: Constructing suitable superharmonic and subharmonic functions with respect to  $\Delta + a^\alpha \lambda^{\alpha-2} \widehat{\Delta}_{d, \lambda}^{\alpha/2}$ .* Let  $\psi$  be a smooth positive function on  $\mathbb{R}^d$  with bounded first and second order partial derivatives such that  $\psi(y) = 2^{p+1} |\tilde{y}|^2 / r_0^2$  for  $|y| < r_0/4$  and  $2^{p+1} \leq \psi(y) \leq 2^{p+2}$  for  $|y| \geq r_0/2$ . Now we consider

$$u_{1, \lambda}(y) := h_\lambda(y) + h_{\lambda, p}(y)$$

and

$$u_{2, \lambda}(y) := h_\lambda(y) + \psi(y) - h_{\lambda, p}(y).$$

Observe that since  $0 \leq h_\lambda \leq 1$  and  $p \geq 1$ , both  $u_{1, \lambda}$  and  $u_{2, \lambda}$  are non-negative. By Taylor's expansion with remainder of order 2,

$$(3.28) \quad \left| \left( \Delta + a^\alpha \lambda^{\alpha-2} \widehat{\Delta}_{d, \lambda}^{\alpha/2} \right) \psi(y) \right| \leq |\Delta \psi(y)| + M^\alpha \left| \widehat{\Delta}_{d, \lambda}^{\alpha/2} \psi(y) \right| \leq c_2(\alpha, M) < \infty.$$

Note that the constant  $c_2$  above is independent of  $\lambda$ . Moreover, since  $\lambda \geq 1$ , by (3.14) and (3.15) there exist  $c_3 = c_3(R, \Lambda) > 0$  and  $\delta_2 = \delta_2(R, \Lambda) \in (0, \delta_1]$  independent of  $\lambda$  such that

$$\widehat{\Delta}_{d, \lambda}^{\alpha/2} h_{\lambda, p}(y) \geq -c_3 \lambda^{p-\alpha} \quad \text{for } y \in D(\lambda, \delta_2, r_0).$$

Thus by using (3.27), the fact that  $p < 2 \wedge (\frac{2}{\alpha} + \alpha - 1)$  and the inequality above, and by choosing  $\delta_2$  smaller if necessary, we get

$$(3.29) \quad \begin{aligned} \left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) h_{\lambda,p}(y) &\geq c_1 \rho_\lambda(y)^{p-2} - M^\alpha c_3 \lambda^{(p-\alpha)+(\alpha-2)} \\ &\geq c_1 \rho_\lambda(y)^{p-2} - M^\alpha c_3 \geq \frac{c_1}{2} \rho_\lambda(y)^{p-2} \end{aligned}$$

for a.e.  $y \in D(\lambda, \delta_2, r_0)$ . Furthermore by (3.26), (3.14) and (3.15), there exist  $c_4 = c_4(M) > 0$  and  $\delta_3 \in (0, \delta_2)$  independent of  $\lambda$  such that for a.e.  $y \in D(\lambda, \delta_3, r_0)$

$$(3.30) \quad \left| \left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) h_\lambda(y) \right| \leq c_4 \left( 1 + \lambda^{(1-\alpha)+(\alpha-2)} \rho_\lambda(y)^{(1-\alpha)\wedge 0} \right).$$

Thus by (3.28)-(3.30) and the fact that  $p < 2 \wedge (3 - \alpha)$ , there exists  $\delta_4 \in (0, \delta_3)$  independent of  $\lambda$  such that

$$(3.31) \quad \left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) u_{2,\lambda}(y) \leq c_2 + c_4 (1 + \rho_\lambda(y)^{(1-\alpha)\wedge 0}) - \frac{c_1}{2} \rho_\lambda(y)^{p-2} \leq -1$$

for a.e.  $y \in D(\lambda, \delta_4, r_0)$ .

On the other hand, we have from (3.15),

$$\left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) h_\lambda(y) \geq -\|\Delta\phi\|_\infty - c_5 M^\alpha \lambda^{(1-\alpha)+(\alpha-2)} \geq -\|\Delta\phi\|_\infty - c_5 M^\alpha$$

for a.e.  $y \in D(\lambda, \delta_4, r_0)$ . Combining the inequality above with (3.29), by choosing  $\delta_4$  smaller if necessary, we have for a.e.  $y \in D(\lambda, \delta_4, r_0)$ ,

$$(3.32) \quad \left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) u_{1,\lambda}(y) \geq -\|\Delta\phi\|_\infty - c_5 M^\alpha + \frac{c_1}{2} \rho_\lambda(y)^{p-2} \geq 0.$$

*Step 2: Translating super-/sub-harmonic functions into super-/sub-martingale properties for  $\widehat{X}^{a\lambda^{(\alpha-2)/\alpha}, \lambda}$ .* For notational convenience, we let

$$\widetilde{X}^{a,\lambda} := \widehat{X}^{a\lambda^{(\alpha-2)/\alpha}, \lambda} \quad \text{and} \quad \widetilde{\tau}_U^{a,\lambda} := \widehat{\tau}_U^{a\lambda^{(\alpha-2)/\alpha}, \lambda}.$$

We claim that the estimates (3.31) and (3.32) imply that

$$(3.33) \quad t \mapsto u_{2,\lambda} \left( \widetilde{X}_{t \wedge \widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a,\lambda}}^{a,\lambda} \right) \text{ is a bounded supermartingale,}$$

$$(3.34) \quad \mathbb{E}_x \left[ \widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a,\lambda} \right] \leq \rho_\lambda(x),$$

and

$$(3.35) \quad t \mapsto u_{1,\lambda} \left( \widetilde{X}_{t \wedge \widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a,\lambda}}^{a,\lambda} \right) \text{ is a bounded submartingale.}$$

Observe that if  $v$  is a bounded  $C^2$ -function on  $\mathbb{R}^d$  with bounded second order partial derivatives, then by Ito's formula and the Lévy system (3.20),

$$(3.36) \quad M_t^v = v(\widetilde{X}_t^{a,\lambda}) - v(\widetilde{X}_0^{a,\lambda}) - \int_0^t \left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) v(\widetilde{X}_s^{a,\lambda}) ds$$

is a martingale (see the proof of Proposition 4.1 in [4] for the derivation of a similar assertion). If the functions  $u_{2,\lambda}$  and  $u_{1,\lambda}$  were  $C^2$  with bounded second order partial derivatives, then the claims (3.33), (3.34) and (3.35) would just follow from (3.36) and the estimates (3.31) and (3.32). However they are not  $C^2$  since  $D$  is  $C^{1,1}$  and they are truncated on the outside of  $B(0, 4r_0) \cap \lambda D$ . So we will use a mollifier. Let  $g$  be a non-negative smooth function with compact support in  $\mathbb{R}^d$  whose value only depends on  $|x|$  such that  $g(x) = 0$  for  $|x| > 1$  and  $\int_{\mathbb{R}^d} g(x) dx = 1$ . For  $k \geq 1$ , define  $g_k(x) = 2^{kd} g(2^k x)$ . Set

$$u_{i,\lambda}^{(k)}(z) := (g_k * u_{i,\lambda})(z) := \int_{\mathbb{R}^d} g_k(y) u_{i,\lambda}(z - y) dy, \quad i = 1, 2.$$

As

$$\left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) u_{i,\lambda}^{(k)} = g_k * \left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) u_{i,\lambda} \quad \text{for } i = 1, 2,$$

we have by (3.31) and (3.32) that

$$\left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) u_{1,\lambda}^{(k)} \geq 0 \quad \text{and} \quad \left( \Delta + a^\alpha \lambda^{(\alpha-2)} \widehat{\Delta}_{d,\lambda}^{\alpha/2} \right) u_{2,\lambda}^{(k)} \leq -1$$

on  $D_k(\lambda, \delta_4, r_0) := \{y : \delta_4 - 2^{-k} > \rho_\lambda(y) > 2^{-k} \text{ and } |\tilde{y}| < r_0 - 2^{-k}\}$ .

Since  $u_{i,\lambda}^{(k)}$ ,  $i = 1, 2$ , are bounded smooth functions on  $\mathbb{R}^d$  with bounded first and second order partial derivatives, it follows from (3.36) that

$$t \mapsto u_{2,\lambda}^{(k)} \left( \widetilde{X}_{t \wedge \widetilde{\tau}_{D_k(\lambda, \delta_4, r_0)}^{a,\lambda}} \right) + t \wedge \widetilde{\tau}_{D_k(\lambda, \delta_4, r_0)}^{a,\lambda} \quad \text{is a positive supermartingale}$$

and

$$t \mapsto u_{1,\lambda}^{(k)} \left( \widetilde{X}_{t \wedge \widetilde{\tau}_{D_k(\lambda, \delta_4, r_0)}^{a,\lambda}} \right) \quad \text{is a bounded submartingale.}$$

Since for  $i = 1, 2$ ,  $u_{i,\lambda}$  is bounded and continuous,  $u_{i,\lambda}^{(k)}$  converges uniformly to  $u_{i,\lambda}$ . Thus

$$(3.37) \quad t \mapsto u_{2,\lambda} \left( \widetilde{X}_{t \wedge \widetilde{\tau}_{D_k(\lambda, \delta_4, r_0)}^{a,\lambda}} \right) + t \wedge \widetilde{\tau}_{D_k(\lambda, \delta_4, r_0)}^{a,\lambda} \quad \text{is a positive supermartingale}$$

and

$$t \mapsto u_{1,\lambda} \left( \widetilde{X}_{t \wedge \widetilde{\tau}_{D_k(\lambda, \delta_4, r_0)}^{a,\lambda}} \right) \quad \text{is a bounded submartingale.}$$

Since  $D_k(\lambda, \delta_4, r_0)$  increases to  $D(\lambda, \delta_4, r_0)$ , we conclude that (3.33) and (3.35) hold. Moreover, for each fixed  $k \geq 1$  and  $t > 0$ , we have from (3.37) that

$$\mathbb{E}_x \left[ u_{2,\lambda} \left( \widetilde{X}_{t \wedge \widetilde{\tau}_{D_k(\lambda, \delta_4, r_0)}^{a,\lambda}} \right) + t \wedge \widetilde{\tau}_{D_k(\lambda, \delta_4, r_0)}^{a,\lambda} \right] \leq u_{2,\lambda}(x).$$

Since  $u_{2,\lambda} \geq 0$ , by first letting  $k \rightarrow \infty$  and then  $t \rightarrow \infty$ , we get  $\mathbb{E}_x \left[ \widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a,\lambda} \right] \leq u_{2,\lambda}(x)$ . Since  $\tilde{x} = 0$ ,  $\psi(x) = 0$  and so  $u_{2,\lambda}(x) \leq \rho_\lambda(x)$ . This proves (3.34).

*Step 3: Deriving the desired exit distribution estimates by utilizing the super-/sub-martingale property.* Since  $\psi \geq 2^{p+1}$  on  $|\tilde{y}| \geq r_0$  and  $\psi(x) = 0$ , we have by (3.33),

$$\begin{aligned} \rho_\lambda(x) &\geq u_{2,\lambda}(x) \\ &\geq \mathbb{E}_x \left[ u_{2,\lambda} \left( \widetilde{X}_{\widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a,\lambda}} \right); \widetilde{X}_{\widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a,\lambda}} \in (\lambda D) \setminus D(\lambda, \infty, r_0) \right] \\ &\geq (2^{p+1} - 1) \mathbb{P}_x \left( \widetilde{X}_{\widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a,\lambda}} \in (\lambda D) \setminus D(\lambda, \infty, r_0) \right). \end{aligned}$$

We also have from (3.35)

$$\rho_\lambda(x) \leq \rho_\lambda(x) + \rho_\lambda(x)^p = u_{1,\lambda}(x) \leq \mathbb{E}_x \left[ u_{1,\lambda} \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \right) \right] \leq 2\mathbb{P}_x \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \in \lambda D \right).$$

Combining the two displays above, we get

$$\begin{aligned} (3.38) \quad & \mathbb{P}_x \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \in D(\lambda, \infty, r_0) \right) \\ &= \mathbb{P}_x \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \in \lambda D \right) - \mathbb{P}_x \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \in (\lambda D) \setminus D(\lambda, \infty, r_0) \right) \\ &\geq \frac{2^{p+1} - 3}{2(2^{p+1} - 1)} \rho_\lambda(x). \end{aligned}$$

By (3.20),

$$\begin{aligned} (3.39) \quad & \mathbb{P}_x \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \in D(\lambda, \infty, r_0) \setminus D(\lambda, 2\delta_4, r_0) \right) \\ &= \mathbb{E}_x \left[ \int_0^{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}} \int_{D(\lambda,\infty,r_0) \setminus D(\lambda,2\delta_4,r_0)} \frac{(a\lambda^{(\alpha-2)/\alpha})^\alpha \mathcal{A}(d,\alpha)}{|\tilde{X}_s^{a,\lambda} - y|^{d+\alpha}} \mathbf{1}_{\{|\tilde{X}_s^{a,\lambda} - y| < \lambda\}} dy ds \right] \\ &\leq \mathbb{E}_x \left[ \int_0^{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}} \int_{D(\lambda,\infty,r_0) \setminus D(\lambda,2\delta_4,r_0)} \frac{a^\alpha \lambda^{\alpha-2} \mathcal{A}(d,\alpha)}{|\tilde{X}_s^{a,\lambda} - y|^{d+\alpha}} dy ds \right] \\ &\leq c_6 \mathcal{A}(d,\alpha) a^\alpha \lambda^{\alpha-2} \left( \int_{D(\lambda,\infty,r_0) \setminus D(\lambda,2\delta_4,r_0)} |y|^{-d-\alpha} dy \right) \mathbb{E}_x \left[ \tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda} \right] \\ &\leq c_7 \mathcal{A}(d,\alpha) a^\alpha \lambda^{\alpha-2} \left( \int_{D(\lambda,2\delta_4,r_0) \setminus D(\lambda,3\delta_4/2,r_0)} |y|^{-d-\alpha} dy \right) \mathbb{E}_x \left[ \tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda} \right] \\ &\leq c_8 \mathbb{E}_x \left[ \int_0^{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}} \int_{D(\lambda,2\delta_4,r_0) \setminus D(\lambda,3\delta_4/2,r_0)} \frac{a^\alpha \lambda^{\alpha-2} \mathcal{A}(d,\alpha)}{|\tilde{X}_s^{a,\lambda} - y|^{d+\alpha}} \mathbf{1}_{\{|\tilde{X}_s^{a,\lambda} - y| < \lambda\}} dy ds \right] \\ &= c_8 \mathbb{P}_x \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \in D(\lambda, 2\delta_4, r_0) \setminus D(\lambda, 3\delta_4/2, r_0) \right). \end{aligned}$$

Thus from (3.38)-(3.39)

$$(3.40) \quad \mathbb{P}_x \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \in D(\lambda, 2\delta_4, r_0) \right) \geq c_9 \rho_\lambda(x).$$

Recall that  $0 \leq h_{\lambda,p} \leq 1$ . If  $|y| > r_0/2$ , then  $\psi(y) \geq 2^{p+1}$ , we have

$$u_{2,\lambda}(y) = \psi(y) + h_\lambda(y) - h_{\lambda,p}(y) \geq \psi(y) - h_{\lambda,p}(y) \geq 2^p \geq 1 \quad \text{for } y \in B(0, r_0/2)^c.$$

Furthermore, for  $y \in B(0, 4r_0)$  such that  $\delta_4 \leq \rho_\lambda(y) < 4r_0$ ,

$$u_{2,\lambda}(y) = \psi(y) + h_\lambda(y) - h_{\lambda,p}(y) \geq \rho_\lambda(y) - \rho_\lambda(y)^p \geq c_{10},$$

where  $c_{10} \in (0, 1)$  depends on  $\delta_4$  and  $R$ . By using the last two observations, it holds that  $u_{2,\lambda} \geq c_{10} > 0$  on  $(\lambda D) \setminus D(\lambda, \delta_4, r_0)$ . Therefore, by (3.33) we get

$$(3.41) \quad \rho_\lambda(x) \geq u_{2,\lambda}(x) \geq \mathbb{E}_x \left[ u_{2,\lambda} \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \right) \right] \geq c_{10} \mathbb{P}_x \left( \tilde{X}_{\tilde{\tau}_{D(\lambda,\delta_4,r_0)}^{a,\lambda}}^{a,\lambda} \in \lambda D \right).$$

Since the process  $\{\lambda(\widehat{X}_{\lambda^{-2}t}^a - \widehat{X}_0^a), t \geq 0\}$  under  $\mathbb{P}_x$  has the same distribution as  $\{\widehat{X}_t^{a\lambda^{(\alpha-2)/\alpha}, \lambda} - \widehat{X}_0^{a\lambda^{(\alpha-2)/\alpha}, \lambda}, t \geq 0\}$  under  $\mathbb{P}_{\lambda x}$ , we have from (3.40) that for  $x \in D_Q(\lambda^{-1}\delta_4, \lambda^{-1}r_0)$

$$\begin{aligned} & \mathbb{P}_x \left( \widehat{X}_{\tau_{D_Q(\lambda^{-1}\delta_4, \lambda^{-1}r_0)}^a}^a \in D \right) \\ & \geq \mathbb{P}_x \left( \widehat{X}_{\tau_{D_Q(\lambda^{-1}\delta_4, \lambda^{-1}r_0)}^a}^a \in D_Q(2\lambda^{-1}\delta_4, \lambda^{-1}r_0) \setminus D_Q(\lambda^{-1}\delta_4, \lambda^{-1}r_0) \right) \\ & = \mathbb{P}_{\lambda x} \left( \widetilde{X}_{\widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a, \lambda}}^{a, \lambda} \in D(\lambda, 2\delta_4, r_0) \setminus D(\lambda, \delta_4, r_0) \right) \\ & \geq c_9 \rho_\lambda(\lambda x) \geq c_{11} \delta_{\lambda D}(\lambda x) = c_{11} \lambda \delta_D(x), \end{aligned}$$

and, from (3.41)

$$\begin{aligned} \mathbb{P}_x \left( \widehat{X}_{\tau_{D_Q(\lambda^{-1}\delta_4, \lambda^{-1}r_0)}^a}^a \in D \right) & = \mathbb{P}_{\lambda x} \left( \widetilde{X}_{\widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a, \lambda}}^{a, \lambda} \in \lambda D \right) \\ & \leq c_{12} \rho_\lambda(\lambda x) \leq c_{13} \delta_{\lambda D}(\lambda x) = c_{13} \lambda \delta_D(x). \end{aligned}$$

Finally by (3.22) and (3.34),

$$\begin{aligned} \mathbb{E}_x \left[ \widehat{\tau}_{D_Q(\lambda^{-1}\delta_4, \lambda^{-1}r_0)}^a \right] & = \int_{D_Q(\lambda^{-1}\delta_4, \lambda^{-1}r_0)} \widehat{G}_{D_Q(\lambda^{-1}\delta_4, \lambda^{-1}r_0)}^{a, 1}(x, y) dy \\ & = \lambda^{d-2} \int_{D_Q(\lambda^{-1}\delta_4, \lambda^{-1}r_0)} \widehat{G}_{D(\lambda, \delta_4, r_0)}^{a\lambda^{(\alpha-2)/\alpha}, \lambda}(\lambda x, \lambda y) dy \\ & = \lambda^{-2} \int_{D(\lambda, \delta_4, r_0)} \widehat{G}_{D(\lambda, \delta_4, r_0)}^{a\lambda^{(\alpha-2)/\alpha}, \lambda}(\lambda x, z) dz \\ & = \lambda^{-2} \mathbb{E}_{\lambda x} \left[ \widetilde{\tau}_{D(\lambda, \delta_4, r_0)}^{a, \lambda} \right] \\ & \leq \lambda^{-2} \rho_\lambda(\lambda x) \leq c_{14} \lambda^{-2} \delta_{\lambda D}(\lambda x) = c_{14} \lambda^{-1} \delta_D(x). \end{aligned}$$

This completes the proof by taking  $\delta_0 = \delta_4$ ,  $C_8 = c_{11}$ , and  $C_9 = \max\{c_{13}, c_{14}\}$ .  $\square$

We now derive exit distribution estimates for the process  $X^a$  from those for  $\widehat{X}^a$  in Lemma 3.4. Recall that  $r_0 = R/(4\sqrt{1 + \Lambda^2})$ .

**Lemma 3.5** *There are constants  $\delta_0 = \delta_0(R, M, \Lambda, \alpha) \in (0, r_0)$ ,  $C_8 = C_8(R, M, \Lambda, \alpha) > 0$  and  $C_{10} = C_{10}(R, M, \Lambda, \alpha) > 0$  such that for every  $a \in (0, M]$ ,  $\lambda \geq 1$ ,  $Q \in \partial D$  and  $x \in D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)$  with  $\tilde{x} = 0$ ,*

$$(3.42) \quad \mathbb{P}_x \left( X_{\tau_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a}^a \in D_Q(2\lambda^{-1}\delta_0, \lambda^{-1}r_0) \right) \geq C_8 \lambda \delta_D(x),$$

$$(3.43) \quad \mathbb{P}_x \left( X_{\tau_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a}^a \in D \right) \leq C_{10} \lambda \delta_D(x)$$

and

$$(3.44) \quad \mathbb{E}_x \left[ \tau_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a \right] \leq C_{10} \lambda^{-1} \delta_D(x).$$

**Proof.** Without loss of generality, we assume  $Q = 0$  and let  $\phi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  be the  $C^{1,1}$ -function satisfying  $\phi(\tilde{0}) = \nabla\phi(\tilde{0}) = 0$ ,  $\|\nabla\phi\|_\infty \leq \Lambda$ ,  $|\nabla\phi(\tilde{y}) - \nabla\phi(\tilde{z})| \leq \Lambda|\tilde{y} - \tilde{z}|$  and  $CS_Q$  be the corresponding coordinate system such that

$$B(Q, R) \cap D = \{(\tilde{y}, y_d) \in CS_Q \cap B(0, R) : y_d > \phi(\tilde{y})\}.$$

Let  $\delta_0, C_8$  and  $C_9$  be the constants from the statement of Lemma 3.4. Since  $\text{diam}(D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)) \leq \frac{1}{2}$ , we have that

$$|x - y|^{-d-\alpha} \mathbf{1}_{\{|x-y|<1\}} = |x - y|^{-d-\alpha} \quad \text{for all } x, y \in D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0).$$

Let

$$j(x) := a^\alpha \mathcal{A}(d, \alpha) |x|^{-(d+\alpha)} \mathbf{1}_{\{|x| \geq 1\}}.$$

Note that  $\int_{\mathbb{R}^d} j(x) dx < \infty$ . Thus we can write  $X_t^a = \hat{X}_t^a + Z_t^a$  where  $Z_t^a$  is a compound Poisson process with the Lévy density  $j(x)$ , independent of  $\hat{X}_t^a$ . Since the jump size of  $Z^a$  is greater than or equal to 1 and  $\text{diam}(D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)) \leq \frac{1}{2}$ , we see from (3.25) that

$$\mathbb{E}_x \left[ \tau_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a \right] \leq \mathbb{E}_x \left[ \hat{\tau}_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a \right] \leq C_9 \lambda^{-1} \delta_D(x).$$

Moreover we have from (3.23) that

$$\begin{aligned} & \mathbb{P}_x \left( X_{\tau_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a}^a \in D_Q(2\lambda^{-1}\delta_0, \lambda^{-1}r_0) \right) \\ &= \mathbb{P}_x \left( \hat{X}_{\hat{\tau}_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a}^a \in D_Q(2\lambda^{-1}\delta_0, \lambda^{-1}r_0) \right) \geq C_8 \lambda \delta_D(x). \end{aligned}$$

We recall the notations from the proof of the previous lemma:

$$\rho_\lambda(x) := y_d - \phi(\lambda^{-1}\tilde{y}),$$

$$D(\lambda, r_1, r_2) := \{y \in CS_Q : r_1 > \rho_\lambda(y) > 0, |\tilde{y}| < r_2\},$$

$$\tilde{X}^{a,\lambda} = \hat{X}^{a\lambda^{(\alpha-2)/\alpha}, \lambda} \quad \text{and} \quad \tilde{\tau}_U^{a,\lambda} := \hat{\tau}_U^{a\lambda^{(\alpha-2)/\alpha}, \lambda}.$$

Let  $a(\lambda) := a\lambda^{(\alpha-2)/\alpha}$ , which is no larger than  $M$ . By (1.3),

$$\begin{aligned} (3.45) \quad & \mathbb{P}_x \left( X_{\tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)}}^{a(\lambda)} \in (\lambda D) \setminus D(\lambda, 2\delta_0, 2r_0) \right) \\ &= \mathbb{E}_x \left[ \int_0^{\tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)}} \int_{(\lambda D) \setminus D(\lambda, 2\delta_0, 2r_0)} \frac{a(\lambda)^\alpha \mathcal{A}(d, \alpha)}{|X_s^{a(\lambda)} - y|^{d+\alpha}} dy ds \right] \\ &\leq c_1 \mathcal{A}(d, \alpha) (a(\lambda))^\alpha \left( \int_{(\lambda D) \setminus D(\lambda, 2\delta_0, 2r_0)} |y|^{-d-\alpha} dy \right) \mathbb{E}_x \left[ \tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)} \right] \\ &\leq c_2 \mathcal{A}(d, \alpha) (a(\lambda))^\alpha \left( \int_{D(\lambda, 2\delta_0, r_0) \setminus D(\lambda, 3\delta_0/2, r_0)} |y|^{-d-\alpha} dy \right) \mathbb{E}_x \left[ \tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)} \right] \\ &\leq c_3 \mathbb{E}_x \left[ \int_0^{\tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)}} \int_{D(\lambda, 2\delta_0, r_0) \setminus D(\lambda, 3\delta_0/2, r_0)} \frac{a(\lambda)^\alpha \mathcal{A}(d, \alpha)}{|X_s^{a(\lambda)} - y|^{d+\alpha}} dy ds \right] \\ &= c_3 \mathbb{P}_x \left( X_{\tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)}}^{a(\lambda)} \in D(\lambda, 2\delta_0, r_0) \setminus D(\lambda, 3\delta_0/2, r_0) \right). \end{aligned}$$

Thus by the above inequality and (3.41), we have

$$\begin{aligned}
(3.46) \quad & \mathbb{P}_x \left( X_{\tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)}}^{a(\lambda)} \in \lambda D \right) \\
&= \mathbb{P}_x \left( X_{\tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)}}^{a(\lambda)} \in (\lambda D) \setminus D(\lambda, 2\delta_0, 2r_0) \right) + \mathbb{P}_x \left( \tilde{X}_{\tau_{D(\lambda, \delta_0, r_0)}^{a, \lambda}}^{a, \lambda} \in D(\lambda, 2\delta_0, 2r_0) \right) \\
&\leq c_3 \mathbb{P}_x \left( X_{\tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)}}^{a(\lambda)} \in D(\lambda, 2\delta_0, r_0) \setminus D(\lambda, 3\delta_0/2, r_0) \right) + \mathbb{P}_x \left( \tilde{X}_{\tau_{D(\lambda, \delta_0, r_0)}^{a, \lambda}}^{a, \lambda} \in \lambda D \right) \\
&= c_3 \mathbb{P}_x \left( \tilde{X}_{\tau_{D(\lambda, \delta_0, r_0)}^{a, \lambda}}^{a, \lambda} \in D(\lambda, 2\delta_0, r_0) \setminus D(\lambda, 3\delta_0/2, r_0) \right) + \mathbb{P}_x \left( \tilde{X}_{\tau_{D(\lambda, \delta_0, r_0)}^{a, \lambda}}^{a, \lambda} \in \lambda D \right) \\
&\leq (c_3 + 1) \mathbb{P}_x \left( \tilde{X}_{\tau_{D(\lambda, \delta_0, r_0)}^{a, \lambda}}^{a, \lambda} \in \lambda D \right) \leq c_4 \rho_\lambda(x).
\end{aligned}$$

Since  $(\lambda X_{\lambda^{-2}t}^a, t \geq 0)$  is the independent sum of a Brownian motion and a symmetric  $\alpha$ -stable process on  $\mathbb{R}^d$  with weight  $a(\lambda)$ , we have from (3.46) that for  $x \in D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)$

$$\begin{aligned}
\mathbb{P}_x \left( X_{\tau_{D_Q(\lambda^{-1}\delta_0, \lambda^{-1}r_0)}^a}^a \in D \right) &= \mathbb{P}_{\lambda x} \left( X_{\tau_{D(\lambda, \delta_0, r_0)}^{a(\lambda)}}^{a(\lambda)} \in \lambda D \right) \\
&\leq c_4 \rho_\lambda(\lambda x) \leq c_5 \delta_{\lambda D}(\lambda x) = c_5 \lambda \delta_D(x).
\end{aligned}$$

The proof is finished by taking  $C_{10} = \max\{C_9, c_5\}$ .  $\square$

## 4 Boundary Harnack principle

In this section, we give the proof of the boundary Harnack principle for the independent sum of a Brownian motion and a symmetric stable process. We first prove the Carleson estimate for the independent sum of a Brownian motion and a symmetric stable process on Lipschitz open sets.

We recall that an open set  $D$  in  $\mathbb{R}^d$  (when  $d \geq 2$ ) is said to be a Lipschitz open set if there exist a localization radius  $R_1 > 0$  and a constant  $\Lambda_1 > 0$  such that for every  $Q \in \partial D$ , there exist a Lipschitz function  $\phi = \phi_Q : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  satisfying  $\phi(0) = 0$ ,  $|\phi(x) - \phi(y)| \leq \Lambda_1|x - y|$ , and an orthonormal coordinate system  $CS_Q: y = (y_1, \dots, y_{d-1}, y_d) =: (\tilde{y}, y_d)$  with its origin at  $Q$  such that

$$B(Q, R_1) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R_1) \text{ in } CS_Q : y_d > \phi(\tilde{y})\}.$$

The pair  $(R_1, \Lambda_1)$  is called the characteristics of the Lipschitz open set  $D$ . Note that a Lipschitz open set can be unbounded and disconnected. For Lipschitz open set  $D$  and every  $Q \in \partial D$  and  $x \in B(Q, R_1) \cap D$ , we define

$$\rho_Q(x) := x_d - \phi_Q(\tilde{x}),$$

where  $(\tilde{x}, x_d)$  is the coordinates of  $x$  in  $CS_Q$ .

We recall that  $X_t^a = X_t^0 + aY_t$  is a Lévy process with characteristic exponent  $\Phi^a(x) = |x|^2 + a^\alpha|x|^\alpha$ . This process may be obtained by subordinating a  $d$ -dimensional Brownian motion  $W = (W_t, t \geq 0)$  by an independent subordinator  $T_t^a := t + a^2T_t$  where  $T = (T_t, t \geq 0)$  is an  $\alpha/2$ -stable subordinator. More precisely, the processes  $X_t^a$  and  $W_{T_t^a}$  have the same distribution. Note that the Laplace exponent corresponding to  $T^a$  is equal to  $\phi^a(\lambda) = \lambda + a^\alpha\lambda^{\alpha/2}$ . Let  $\mathcal{M}_{\alpha/2}(t) =$

$\sum_{n=0}^{\infty} t^{n\alpha/2}/\Gamma(1+n\alpha/2)$  be the Mittag-Leffler function. It follows by a straightforward integration that

$$\int_0^{\infty} e^{-\lambda t} \mathcal{M}_{1-\alpha/2}(-a^{2/(2-\alpha)}t) dt = \frac{1}{\phi^a(\lambda)},$$

which shows that the potential density  $u^a$  of the subordinator  $T^a$  is given by

$$u^a(t) = \mathcal{M}_{1-\alpha/2}(-a^{2/(2-\alpha)}t).$$

It is well known that  $a \mapsto u^a(t)$  is a decreasing function. Therefore, if  $0 < a_1 < a_2$ , then  $u^{a_1}(t) \geq u^{a_2}(t)$  for all  $t > 0$ . We will need this fact in the proof of next lemma.

**Lemma 4.1** *Let  $D \subset \mathbb{R}^d$  be a Lipschitz open set with the characteristics  $(R_1, \Lambda_1)$ . There exists a constant  $\delta = \delta(R_1, \Lambda_1, M) > 0$  such that for all  $a \in [0, M]$  and  $Q \in \partial D$ ,  $x \in D$  with  $\rho_Q(x) < R_1/2$ ,*

$$\mathbb{P}_x(X_{\tau(x)}^a \in D^c) \geq \delta,$$

where  $\tau(x) := \tau_{D \cap B(x, 2\rho_Q(x))}^a = \inf\{t > 0 : X_t^a \notin D \cap B(x, 2\rho_Q(x))\}$ .

**Proof.** Clearly,

$$\mathbb{P}_x \left( X_{\tau(x)}^a \in D^c \right) \geq \mathbb{P}_x \left( X_{\tau(x)}^a \in D^c \cap B(x, 2\rho_Q(x)) \right) \geq \mathbb{P}_x \left( X_{\tau(x)}^a \in \partial D \cap B(x, 2\rho_Q(x)) \right).$$

Let  $D_x := D \cap B(x, 2\rho_Q(x))$  and  $W^{D_x}$  be the subprocess of Brownian motion  $W$  killed upon leaving  $D_x$ . The process  $Z^a$  defined by  $Z_t^a := W^{D_x}(T_t^a)$ , where  $T_t^a$  is an independent subordinator described in the paragraph before the statement of the lemma, is called a subordinate killed Brownian motion in  $D_x$ . We will use  $\zeta$  to denote the lifetime of  $Z^a$ . It is known from [18] that

$$\begin{aligned} \mathbb{P}_x \left( X_{\tau(x)}^a \in \partial D \cap B(x, 2\rho_Q(x)) \right) &\geq \mathbb{P}_x \left( Z_{\zeta^-}^a \in \partial D \cap B(x, 2\rho_Q(x)) \right) \\ &= \mathbb{E}_x \left[ u^a(\tilde{\tau}_{D_x}); W_{\tilde{\tau}_{D_x}} \in \partial D \cap B(x, 2\rho_Q(x)) \right]. \end{aligned}$$

Here and below,  $\tilde{\tau}_U := \inf\{t > 0 : W_t \notin U\}$  is the exit time of  $W$  from  $U$ . Denote  $C_x := \partial D \cap B(x, 2\rho_Q(x))$ . Then

$$(4.1) \quad \begin{aligned} \mathbb{E}_x \left[ u^a(\tilde{\tau}_{D_x}); W_{\tilde{\tau}_{D_x}} \in C_x \right] &\geq \mathbb{E}_x \left[ u^a(\tilde{\tau}_{D_x}); W_{\tilde{\tau}_{D_x}} \in C_x, \tilde{\tau}_{D_x} \leq t \right] \\ &\geq u^a(t) \mathbb{P}_x \left[ W_{\tilde{\tau}_{D_x}} \in C_x, \tilde{\tau}_{D_x} \leq t \right] \\ &\geq u^a(t) \left( \mathbb{P}_x(W_{\tilde{\tau}_{D_x}} \in C_x) - \mathbb{P}_x(\tilde{\tau}_{D_x} > t) \right), \end{aligned}$$

where  $t > 0$  will be chosen later.

Since  $D$  is a Lipschitz open set with characteristics  $(R_1, \Lambda_1)$ , there exist  $\eta = \eta(\Lambda, R_1) > 0$  and a cone

$$(4.2) \quad \mathcal{C} := \left\{ y = (y_1, \dots, y_d) \in \mathbb{R}^d : y_d < 0, (y_1^2 + \dots + y_{d-1}^2)^{1/2} < \eta|y_d| \right\}$$

such that for every  $z \in \partial D$ , there is a cone  $\mathcal{C}_z$  with vertex  $z$ , isometric to  $\mathcal{C}$ , satisfying  $\mathcal{C}_z \cap B(Q, R_1) \subset D^c$ . Then by the scaling property of  $W$  and symmetry considerations, we have

$$\begin{aligned} \mathbb{P}_x(W_{\tilde{\tau}_{D_x}} \in C_x) &\geq \mathbb{P}_x(W_{\tilde{\tau}_{B(x, 2\rho_Q(x))}} \in \partial B(x, 2\rho_Q(x)) \cap \mathcal{C}_{(\tilde{x}, \phi_Q(\tilde{x}))}) \\ &\geq \mathbb{P}_0(W_{\tilde{\tau}_{B(0,2)}} \in \partial B(0, 2) \cap (\mathcal{C} + (\tilde{0}, -1))), \end{aligned}$$

which is strictly positive. Hence we can conclude that there exists  $c_1 = c_1(D) > 0$  such that

$$(4.3) \quad \mathbb{P}_x(W_{\tilde{\tau}_{D_x}} \in C_x) \geq c_1.$$

Next,

$$(4.4) \quad \mathbb{P}_x(\tilde{\tau}_{D_x} > t) \leq \frac{\mathbb{E}_x[\tilde{\tau}_{D_x}]}{t} \leq \frac{\mathbb{E}_x[\tilde{\tau}_{B(x, 2\rho_Q(x))}]}{t} \leq c_2 \frac{(\rho_Q(x))^2}{t} \leq c_2 \frac{R_1^2}{t},$$

for some constant  $c_2 > 0$ . By using (4.3) and (4.4) in (4.1), we obtain that

$$\mathbb{E}_x[u^a(\tilde{\tau}_{D_x}); W_{\tilde{\tau}_{D_x}} \in C_x] \geq u^a(t) \left( c_1 - c_2 \frac{R_1^2}{t} \right).$$

Now choose  $t = t(R_1, \Lambda_1) > 0$  large enough so that  $c_1 - c_2 R_1^2/t \geq c_1/2$ . Then

$$\mathbb{E}_x[u(\tilde{\tau}_{D_x}); W_{\tilde{\tau}_{D_x}} \in C_x] \geq c_1 u^a(t)/2 \geq c_1 u^M(t)/2 =: \delta.$$

The lemma is thus proved.  $\square$

Suppose that  $D$  is an open set and that  $U$  and  $V$  are bounded open sets with  $V \subset \bar{V} \subset U$  and  $D \cap V \neq \emptyset$ . If  $u$  vanishes continuously on  $D^c \cap U$ , then by a finite covering argument, it is easy to see that  $u$  is bounded in an open neighborhood of  $\partial D \cap V$ .

**Lemma 4.2** *Let  $D$  be an open set and  $U$  and  $V$  be bounded open sets with  $V \subset \bar{V} \subset U$  and  $D \cap V \neq \emptyset$ . Suppose  $u$  is a nonnegative function in  $\mathbb{R}^d$  that is harmonic in  $D \cap U$  with respect to  $X^a$  and vanishes continuously on  $D^c \cap U$ . Then  $u$  is regular harmonic in  $D \cap V$  with respect to  $X^a$ , i.e.,*

$$(4.5) \quad u(x) = \mathbb{E}_x \left[ u(X_{\tau_{D \cap V}^a}^a) \right] \quad \text{for all } x \in D \cap V.$$

**Proof.** For  $n \geq 1$ , let  $B_n = \{y \in D \cap V : \delta_D(y) > 1/n\}$ . Then for large  $n$ ,  $B_n$  is a non-empty open subset of  $D \cap V$  whose closure is contained in  $D \cap U$ . Since  $u$  is harmonic in  $D \cap U$  with respect to  $X^a$ , for  $x \in D \cap V$  and  $n$  large enough so that  $x \in B_n$ , we have that

$$u(x) = \mathbb{E}_x \left[ u(X_{\tau_{B_n}^a}^a) \right] = \mathbb{E}_x \left[ u(X_{\tau_{B_n}^a}^a); \tau_{B_n}^a < \tau_{D \cap V}^a \right] + \mathbb{E}_x \left[ u(X_{\tau_{B_n}^a}^a); \tau_{B_n}^a = \tau_{D \cap V}^a \right].$$

Hence

$$(4.6) \quad \left| u(x) - \mathbb{E}_x \left[ u(X_{\tau_{D \cap V}^a}^a) \right] \right| \leq \mathbb{E}_x \left[ u(X_{\tau_{B_n}^a}^a); \tau_{B_n}^a < \tau_{D \cap V}^a \right] + \mathbb{E}_x \left[ u(X_{\tau_{D \cap V}^a}^a); \tau_{B_n}^a < \tau_{D \cap V}^a \right].$$

Since  $\lim_{n \rightarrow \infty} \tau_{B_n}^a = \tau_{D \cap V}^a$   $\mathbb{P}_x$ -a.s., the second term in (4.6) converges to  $\mathbb{E}_x[u(X_{\tau_{D \cap V}^a}^a); A]$  where  $A := \cap_{n=1}^{\infty} \{\tau_{B_n}^a < \tau_{D \cap V}^a\}$ . Note that

$$X_{\tau_{D \cap V}^a}^a \in \partial D \cap V \quad \text{on } A.$$

Hence  $u(X_{\tau_{D \cap V}^a}^a) = 0$  on  $A$ , as  $u$  is assumed to vanish on  $D^c \cap U$ . Consequently

$$\lim_{n \rightarrow \infty} \mathbb{E}_x \left[ u(X_{\tau_{D \cap V}^a}^a); \tau_{B_n}^a < \tau_{D \cap V}^a \right] = 0.$$

For the first term in (4.6), note that  $\delta_D(X_{\tau_{B_n}^a}^a) \leq 1/n$  on  $\{\tau_{B_n}^a < \tau_{D \cap V}^a\}$ . Therefore, by the assumption that  $u$  vanishes continuously on  $D^c \cap U$ , one has  $\lim_{n \rightarrow \infty} u(X_{\tau_{B_n}^a}^a) = 0$ . Moreover, since  $u$  vanishes continuously on  $(\partial D) \cap U$ , there is  $n_0 \geq 1$  so that  $u$  is bounded in  $D \cap V \setminus B_{n_0}$ . So by the bounded convergence theorem we have

$$\lim_{n \rightarrow \infty} \mathbb{E}_x \left[ u(X_{\tau_{B_n}^a}^a); \tau_{B_n}^a < \tau_{D \cap V}^a \right] = 0.$$

This proves the lemma.  $\square$

**Proof of Proposition 1.3.** We know from the parabolic Harnack inequality from Theorem 6.7 of [9] that Harnack inequality holds for the process  $X := X^1$ . That is, there exists a constant  $c_1 = c_1(\alpha, M) > 0$  such that for any  $r \in (0, M^{\alpha/(2-\alpha)}]$ ,  $x_0 \in \mathbb{R}^d$  and any function  $v \geq 0$  harmonic in  $B(x_0, r)$  with respect to  $X$ , we have

$$(4.7) \quad v(x) \leq c_1 v(y) \quad \text{for all } x, y \in B(x_0, \frac{r}{2}).$$

Now the proposition is an easy consequence of (4.7). In fact, note that for any  $a \in (0, M]$ ,  $X^a$  has the same distribution as  $\lambda X_{\lambda^{-2}t}$ , where  $\lambda = a^{\alpha/(\alpha-2)} \geq M^{\alpha/(\alpha-2)}$ . Consequently, if  $u$  is harmonic in  $B(x_0, r)$  with respect to  $X^a$  where  $r \in (0, 1]$ , then  $v(x) := u(\lambda x)$  is harmonic in  $B(\lambda^{-1}x_0, \lambda^{-1}r)$  with respect to  $X$  and  $\lambda^{-1}r \leq M^{\alpha/(2-\alpha)}$ . So by (4.7)

$$u(\lambda x) = v(x) \leq c_1 v(y) = c_1 u(\lambda y) \quad \text{for all } x, y \in B(\lambda^{-1}x_0, \lambda^{-1}r/2).$$

That is,

$$u(x) \leq c_1 u(y) \quad \text{for all } x, y \in B(x_0, r/2).$$

$\square$

**Theorem 4.3 (Carleson estimate)** *Let  $D \subset \mathbb{R}^d$  be a Lipschitz open set with the characteristics  $(R_1, \Lambda_1)$ . Then there exists a positive constant  $A = A(\alpha, \Lambda_1, R_1, M)$  such that for  $a \in (0, M]$ ,  $Q \in \partial D$ ,  $0 < r < R_1/2$ , and any nonnegative function  $u$  in  $\mathbb{R}^d$  that is harmonic in  $D \cap B(Q, r)$  with respect to  $X^a$  and vanishes continuously on  $D^c \cap B(Q, r)$ , we have*

$$(4.8) \quad u(x) \leq Au(x_0) \quad \text{for } x \in D \cap B(Q, r/2).$$

where  $x_0 \in D \cap B(Q, r)$  with  $\rho_Q(x_0) = r/2$ .

**Proof.** Fix  $a \in (0, M]$ . Since  $D$  is Lipschitz and  $r < R_1/2$ , by the uniform Harnack principle in Proposition 1.3 and a standard chain argument, it suffices to prove (4.8) for  $x \in D \cap B(Q, r/12)$  and  $\tilde{x}_0 = \tilde{Q}$ . Without loss of generality, we may assume that  $u(x_0) = 1$ . In this proof, constants  $\delta, \beta, \eta$  and  $c_i$ 's are always independent of  $r$  and  $a$ .

Choose  $0 < \gamma < \alpha/(d + \alpha)$  and let

$$B_0 = D \cap B(x, 2\rho_Q(x)), \quad B_1 = B(x, r^{1-\gamma}\rho_Q(x)^\gamma).$$

Further, set

$$B_2 = B(x_0, \rho_Q(x_0)/3), \quad B_3 = B(x_0, 2\rho_Q(x_0)/3)$$

and

$$\tau_0 = \inf\{t > 0 : X_t^a \notin B_0\}, \quad \tau_2 = \inf\{t > 0 : X_t^a \notin B_2\}.$$

By Lemma 4.1, there exists  $\delta = \delta(R_1, \Lambda_1, M) > 0$  such that

$$(4.9) \quad \mathbb{P}_x(X_{\tau_0}^a \in D^c) \geq \delta, \quad x \in B(Q, r/4).$$

By the uniform Harnack principle in Proposition 1.3 and a chain argument, there exists  $\beta$  such that

$$(4.10) \quad u(x) < (\rho_Q(x)/r)^{-\beta} u(x_0), \quad x \in D \cap B(Q, r/4).$$

In view of Lemma 4.2,  $u$  is regular harmonic in  $B_0$  with respect to  $X^a$ . So

$$(4.11) \quad u(x) = \mathbb{E}_x[u(X_{\tau_0}^a); X_{\tau_0}^a \in B_1] + \mathbb{E}_x[u(X_{\tau_0}^a); X_{\tau_0}^a \notin B_1] \quad \text{for } x \in B(Q, r/4).$$

We first show that there exists  $\eta > 0$  such that

$$(4.12) \quad \mathbb{E}_x[u(X_{\tau_0}^a); X_{\tau_0}^a \notin B_1] \leq u(x_0) \quad \text{if } x \in D \cap B(Q, r/12) \text{ with } \rho_Q(x) < \eta r.$$

Let  $\eta_0 := 2^{-2(d+\alpha)/d}$ , then for  $\rho_Q(x) < \eta_0 r$ ,

$$(\rho_Q(x))^{d/(\alpha+d)} < 1/4 \quad \text{and} \quad 2\rho_Q(x) \leq r^{1-\gamma} \rho_Q(x)^\gamma - 2\rho_Q(x).$$

Thus if  $x \in D \cap B(Q, r/12)$  with  $\rho_Q(x) < \eta_0 r$ , then  $|x - y| \leq 2|z - y|$  for  $z \in B_0, y \notin B_1$ . Thus we have by (1.3) and Lemma 3.1

$$(4.13) \quad \begin{aligned} & \mathbb{E}_x[u(X_{\tau_0}^a); X_{\tau_0}^a \notin B_1] \\ &= \mathcal{A}(d, \alpha) \int_{B_0} G_{B_0}^a(x, z) \int_{|y-x| > r^{1-\gamma} \rho_Q(x)^\gamma} a^\alpha |z - y|^{-d-\alpha} u(y) dy dz \\ &\leq 2^{d+\alpha} \mathcal{A}(d, \alpha) \int_{B_0} G_{B_0}^a(x, z) dz \int_{|y-x| > r^{1-\gamma} \rho_Q(x)^\gamma} a^\alpha |x - y|^{-d-\alpha} u(y) dy \\ &\leq 2^{d+\alpha} \mathcal{A}(d, \alpha) \mathbb{E}_x[\tau_{B(x, 2\rho_Q(x))}] \int_{|y-x| > r^{1-\gamma} \rho_Q(x)^\gamma} a^\alpha |x - y|^{-d-\alpha} u(y) dy \\ &\leq 2^{d+\alpha} \mathcal{A}(d, \alpha) c_1 \rho_Q(x)^2 \left( \int_{|y-x| > r^{1-\gamma} \rho_Q(x)^\gamma, |y-x_0| > 2\rho_Q(x_0)/3} a^\alpha |x - y|^{-d-\alpha} u(y) dy \right. \\ &\quad \left. + \int_{|y-x_0| \leq 2\rho_Q(x_0)/3} a^\alpha |x - y|^{-d-\alpha} u(y) dy \right) \\ &=: c_2 \rho_Q(x)^2 (I_1 + I_2). \end{aligned}$$

On the other hand, for  $z \in B_2$  and  $y \notin B_3$ , we have  $|z - y| \leq |z - x_0| + |x_0 - y| \leq \rho_Q(x_0)/3 + |x_0 - y| \leq 2|x_0 - y|$ . we have again by (1.3) and Lemma 3.1

$$(4.14) \quad \begin{aligned} u(x_0) &\geq \mathbb{E}_{x_0}[u(X_{\tau_2}^a), X_{\tau_2}^a \notin B_3] \\ &\geq \mathcal{A}(d, \alpha) \int_{B_2} G_{B_2}^a(x_0, z) \int_{|y-x_0| > 2\rho_Q(x_0)/3} a^\alpha |z - y|^{-d-\alpha} u(y) dy dz \\ &\geq 2^{-d-\alpha} \mathcal{A}(d, \alpha) \int_{B_2} G_{B_2}^a(x_0, z) dz \int_{|y-x_0| > 2\rho_Q(x_0)/3} a^\alpha |x_0 - y|^{-d-\alpha} u(y) dy \\ &\geq 2^{-d-\alpha} \mathcal{A}(d, \alpha) c_3 (\rho_Q(x_0)/3)^2 \int_{|y-x_0| > 2\rho_Q(x_0)/3} a^\alpha |x_0 - y|^{-d-\alpha} u(y) dy \\ &= c_4 \rho_Q(x_0)^2 \int_{|y-x_0| > 2\rho_Q(x_0)/3} a^\alpha |x_0 - y|^{-d-\alpha} u(y) dy. \end{aligned}$$

Suppose now that  $|y - x| \geq r^{1-\gamma} \rho_Q(x)^\gamma$  and  $x \in B(Q, r/4)$ . Then

$$|y - x_0| \leq |y - x| + r \leq |y - x| + r^\gamma \rho_Q(x)^{-\gamma} |y - x| \leq 2r^\gamma \rho_Q(x)^{-\gamma} |y - x|.$$

Therefore

$$\begin{aligned}
(4.15) \quad I_1 &= \int_{|y-x| > r^{1-\gamma} \rho_Q(x)^\gamma, |y-x_0| > 2\rho_Q(x_0)/3} a^\alpha |x - y|^{-d-\alpha} u(y) dy \\
&\leq \int_{|y-x_0| > 2\rho_Q(x_0)/3} (2^{-1}(\rho_Q(x)/r)^\gamma)^{-d-\alpha} a^\alpha |y - x_0|^{-d-\alpha} u(y) dy \\
&= 2^{d+\alpha} (\rho_Q(x)/r)^{-\gamma(d+\alpha)} \int_{|y-x_0| > 2\rho_Q(x_0)/3} a^\alpha |y - x_0|^{-d-\alpha} u(y) dy \\
&\leq 2^{d+\alpha} (\rho_Q(x)/r)^{-\gamma(d+\alpha)} c_4^{-1} \rho_Q(x_0)^{-2} u(x_0) \\
&= c_5 (\rho_Q(x)/r)^{-\gamma(d+\alpha)} \rho_Q(x_0)^{-2} u(x_0),
\end{aligned}$$

where the last inequality is due to (4.14).

If  $|y - x_0| < 2\rho_Q(x_0)/3$ , then  $|y - x| \geq |x_0 - Q| - |x - Q| - |y - x_0| > \rho_Q(x_0)/6$ . This together with the uniform Harnack principle in Proposition 1.3 implies that

$$\begin{aligned}
(4.16) \quad I_2 &= \int_{|y-x_0| \leq 2\rho_Q(x_0)/3} a^\alpha |x - y|^{-d-\alpha} u(y) dy \\
&\leq c_6 \int_{|y-x_0| \leq 2\rho_Q(x_0)/3} a^\alpha |x - y|^{-d-\alpha} u(x_0) dy \\
&\leq c_6 u(x_0) \int_{|y-x| > \rho_Q(x_0)/6} a^\alpha |x - y|^{-d-\alpha} dy = c_7 a^\alpha \rho_Q(x_0)^{-\alpha} u(x_0).
\end{aligned}$$

Combining (4.13)-(4.16) we obtain

$$\begin{aligned}
(4.17) \quad &\mathbb{E}_x [u(X_{\tau_0}^a); X_{\tau_0}^a \notin B_1] \\
&\leq c_2 \rho_Q(x)^2 \left( c_5 (\rho_Q(x)/r)^{-\gamma(d+\alpha)} \rho_Q(x_0)^{-2} u(x_0) + c_7 a^\alpha \rho_Q(x_0)^{-\alpha} u(x_0) \right) \\
&\leq c_8 u(x_0) \left( \rho_Q(x)^2 (\rho_Q(x)/r)^{-\gamma(d+\alpha)} \rho_Q(x_0)^{-2} + a^\alpha \rho_Q(x)^2 \rho_Q(x_0)^{-\alpha} \right) \\
&\leq c_9 u(x_0) \left( (\rho_Q(x)/r)^{2-\gamma(d+\alpha)} + a^\alpha \rho_Q(x)^2 r^{-\alpha} \right),
\end{aligned}$$

where in the last inequality we used the fact that  $\rho_Q(x_0) = r/2$ . Choose now  $\eta \in (0, \eta_0)$  so that

$$c_9 \left( \eta^{2-\gamma(d+\alpha)} + \eta^2 M^\alpha \right) \leq 1.$$

Then for  $x \in D \cap B(Q, r/12)$  with  $\rho_Q(x) < \eta r$ , we have by (4.17)

$$\begin{aligned}
\mathbb{E}_x [u(X_{\tau_0}^a); X_{\tau_0}^a \notin B_1] &\leq c_9 u(x_0) \left( \eta^{2-\gamma(d+\alpha)} + \eta^2 r^{2-\alpha} M^\alpha \right) \\
&\leq c_9 \left( \eta^{2-\gamma(d+\alpha)} + \eta^2 M^\alpha \right) u(x_0) \leq u(x_0).
\end{aligned}$$

We now prove the Carleson estimate (4.8) for  $x \in D \cap B(Q, r/12)$  by a method of contradiction. Recall that  $u(x_0) = 1$ . Suppose that there exists  $x_1 \in D \cap B(x, r/12)$  such that  $u(x_1) \geq K > \eta^{-\beta} \vee (1 + \delta^{-1})$ , where  $K$  is a constant to be specified later. By (4.10) and the assumption

$u(x_1) \geq K > \eta^{-\beta}$ , we have  $(\rho_Q(x_1)/r)^{-\beta} > u(x_1) \geq K > \eta^{-\beta}$ , and hence  $\rho_Q(x_1) < \eta r$ . Let  $B_0$ ,  $B_1$  and  $\tau_0$  be now defined with respect to the point  $x_1$  instead of  $x$ . Then by (4.11), (4.12) and  $K > 1 + \delta^{-1}$ ,

$$K \leq u(x_1) \leq \mathbb{E}_{x_1} [u(X_{\tau_0}^a); X_{\tau_0}^a \in B_1] + 1,$$

and hence

$$\mathbb{E}_{x_1} [u(X_{\tau_0}^a); X_{\tau_0}^a \in B_1] \geq u(x_1) - 1 > \frac{1}{1 + \delta} u(x_1).$$

In the last inequality of the display above we used the assumption that  $u(x_1) \geq K > 1 + \delta^{-1}$ . If  $K \geq 2^{\beta/\gamma}$ , then  $D^c \cap B_1 \subset D^c \cap B(Q, r)$ . By using the assumption that  $u = 0$  on  $D^c \cap B(Q, r)$ , we get from (4.9)

$$\mathbb{E}_{x_1} [u(X_{\tau_0}^a), X_{\tau_0}^a \in B_1] = \mathbb{E}_{x_1} [u(X_{\tau_0}^a), X_{\tau_0}^a \in B_1 \cap D] \leq \mathbb{P}_x(X_{\tau_0}^a \in D) \sup_{B_1} u \leq (1 - \delta) \sup_{B_1} u.$$

Therefore,  $\sup_{B_1} u > u(x_1)/((1 + \delta)(1 - \delta))$ , i.e., there exists a point  $x_2 \in D$  such that

$$|x_1 - x_2| \leq r^{1-\gamma} \rho_Q(x_1)^\gamma \quad \text{and} \quad u(x_2) > \frac{1}{1 - \delta^2} u(x_1) \geq \frac{1}{1 - \delta^2} K.$$

By induction, if  $x_k \in D \cap B(Q, r/12)$  with  $u(x_k) \geq K/(1 - \delta^2)^{k-1}$  for  $k \geq 2$ , then there exists  $x_{k+1} \in D$  such that

$$(4.18) \quad |x_k - x_{k+1}| \leq r^{1-\gamma} \rho_Q(x_k)^\gamma \quad \text{and} \quad u(x_{k+1}) > \frac{1}{1 - \delta^2} u(x_k) > \frac{1}{(1 - \delta^2)^k} K.$$

From (4.10) and (4.18) it follows that  $\rho_Q(x_k)/r \leq (1 - \delta^2)^{(k-1)/\beta} K^{-1/\beta}$ , for every  $k \geq 1$ . Therefore,

$$\begin{aligned} |x_k - Q| &\leq |x_1 - Q| + \sum_{j=1}^{k-1} |x_{j+1} - x_j| \leq \frac{r}{12} + \sum_{j=1}^{\infty} r^{1-\gamma} \rho_Q(x_j)^\gamma \\ &\leq \frac{r}{12} + r^{1-\gamma} \sum_{j=1}^{\infty} (1 - \delta^2)^{(j-1)\gamma/\beta} K^{-\gamma/\beta} r^\gamma \\ &= \frac{r}{12} + r^{1-\gamma} r^\gamma K^{-\gamma/\beta} \sum_{j=0}^{\infty} (1 - \delta^2)^{j\gamma/\beta} \\ &= \frac{r}{12} + r K^{-\gamma/\beta} \frac{1}{1 - (1 - \delta^2)^{\gamma/\beta}}. \end{aligned}$$

Choose  $K = \eta \vee (1 + \delta^{-1}) \vee 12^{\beta/\gamma} (1 - (1 - \delta^2)^{\gamma/\beta})^{-\beta/\gamma}$ . Then  $K^{-\gamma/\beta} (1 - (1 - \delta^2)^{\gamma/\beta})^{-1} \leq 1/12$ , and hence  $x_k \in D \cap B(Q, r/6)$  for every  $k \geq 1$ . Since  $\lim_{k \rightarrow \infty} u(x_k) = +\infty$ , this contradicts the fact that  $u$  is bounded on  $B(Q, r/2)$ . This contradiction shows that  $u(x) < K$  for every  $x \in D \cap B(Q, r/12)$ . This completes the proof of the theorem.  $\square$

**Proof of Theorem 1.4 .** Since  $D$  is a  $C^{1,1}$  open set and  $r < R$ , by the uniform Harnack principle in Proposition 1.3 and a standard chain argument, it suffices to prove (1.5) for  $x, y \in D \cap B(Q, r/8)$ . In this proof, constants  $\eta$  and  $c_i$ 's are always independent of  $r$  and  $a$ .

We recall that  $r_0 = \frac{R}{4\sqrt{1+\Lambda^2}}$  and  $\delta_0 \in (0, r_0)$  is the constant in the statement of Lemma 3.5.

For any  $r \in (0, R]$  and  $x \in D \cap B(Q, r/8)$ , let  $Q_x$  be the point  $Q_x \in \partial D$  so that  $|x - Q_x| = \delta_D(x)$  and let  $x_0 := Q_x + \frac{r}{8}(x - Q_x)/|x - Q_x|$ . We choose a  $C^{1,1}$ -function  $\phi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  satisfying

$\phi(0) = \nabla\phi(0) = 0$ ,  $\|\nabla\phi\|_\infty \leq \Lambda$ ,  $|\nabla\phi(y) - \nabla\phi(z)| \leq \Lambda|y - z|$ , and an orthonormal coordinate system  $CS$  with its origin at  $Q_x$  such that

$$B(Q_x, R) \cap D = \{y = (\tilde{y}, y_d) \in B(0, R) \text{ in } CS : y_d > \phi(\tilde{y})\}.$$

In the coordinate system  $CS$  we have  $\tilde{x} = \tilde{0}$  and  $x_0 = (\tilde{0}, r/8)$ . For any  $b_1, b_2 > 0$ , we define

$$D(b_1, b_2) := \{y = (\tilde{y}, y_d) \text{ in } CS : 0 < y_d - \phi(\tilde{y}) < b_1 r \delta_0 / 8, |\tilde{y}| < b_2 r r_0 / 8\}.$$

It is easy to see that  $D(2, 2) \subset D \cap B(Q, r/2)$ . In fact, since  $r_0 \leq \frac{1}{8\Lambda}$  and  $\delta_0 \leq \frac{1}{8\Lambda}$ , for every  $y \in D(2, 2)$

$$|y - Q| \leq |x - Q| + |x - Q_x| + |Q_x - y| \leq \frac{r}{4} + |\tilde{y}| + |y_d - \phi(\tilde{y})| + |\phi(\tilde{y})| < \frac{r}{4}(1 + r_0(\Lambda + 1) + \delta_0) \leq \frac{r}{2}.$$

Thus if  $u$  is a nonnegative function on  $\mathbb{R}^d$  that is harmonic in  $D \cap B(Q, r)$  with respect to  $X^a$  and vanishes continuously in  $D^c \cap B(Q, r)$ , then, by Lemma 4.2,  $u$  is regular harmonic in  $D \cap B(Q, r/2)$  with respect to  $X^a$ , hence also in  $D(2, 2)$ . Thus by the uniform Harnack principle in Proposition 1.3, we have

$$(4.19) \quad \begin{aligned} u(x) &= \mathbb{E}_x \left[ u(X_{\tau_{D(1,1)}^a}^a) \right] \geq \mathbb{E}_x \left[ u(X_{\tau_{D(1,1)}^a}^a); X_{\tau_{D(1,1)}^a}^a \in D(2, 1) \right] \\ &\geq c_1 u(x_0) \mathbb{P}_x \left( X_{\tau_{D(1,1)}^a}^a \in D(2, 1) \right) \geq c_2 u(x_0) \delta_D(x) / r. \end{aligned}$$

In the last inequality above we have used (3.42).

Let  $x_1 = (\tilde{0}, r r_0 / 16)$ . Then it is easy to see that there exists a constant  $\eta = \eta(\Lambda, r_0, \delta_0) \in (0, 1)$  such that  $B(x_1, \eta r r_0 / 16) \in D(1, 1)$ . By (1.3) and Lemma 3.1,

$$\begin{aligned} u(x_1) &\geq \mathbb{E}_{x_1} \left[ u(X_{\tau_{D(1,1)}^a}^a); X_{\tau_{D(1,1)}^a}^a \notin D(2, 2) \right] \\ &= \mathcal{A}(d, \alpha) a^\alpha \int_{D(1,1)} G_{D(1,1)}^a(x_1, z) \int_{\mathbb{R}^d \setminus D(2,2)} \frac{u(y)}{|z - y|^{d+\alpha}} dy dz \\ &\geq c_3 a^\alpha \mathbb{E}_{x_1} [\tau_{B(x_1, \eta r r_0 / (16))}^a] \int_{\mathbb{R}^d \setminus D(2,2)} \frac{u(y)}{|x_1 - y|^{d+\alpha}} dy \\ &\geq c_4 a^\alpha r^2 \int_{\mathbb{R}^d \setminus D(2,2)} \frac{u(y)}{|x_1 - y|^{d+\alpha}} dy. \end{aligned}$$

Hence by (3.44),

$$\begin{aligned} &\mathbb{E}_x \left[ u \left( X_{\tau_{D(1,1)}^a}^a \right); X_{\tau_{D(1,1)}^a}^a \notin D(2, 2) \right] \\ &= \mathcal{A}(d, \alpha) a^\alpha \int_{D(1,1)} G_{D(1,1)}^a(x, z) \int_{\mathbb{R}^d \setminus D(2,2)} \frac{u(y)}{|z - y|^{d+\alpha}} dy dz \\ &\leq c_5 a^\alpha \mathbb{E}_x [\tau_{D(1,1)}^a] \int_{\mathbb{R}^d \setminus D(2,2)} \frac{u(y)}{|x_1 - y|^{d+\alpha}} dy \\ &\leq c_6 a^\alpha \delta_D(x) r \int_{\mathbb{R}^d \setminus D(2,2)} \frac{u(y)}{|x_1 - y|^{d+\alpha}} dy \leq \frac{c_6 \delta_D(x)}{c_4 r} u(x_1). \end{aligned}$$

On the other hand, by the uniform Harnack principle (Proposition 1.3) and the Carleson estimate (Theorem 4.3), we have

$$\mathbb{E}_x \left[ u \left( X_{\tau_{D(1,1)}^a}^a \right); X_{\tau_{D(1,1)}^a}^a \in D(2, 2) \right] \leq c_7 u(x_0) \mathbb{P}_x \left( X_{\tau_{D(1,1)}^a}^a \in D(2, 2) \right) \leq c_8 u(x_0) \delta_D(x) / r.$$

In the last inequality above we have used (3.43). Combining the two inequalities above, we get

$$\begin{aligned}
(4.20) \quad u(x) &= \mathbb{E}_x \left[ u \left( X_{\tau_{D(1,1)}^a}^a \right); X_{\tau_{D(1,1)}^a}^a \in D(2, 2) \right] \\
&\quad + \mathbb{E}_x \left[ u \left( X_{\tau_{D(1,1)}^a}^a \right); X_{\tau_{D(1,1)}^a}^a \notin D(2, 2) \right] \\
&\leq \frac{c_8}{r} \delta_D(x) u(x_0) + \frac{c_6 \delta_D(x)}{c_4 r} u(x_1) \\
&\leq \frac{c_9}{r} \delta_D(x) (u(x_0) + u(x_1)) \\
&\leq \frac{c_{10}}{r} \delta_D(x) u(x_0).
\end{aligned}$$

In the last inequality above we have used the uniform Harnack principle (Proposition 1.3).

From (4.19)-(4.21), we have that for every  $x, y \in D \cap B(Q, r/8)$ ,

$$\frac{u(x)}{u(y)} \leq \frac{c_{10}}{c_2} \frac{\delta_D(x)}{\delta_D(y)},$$

which proves the theorem. □

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