Potential theory of subordinate Brownian motions with Gaussian components

Panki Kim* **Renming Song** and **Zoran Vondraček†**

Abstract

In this paper we study a subordinate Brownian motion with a Gaussian component and a rather general discontinuous part. The assumption on the subordinator is that its Laplace exponent is a complete Bernstein function with a Lévy density satisfying a certain growth condition near zero. The main result is a boundary Harnack principle with explicit boundary decay rate for non-negative harmonic functions of the process in $C^{1,1}$ open sets. As a consequence of the boundary Harnack principle, we establish sharp two-sided estimates on the Green function of the subordinate Brownian motion in any bounded $C^{1,1}$ open set D and identify the Martin boundary of D with respect to the subordinate Brownian motion with the Euclidean boundary.

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1 Introduction

The infinitesimal generator of a d-dimensional rotationally invariant Lévy process is a non-local operator of the form $\mathcal{L} = b\Delta + \mathcal{A}$ where $b \geq 0$ and

$$\mathcal{A}f(x) = \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \nabla f(x) \cdot y \mathbf{1}_{\{|y|\leq 1\}}\right) \nu(dy) = \lim_{\epsilon \to 0} \int_{\{|y|>\epsilon\}} (f(x+y) - f(x)) \, \nu(dy) \, .$$

The measure ν on $\mathbb{R}^d \setminus \{0\}$ is invariant under rotations around origin and satisfies $\int_{\mathbb{R}^d} (1 \wedge |y|^2) \, \nu(dy) < \infty$. When $\nu = 0$, the operator $\mathcal{L}$ is proportional to the Laplacian, hence a local operator, while when $b = 0$, the operator $\mathcal{L}$ is a purely non-local integro-differential operator. In particular, if $b = 0$ and $\nu(dx) = c|x|^{-d-\alpha}dx$, $\alpha \in (0, 2)$, then $\mathcal{A}$ becomes the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. Lévy processes are of intrinsic importance in probability theory, while integro-differential operator are important in the theory of partial differential equations. Most of the research in the last fifteen years concentrates on purely discontinuous Lévy processes, such as rotationally invariant stable

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processes, or equivalently, on purely non-local operators of the type $\mathcal{A}$. For summary of some recent results from a probabilistic point of view one can consult [4, 8, 21, 22] and references therein. We refer the readers to [5, 6, 7] for a sample of recent progress in the PDE literature, mostly for the case of a fractional Laplacian $\Delta^{\alpha/2}$, $\alpha \in (0, 2)$.

In many situations one would like to study operators that have both local and non-local parts. From a probabilistic point of view, this corresponds to processes with both a Gaussian component and a jump component. The fact that such a process X has both Gaussian and jump components is the source of many difficulties in investigating the potential theory of X. The main difficulty in studying X stems from the fact that it runs on two different scales: on the small scale the diffusion corresponding to the Gaussian part dominates, while on the large scale the jumps take over. Another difficulty is encountered when looking at the exit of X from an open set: for diffusions, the exit is through the boundary, while for the pure jump processes, typically the exit happens by jumping out from the open set. For the process X, both cases will occur which makes the process X much more difficult to study.

Despite the difficulties mentioned above, in the last few years significant progress has been made in understanding the potential theory of such processes. Green function estimates (for the whole space) and the Harnack inequality for a class of processes with both diffusion and jump components were established in [23, 27]. The parabolic Harnack inequality and heat kernel estimates were studied in [28] for Lévy processes in $\mathbb{R}^d$ that are independent sums of Brownian motions and symmetric stable processes, and in [14] for much more general symmetric diffusions with jumps. Moreover, an a priori Hölder estimate was established in [14] for bounded parabolic functions. For earlier results on second order integro-differential operators, one can see [15] and the references therein.

Important progress has been made in two recent papers [10, 11] which consider operators of the type $\Delta + a^\alpha \Delta^{\alpha/2}$ for $a \in [0, M]$. The process corresponding to such an operator is an independent sum of a Brownian motion and an α-stable rotationally invariant process with weight a. In [10] the authors established a (uniform in a) boundary Harnack principle (BHP) with explicit boundary decay rate for non-negative harmonic functions with respect to $\Delta + a^\alpha \Delta^{\alpha/2}$ in $C^{1,1}$ open sets. By using the BHP, the second paper [11] established sharp Green function estimates in bounded $C^{1,1}$ open sets D, and identified the Martin boundary of D for the operator $\Delta + a^\alpha \Delta^{\alpha/2}$ with its Euclidean boundary.

The purpose of the current paper is to extend the results in [10, 11] to more general operators than $\Delta + a^\alpha \Delta^{\alpha/2}$. Analytically, the operators that we consider are certain functions of the Laplacian. To be more precise, we consider a Bernstein function $\phi : (0, \infty) \to (0, \infty)$ with $\phi(0+) = 0$, i.e., ϕ is of the form

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)\,, \quad \lambda > 0\,, \tag{1.1}$$

where $b \geq 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$. μ is called the Lévy measure of ϕ. By Bochner's functional calculus one can define the operator $\phi(\Delta) := -\phi(-\Delta)$ which on $C_b^2(\mathbb{R}^d)$, the collection of bounded C^2 functions in $\mathbb{R}^d$ with bounded derivatives, turns out to be an integro-differential operator of the type

$$b\Delta f(x) + \int_{\mathbb{R}^d} \left(f(x + y) - f(x) - \nabla f(x) \cdot y \mathbf{1}_{\{|y| \leq 1\}} \right) \nu(dy)\,,$$

where the measure ν has the form $\nu(dy) = j(|y|)\,dy$ with $j:(0,\infty)\to(0,\infty)$ given by

$$j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)}\,\mu(dt)\,.$$

In order for the operator to have both local and non-local parts we will assume that $b>0$ and $\mu\neq 0$. In fact, without loss of generality, throughout the paper we always suppose that $b=1$. Note that by taking $\phi(\lambda)=\lambda+a^\alpha\lambda^{\alpha/2}$ we are back to the operator $\Delta+a^\alpha\Delta^{\alpha/2}$.

The operator $\phi(\Delta)$ is the infinitesimal generator of the Lévy process X that can be constructed as follows: Recall that a one-dimensional Lévy process $S=(S_t:t\geq 0)$ is called a subordinator if $S_0=0$ and if $t\to S_t(\omega)$ is non-negative. A subordinator S can be characterized by its Laplace exponent ϕ through the equality

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \quad t>0, \lambda>0\,.$$

A function ϕ is the Laplace exponent of a subordinator if and only if it is a Bernstein function satisfying $\phi(0+)=0$. We will assume that ϕ is given by (1.1) with $b=1$. Suppose that $W=(W_t:t\geq 0)$ is a d-dimensional Brownian motion and $S=(S_t:t\geq 0)$ is a subordinator, independent of W, with Laplace exponent ϕ. The process $X=(X_t:t\geq 0)$ defined by $X_t=W_{S_t}$ is called a subordinate Brownian motion and its infinitesimal generator is $\phi(\Delta)$. It is a sum of a Brownian motion and an independent purely discontinuous (rotationally invariant) Lévy process.

Potential theory of one-dimensional subordinate Brownian motions in this setting was studied in [20]. In the current paper we look at the case when $d\geq 2$. In order for our methods to work we need additional assumptions on the Bernstein function ϕ. We will assume that ϕ is a complete Bernstein function, namely that the Lévy measure μ has a completely monotone density. By a slight abuse of notation we will denote the density by $\mu(t)$. For the Lévy density μ we assume a growth condition near zero: For any $K>0$, there exists $c=c(K)>1$ such that

$$\mu(r)\leq c\,\mu(2r), \qquad \forall r\in(0,K)\,. \tag{1.2}$$

We will later explain the role of these additional assumptions.

To state our main result, we first recall that an open set D in $\mathbb{R}^d$ (when $d\geq 2$) is said to be $C^{1,1}$ if there exist a localization radius $R>0$ and a constant $\Lambda>0$ such that for every $Q\in\partial D$, there exist a $C^{1,1}$-function $\varphi=\varphi_Q:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying $\varphi(0)=0$, $\nabla\varphi(0)=(0,\dots 0)$, $\|\nabla\varphi\|_\infty\leq\Lambda$, $|\nabla\varphi(x)-\nabla\varphi(y)|\leq\Lambda|x-y|$, and an orthonormal coordinate system CS_Q: $y=(y_1,\cdots,y_{d-1},y_d)=:(\widetilde{y},\,y_d)$ with its origin at Q such that

$$B(Q,R)\cap D=\{y=(\widetilde{y},y_d)\in B(0,R) \text{ in } CS_Q: y_d>\varphi(\widetilde{y})\}.$$

The pair (R,Λ) is called the characteristics of the $C^{1,1}$ open set D. Note that a $C^{1,1}$ open set can be unbounded and disconnected.

For any $x\in D$, $\delta_D(x)$ denotes the Euclidean distance between x and D^c. For any $x\notin D$, $\delta_{\partial D}(x)$ denotes the Euclidean distance between x and ∂D. It is well known that, if D is a $C^{1,1}$ open set D with characteristics (R,Λ), there exists $\widetilde{R}\leq R$ such that D satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition* with radius $\widetilde{R}$: for every $x\in D$ with $\delta_D(x)<\widetilde{R}$ and $y\in\mathbb{R}^d\setminus\overline{D}$ with $\delta_{\partial D}(y)<\widetilde{R}$, there are $z_x,z_y\in\partial D$ so that $|x-z_x|=\delta_D(x)$, $|y-z_y|=\delta_{\partial D}(y)$ and that $B(x_0,\widetilde{R})\subset D$ and $B(y_0,\widetilde{R})\subset\mathbb{R}^d\setminus\overline{D}$ where $x_0=z_x+\widetilde{R}(x-z_x)/|x-z_x|$

and $y_0 = z_y + \widetilde{R}(y - z_y)/|y - z_y|$. Without loss of generality, throughout this paper, we assume that the characteristics (R, Λ) of a $C^{1,1}$ open set satisfies $R = \widetilde{R} \le 1$ and $\Lambda \ge 1$.

For any open set $D \subset \mathbb{R}^d$, $\tau_D := \inf\{t > 0 : X_t \notin D\}$ denotes the first exit time from D by X.

Definition 1.1 A real-valued function f defined on $\mathbb{R}^d$ is said to be harmonic in $D \subset \mathbb{R}^d$ with respect to X if for every open set B whose closure is a compact subset of D,

$$\mathbb{E}_x \left[\left| f(X_{\tau_B}) \right| \right] < \infty \quad \text{and} \quad f(x) = \mathbb{E}_x \left[f(X_{\tau_B}) \right] \qquad \text{for every } x \in B. \tag{1.3}$$

It follows from [14, Theorem 1.2] that all harmonic functions in D with respect to X are continuous on D, since every harmonic function in D with respect to X can be approximated locally uniformly in D by functions that are bounded on $\mathbb{R}^d$ and harmonic with respect to X in relatively compact open subsets of D. We will use $C_c^\infty(\mathbb{R}^d)$ to denote the space of infinitely differentiable functions with compact support and $W^{1,2}(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d; dx) : \partial_i u \in L^2(\mathbb{R}^d; dx),\ \forall 1 \le i \le d\}$. A function u is said to be in $W^{1,2}_{\text{loc}}(D)$ if for every relatively compact subset B with $\overline{B} \subset D$, there is a function $f \in W^{1,2}(\mathbb{R}^d)$ such that $u = f$ a.e. on B. Using (2.5)–(2.6) which are consequences of (1.2), and by the same argument used to derive to [10, Proposition 1.2], we can show that u is harmonic in D with respect to X, if and only if u is locally bounded in D, $\int_{\mathbb{R}^d} |u(y)|(1 \wedge j(|y|))dy < \infty$, $u \in W^{1,2}_{\text{loc}}(D)$, and for every $\phi \in C_c^\infty(D)$,

$$\int_{\mathbb{R}^d} \nabla u(x) \cdot \nabla \phi(x) \, dx + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))(\phi(x) - \phi(y)) j(|x - y|) \, dx dy = 0.$$

Let $Q \in \partial D$. We will say that a function $f : \mathbb{R}^d \to \mathbb{R}$ vanishes continuously on $D^c \cap B(Q, r)$ if $f = 0$ on $D^c \cap B(Q, r)$ and f is continuous at every point of $\partial D \cap B(Q, r)$. The following is the main result of this paper.

Theorem 1.2 *Suppose that the Laplace exponent ϕ of the subordinator S, independent of the Brownian motion W, is a complete Bernstein function and that the Lévy density of S satisfies* (1.2)*. Let $X = (X_t : t \ge 0)$ be the subordinate Brownian motion defined by $X_t = W(S_t)$. For any $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (R, Λ), there exists a positive constant $C = C(d, \Lambda, R)$ such that for $r \in (0, R]$, $Q \in \partial D$ and any nonnegative function f in $\mathbb{R}^d$ which is harmonic in $D \cap B(Q, r)$ with respect to X and vanishes continuously on $D^c \cap B(Q, r)$, we have*

$$\frac{f(x)}{\delta_D(x)} \le C \, \frac{f(y)}{\delta_D(y)} \qquad \textit{for every } x, y \in D \cap B(Q, r/2). \tag{1.4}$$

We note that (1.4) is a strengthened version of the usual boundary Harnack principle stated for the ratio of two non-negative functions, f and g, harmonic in $D \cap B(Q, r)$ with respect to X, and which says that

$$\frac{f(x)}{g(x)} \le C \, \frac{f(y)}{g(y)} \qquad \text{for every } x, y \in D \cap B(Q, r/2).$$

The above inequality is clearly a consequence of (1.4) which gives the precise boundary decay of non-negative harmonic functions. We note that the function $x \mapsto \delta_D(x)$ is not harmonic in $D \cap B(Q, r)$ with respect to X.

Remark 1.3 The same type of boundary Harnack principle in $C^{1,1}$ domains is valid also for Brownian motions, namely the boundary decay rate is of the order $\delta_D(x)$. Since on the small scale the diffusion part of X dominates, one would expect that harmonic functions of X and of Brownian motion have the same decay rate at the boundary. For this reason, some people might expect that some kind of perturbation methods can be used to prove the BHP for X. We note that it is unlikely that any perturbation method would work because of the following: Suppose that instead of X we consider a process X^a with the infinitesimal generator

$$\mathcal{L}^a f(x) = \Delta f(x) + \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \nabla f(x) \cdot y \mathbf{1}_{\{|y| \le 1\}} \right) \nu^a(dy) \,,$$

where $\nu^a(dy) = \mathbf{1}_{\{|y| \le a\}} \, \nu(dy)$ with $0 < a < \infty$. Thus X^a is the process X with jumps of size larger than a suppressed. In Section 6 we present an example of a (bounded) $C^{1,1}$ domain D on which the boundary Harnack principle for X^a fails, even for regular harmonic functions vanishing on D^c. Note that if we think of X as a perturbation of Brownian motion, then X^a is an even smaller perturbation of the same Brownian motion. The counterexample in Section 6 shows that, despite the (seemingly) local nature of the BHP, one needs some information of the structure of large jumps of X.

We will use X^D to denote the process defined by $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \ge \tau_D(\omega)$, where ∂ is a cemetery point. The Green function of X^D will be denoted by $G_D(x, y)$. For the precise definition of G_D, see Section 2.

To state our result on Green function estimates, we introduce a function g_D first. For $d \ge 2$, we define for $x, y \in D$,

$$g_D(x,y) = \begin{cases} \frac{1}{|x-y|^{d-2}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \right) & \text{when } d \ge 3, \\ \log \left(1 + \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \right) & \text{when } d = 2. \end{cases}$$

Theorem 1.4 *Suppose that the Laplace exponent ϕ of S is a complete Bernstein function and that the Lévy density of S satisfies* (1.2). *For any bounded $C^{1,1}$ open set $D \subset \mathbb{R}^d$, there exists $C = C(D) > 1$ such that for all $x, y \in D$*

$$C^{-1} \, g_D(x,y) \le G_D(x,y) \le C \, g_D(x,y). \tag{1.5}$$

Finally, we state the result about the Martin boundary of a bounded $C^{1,1}$ open set D with respect to X^D. Fix $x_0 \in D$ and define

$$M_D(x,y) := \frac{G_D(x,y)}{G_D(x_0,y)}, \qquad x, y \in D, \ y \ne x_0.$$

A function f is called a harmonic function for X^D if it is harmonic for X in D and vanishes outside D. A positive harmonic function f for X^D is minimal if, whenever g is a positive harmonic function for X^D with $g \le f$ on D, one must have $f = cg$ for some constant c.

Theorem 1.5 *Suppose that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. For every $z \in \partial D$, there exists $M_D(x,z) := \lim_{y \to z} M_D(x,y)$. Further, for every $z \in \partial D$, $M_D(\cdot, z)$ is a minimal harmonic function for X^D and $M_D(\cdot, z_1) \ne M_D(\cdot, z_1)$ if $z_1 \ne z_2$. Thus the minimal Martin boundary of D can be identified with the Euclidean boundary.*

Let us now describe the main ingredients of the proof of Theorem 1.2, the boundary Harnack principle. We follow the general strategy for proving the boundary Harnack principle in different settings which requires the Carleson estimate, and upper and lower estimates on exit probabilities and exit times from certain sets usually called "boxes". In Theorem 5.4 we prove the Carleson estimate for a Lipschitz open set by modifying the proof in [10]. In order to obtain the upper exit probability and exit times estimates, we follow the approach from [10], the so-called "test function" method (which was modeled after some earlier ideas, see [3, 16]), but have to make major modifications. In [10], the test functions are power functions of the form $x \mapsto (x_d)^p$ which are either superharmonic or subharmonic for the corresponding process, and the values of the generator on these test functions is computed in detail. In our setting, the power functions are neither superharmonic nor subharmonic, and explicit calculations cannot be carried out because of the lack of explicit form of the Lévy measure. Instead we use the approach developed in [22] for the case of certain pure-jump subordinate Brownian motions, which seems to be quite versatile to cover various other cases.

The first ingredient in [22] comes from the fluctuation theory of one-dimensional Lévy processes. Its purpose is to identify a correct boundary decay rate by finding an appropriate harmonic function. Let $Z = (Z_t : t \geq 0)$ be the one-dimensional subordinate Brownian motion defined by $Z_t := W^d_{S_t}$, and let V be the renewal function of the ladder height process of Z. The function V is harmonic for the process Z killed upon exiting $(0, \infty)$, and the function $w(x) := V(x_d)\mathbf{1}_{\{x_d>0\}}$, $x \in \mathbb{R}^d$, is harmonic for the process X killed upon exiting the half space $\mathbb{R}^d_+ := \{x = (x_1, \ldots, x_{d-1}, x_d) \in \mathbb{R}^d : x_d > 0\}$ (Theorem 3.2). Therefore, w gives the correct rate of decay of harmonic functions near the boundary of $\mathbb{R}^d_+$. We will use the function w as our test function. Note that the assumption that ϕ is a complete Bernstein function implies that w is smooth. Using smoothness and harmonicity of w together with the characterization of harmonic functions recently established in [9], we show that $(\Delta+\mathcal{A})w \equiv 0$ on the half space (Theorem 3.4). Consequently we prove the following fact in Lemma 4.1, which is the key to proving upper estimates: If D is a $C^{1,1}$ open set with characteristics (R, Λ), $Q \in \partial D$ and $h(y) = V(\delta_D(y))\mathbf{1}_{D\cap B(Q,R)}$, then $(\Delta + \mathcal{A})h(y)$ is a.e. well defined and bounded for $y \in D$ close enough to the boundary point Q. Using this lemma, we give necessary exit distribution estimates in Lemma 4.3. Here we modify the test function h by adding a quadratic type function (in one variable) – this is necessary due to the presence of the Laplacian. The desired exit distribution estimates are directly derived by applying Dynkin's formula to the new test function. The reader will note that our proof is even shorter than the one in [10], partly because, in [10], the uniformity of the boundary Harnack principle for $\Delta + a^\alpha \Delta^{\alpha/2}$ in the weight $a \in (0, M]$ is established.

In order to prove the lower bound for the exit probabilities we compare the process X killed upon exiting a certain box $\widehat{D}$ with the so-called subordinate killed Brownian motion obtained by first killing Brownian motion upon exiting the box $\widehat{D}$, and then by subordinating the obtained process. If the latter process is denoted by $Y^{\widehat{D}}$, then its infinitesimal generator is equal to $-\phi(-\Delta|_{\widehat{D}})$. Here $\Delta|_{\widehat{D}}$ is the Dirichlet Laplacian and $-\phi(-\Delta|_{\widehat{D}})$ is constructed by Bochner's subordination. The advantage of this approach is that the exit probabilities of $X^{\widehat{D}}$ dominate from the above those of the process $Y^{\widehat{D}}$, while the latter can be rather easily computed, see [29]. This idea is carried out in Lemma 4.4 (as well as for some other lower bounds throughout the paper).

Once the boundary Harnack principle has been established, proofs of Theorems 1.4 and 1.5 are similar to the corresponding proofs in [11] for the operator $\Delta + a^\alpha \Delta^\alpha$. Therefore we do not give complete proofs of these two theorems in this paper, only indicate the necessary changes to the proofs in [11].

Organization of the paper: In the next section we precisely describe the settings and recall necessary preliminary results. Section 3 is devoted to the analysis of the process and harmonic functions in the half-space. Section 4 is on the analysis in $C^{1,1}$ open sets, and is central to the paper, and this is where most of the new ideas appear. In this rather technical section we establish the upper and lower bounds on the exit probabilities and exit times. In Section 5 we first prove the Carleson estimate for Lipschitz open sets and then prove the main Theorem 1.2. In Section 6 we provide the counterexample already mentioned in Remark 6. Finally, in Section 7 we explain the differences between the proofs of Theorems 1.4 and 1.5 and their counterparts from [11].

Throughout this paper, the constants C_1, C_2, R, R_1, R_2, R_3 will be fixed. The lowercase constants $c_1, c_2, \cdots$ will denote generic constants whose exact values are not important and can change from one appearance to another. The dependence of the lower case constants on the dimension d may not be mentioned explicitly. We will use ":=" to denote a definition, which is read as "is defined to be". For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For every function f, let $f^+ := f \vee 0$. We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$ and, for a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure.

2 Setting and Preliminary Results

A C^∞ function $\phi : (0, \infty) \to [0, \infty)$ is called a Bernstein function if $(-1)^n D^n \phi \le 0$ for every positive integer n. Every Bernstein function has a representation $\phi(\lambda) = a + b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt)$ where $a, b \ge 0$ and μ is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)} (1 \wedge t)\, \mu(dt) < \infty$; a is called the killing coefficient, b the drift and μ the Lévy measure of the Bernstein function. A Bernstein function ϕ is called a complete Bernstein function if the Lévy measure μ has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu(t) \ge 0$ for every non-negative integer n and all $t > 0$. Here and below, by abuse of notation we denote the Lévy density by $\mu(t)$. For more on Bernstein and complete Bernstein functions we refer the readers to [25].

A Bernstein function ϕ on $(0, \infty)$ is the Laplace exponent of a subordinator if and only if $\phi(0+) = 0$. Suppose that S is a subordinator with Laplace exponent ϕ. S is called a complete subordinator if ϕ is a complete Bernstein function. The potential measure U of S is defined by

$$U(A) = \mathbb{E} \int_0^\infty \mathbf{1}_{\{S_t \in A\}}\, dt, \quad A \subset [0, \infty). \tag{2.1}$$

Note that $U(A)$ is the expected time the subordinator S spends in the set A.

Throughout the remainder of this paper, we assume that $S = (S_t : \ t \ge 0)$ is a complete subordinator with a positive drift and, without loss of generality, we shall assume that the drift of S is equal to 1. Thus the Laplace exponent of S can be written as

$$\phi(\lambda) := \lambda + \psi(\lambda) \qquad \text{where} \quad \psi(\lambda) := \int_{(0,\infty)} (1 - e^{-\lambda t})\, \mu(dt).$$

We will exclude the trivial case of $S_t = t$, that is the case of $\psi \equiv 0$. Since the drift of S is equal to 1, the potential measure U of S has a completely monotone density u (cf. [4, Corollary 5.4 and Corollary 5.5]).

Suppose that $W = (W_t : t \geq 0)$ is a Brownian motion in $\mathbb{R}^d$ independent of S and with

$$\mathbb{E}_x[e^{i\theta\cdot(W_t - W_0)}] = e^{-t|\theta|^2}, \qquad \text{for all } x, \theta \in \mathbb{R}^d.$$

The process $X = (X_t : t \geq 0)$ defined by $X_t = W_{S_t}$ is called a subordinate Brownian motion. It follows from [4, Chapter 5] that X is a Lévy process with Lévy exponent $\phi(|\theta|^2) = |\theta|^2 + \psi(|\theta|^2)$:

$$\mathbb{E}_x[e^{i\theta\cdot(X_t - X_0)}] = e^{-t\phi(|\theta|^2)}, \qquad \text{for all } x, \theta \in \mathbb{R}^d.$$

The Lévy measure of the process X has a density J, called the Lévy density, given by $J(x) = j(|x|)$ where

$$j(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t)\, dt, \qquad r > 0. \tag{2.2}$$

Note that the function $r \mapsto j(r)$ is continuous and decreasing on $(0, \infty)$. We will sometimes use the notation $J(x, y)$ for $J(x - y)$.

The function $J(x, y)$ is the Lévy intensity of X. It determines a Lévy system for X, which describes the jumps of the process X: For any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, y, y) = 0$ for all $y \in \mathbb{R}^d$, any stopping time T (with respect to the filtration of X) and any $x \in \mathbb{R}^d$,

$$\mathbb{E}_x\left[\sum_{s \leq T} f(s, X_{s-}, X_s)\right] = \mathbb{E}_x\left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X_s, y) J(X_s, y) dy\right) ds\right]. \tag{2.3}$$

(See, for example, [12, Proof of Lemma 4.7] and [13, Appendix A].)

We will need some control on the behavior of j near the origin. For this, we will assume that for any $K > 0$, there exists $c = c(K) > 1$ such that

$$\mu(r) \leq c\, \mu(2r), \qquad \forall r \in (0, K). \tag{2.4}$$

On the other hand, since ϕ is a complete Bernstein function, it follows from [22, Lemma 2.1] that there exists $c > 1$ such that $\mu(t) \leq c\mu(t + 1)$ for every $t > 1$. Thus by repeating the proof of [23, Lemma 4.2] (see also [21, Proposition 1.3.5]), we can show that for any $K > 0$, there exists $c = c(K) > 1$ such that

$$j(r) \leq c\, j(2r), \qquad \forall r \in (0, K), \tag{2.5}$$

and, there exists $c > 1$ such that

$$j(r) \leq c\, j(r + 1), \qquad \forall r > 1. \tag{2.6}$$

Note that, as a consequence of (2.5), we have that, for any $K > 0$,

$$j(ar) \leq c\, a^{-\nu} j(r), \qquad \forall r \in (0, K) \quad \text{and} \quad a \in (0, 1) \tag{2.7}$$

where $c = c(K)$ is the constant in (2.5) and $\nu = \nu(K) := \log_2 c$.

Definition 2.1 *Let D be an open subset of $\mathbb{R}^d$. A function h defined on $\mathbb{R}^d$ is said to be*

(1) regular harmonic in D for X if it is harmonic in D with respect to X and for each $x \in D$, $h(x) = \mathbb{E}_x[h(X_{\tau_D})]$;

(2) invariant for X^D if for each $x \in D$ and each $t \ge 0$, $h(x) = \mathbb{E}_x\left[h(X_t^D)\right]$;

(3) harmonic for X^D if it is harmonic for X in D and vanishes outside D.

The following Harnack inequality will be used to prove the main result of this paper.

Proposition 2.2 (Harnack inequality) ([23, Theorem 4.5]) *There exists a constant $c > 0$ such that for any $r \in (0, 1]$ and $x_0 \in \mathbb{R}^d$ and any function f which is nonnegative in $\mathbb{R}^d$ and harmonic in $B(x_0, r)$ with respect to X we have*

$$f(x) \le cf(y) \qquad \text{for all } x, y \in B(x_0, r/2).$$

Note that, even though [23, Theorem 4.5] is stated for bounded harmonic function, using the same argument as in the proof of [23, Corollary 4.7], one can easily see that [23, Theorem 4.5] is, in fact, true without the boundedness assumption.

It follows from [4, Chapter 5] that the process X has a transition density $p(t, x, y)$, which is jointly continuous. Using this and the strong Markov property, one can easily check that

$$p_D(t, x, y) := p(t, x, y) - \mathbb{E}_x[t > \tau_D, p(t - \tau_D, X_{\tau_D}, y)], \quad x, y \in D$$

is continuous and is the transition density of X^D. For any bounded open set $D \subset \mathbb{R}^d$, we will use G_D to denote the Green function of X^D, i.e.,

$$G_D(x, y) := \int_0^\infty p_D(t, x, y)dt, \quad x, y \in D.$$

Note that $G_D(x, y)$ is continuous on $\{(x, y) \in D \times D : x \ne y\}$.

3 Analysis on half-space

Recall that $X = (X_t : t \ge 0)$ is the d-dimensional subordinate Brownian motion defined by $X_t = W_{S_t}$, where $W = (W^1, \dots, W^d)$ is a d-dimensional Brownian motion and $S = (S_t : t \ge 0)$ an independent complete subordinator whose drift is equal to 1 and whose Lévy density satisfies (1.2).

Let $Z = (Z_t : t \ge 0)$ be the one-dimensional subordinate Brownian motion defined as $Z_t := W^d_{S_t}$. Let $\overline{Z}_t := \sup\{0 \vee Z_s : 0 \le s \le t\}$ be the supremum process of Z and let $L = (L_t : t \ge 0)$ be a local time of $\overline{Z} - Z$ at 0. L is also called a local time of the process Z reflected at the supremum. The right continuous inverse L_t^{-1} of L is a subordinator and is called the ladder time process of Z. The process $H_t = \overline{Z}_{L_t^{-1}}$ is also a subordinator and is called the ladder height process of Z. (For the basic properties of the ladder time and ladder height processes, we refer our readers to [1, Chapter 6].) The ladder height process H has a drift ([20, Lemma 2.1]). The potential measure of the subordinator H will be denoted by V. Let $V(t) := V((0, t))$ be the renewal function of H.

By [1, Theorem 5, page 79] and [20, Lemma 2.1], V is absolutely continuous and has a continuous and strictly positive density v such that $v(0+) = 1$.

Lemma 3.1 ([20, Lemma 2.2]) *Let $R > 0$. There exists a constant $c = c(R) > 1$ such that for all $x \in (0, R]$, we have $c^{-1} \le v(x) \le c$ and $c^{-1}x \le V(x) \le cx$.*

By [22, Proposition 2.4], the Laplace exponent χ of the ladder height process H of Z_t is also a complete Bernstein function. Using this and the fact that χ has a drift, we see from [21, Corollary 2.3], that v is completely monotone. In particular, v and the renewal function V are C^∞ functions.

We will use $\mathbb{R}^d_+$ to denote the half-space $\{x = (x_1, \dots, x_{d-1}, x_d) := (\tilde{x}, x_d) \in \mathbb{R}^d : x_d > 0\}$. Define $w(x) := V((x_d)^+)$. The proof of the next theorem is taken from [22, Theorem 4.1].

Theorem 3.2 *The function w is harmonic in $\mathbb{R}^d_+$ with respect to X and, for any $r > 0$, regular harmonic in $\mathbb{R}^{d-1} \times (0, r)$ with respect to X.*

Proof. Since $Z_t := W^d_{S_t}$ has a transition density, it satisfies the condition ACC in [26], namely the resolvent kernels are absolutely continuous. The assumption in [26] that 0 is regular for $(0,\infty)$ is also satisfied since X is of unbounded variation. Further, by symmetry of Z, the notions of coharmonic and harmonic functions coincide. Let $Z^{(0,\infty)}$ (respectively $X^{\mathbb{R}^d_+}$) denote the process Z killed upon exiting $(0,\infty)$ (respectively X killed upon exiting $\mathbb{R}^d_+$). By [26, Theorem 2], the renewal function V of the ladder height process of Z is invariant for $Z^{(0,\infty)}$. Thus w is invariant for $X^{\mathbb{R}^d_+}$. Being invariant for $X^{\mathbb{R}^d_+}$, w is also harmonic for $X^{\mathbb{R}^d_+}$, and consequently, harmonic in $\mathbb{R}^d_+$ with respect to X. We show now that w is regular harmonic for X in $\mathbb{R}^{d-1} \times (0, r)$ for any $r > 0$. First note that since V is continuous at zero and $V(0) = 0$, it follows that

$$\lim_{x_d \to 0} w(x) = \lim_{x_d \to 0} w(\tilde{x}, x_d) = \lim_{x_d \to 0} V(x_d) = 0\,. \tag{3.1}$$

Thus, by harmonicity of w and (3.1)

$$w(x) = w(\tilde{x}, x_d) = \lim_{\varepsilon \to 0} \mathbb{E}_x \left[w\left(X_{\tau_{\mathbb{R}^{d-1} \times (\varepsilon, r)}} \right) \right] = \mathbb{E}_x \left[w\left(X_{\tau_{\mathbb{R}^{d-1} \times (0, r)}} \right) \right], \quad x_d > 0\,.$$

□

Unlike [22, Proposition 4.2], we prove the next result without using the boundary Harnack principle.

Proposition 3.3 *For all positive constants r_0 and L, we have*

$$\sup_{x \in \mathbb{R}^d :\, 0 < x_d < L} \int_{B(x, r_0)^c \cap \mathbb{R}^d_+} w(y) j(|x-y|)\, dy < \infty\,.$$

Proof. Without loss of generality, we assume $\widetilde{x} = 0$. We consider two separate cases.
(a) Suppose $L > x_d \geq r_0/4$. By (2.3) and Theorem 3.2, for every $x \in \mathbb{R}^d_+$,

$$\begin{aligned} w(x) &\geq \mathbb{E}_x \left[w\big(X_{\tau_{B(x, r_0/2) \cap \mathbb{R}^d_+}}\big) : X_{\tau_{B(x, r_0/2) \cap \mathbb{R}^d_+}} \in B(x, r_0)^c \cap \mathbb{R}^d_+ \right] \\ &= \mathbb{E}_x \left[\int_0^{\tau_{B(x, r_0/2) \cap \mathbb{R}^d_+}} \int_{B(x, r_0)^c \cap \mathbb{R}^d_+} j(|X_t - y|) w(y)\, dy dt \right]. \end{aligned} \tag{3.2}$$

Since $|z - y| \leq |x - z| + |x - y| \leq r_0 + |x - y| \leq 2|x - y|$ for $(z, y) \in B(x, r_0/2) \times B(x, r_0)^c$, using (2.5) and (2.6), we have $j(|z - y|) \geq c_1 j(|x - y|)$. Thus, combining this with (3.2), we obtain that

$$\int_{B(x, r_0)^c \cap \mathbb{R}^d_+} w(y) j(|x-y|) dy \leq c_1^{-1} \frac{w(x)}{\mathbb{E}_x[\tau_{B(x, r_0/2) \cap \mathbb{R}^d_+}]} \leq c_1^{-1} \frac{V(L)}{\mathbb{E}_0[\tau_{B(0, r_0/4)}]}.$$

(b) Suppose $x_d < r_0/4$. Note that if $|y-x| > r_0$, then $|y| \ge |y-x| - |x| > 3r_0/4$ and $|y| \le |y-x| + |x| \le |y-x| + r_0/4 \le |y-x| + |y-x|/4$. Thus, using (2.5) and (2.6), we have $j(|y|) \ge c_2 j(|x-y|)$ and

$$\sup_{x\in\mathbb{R}^d:\, 0<x_d<r_0/4} \int_{B(x,r_0)^c\cap\mathbb{R}^d_+} w(y)j(|x-y|)dy \le c_3 \int_{B(0,r_0/2)^c\cap\mathbb{R}^d_+} w(y)j(|y|)dy. \tag{3.3}$$

Let $x_1 := (\widetilde{0}, r_0/8)$. By Theorem 3.2 and (2.3),

$$\begin{aligned}\infty > w(x_1) &\ge \mathbb{E}_{x_1}\left[w(X_{\tau_{B(0,r_0/4)\cap\mathbb{R}^d_+}}) : X_{\tau_{B(0,r_0/4)\cap\mathbb{R}^d_+}} \in B(x,r_0/2)^c\cap\mathbb{R}^d_+\right] \\ &= \mathbb{E}_{x_1}\left[\int_0^{\tau_{B(0,r_0/4)\cap\mathbb{R}^d_+}} \int_{B(0,r_0/2)^c\cap\mathbb{R}^d_+} j(|X_t-y|)w(y)\,dy\,dt\right]. \end{aligned}\tag{3.4}$$

Since $|z-y| \le |z| + |y| \le (r_0/4) + |y| \le 2|y|$ for $(z,y) \in B(0,r_0/4)\times B(0,r_0/2)^c$, using (2.5) and (2.6), we have $j(|z-y|) \ge c_3 j(|y|)$. Thus, combining this with (3.4), we obtain that

$$\begin{aligned}\infty > w(x_1) &> c_3\mathbb{E}_{x_1}\left[\int_0^{\tau_{B(0,r_0/4)\cap\mathbb{R}^d_+}} \int_{B(0,r_0/2)^c\cap\mathbb{R}^d_+} j(|y|)w(y)\,dy\,dt\right] \\ &= c_3\mathbb{E}_{x_1}[\tau_{B(0,r_0/4)\cap\mathbb{R}^d_+}] \int_{B(0,r_0/2)^c\cap\mathbb{R}^d_+} j(|y|)w(y)dy. \end{aligned}\tag{3.5}$$

Combining (3.3) and (3.5), we conclude that

$$\sup_{x\in\mathbb{R}^d:\, 0<x_d<r_0/4} \int_{B(x,r_0)^c\cap\mathbb{R}^d_+} w(y)j(|x-y|)dy \le c_4\frac{V(r_0/8)}{\mathbb{E}_0[\tau_{B(0,r_0/8)}]} < \infty.$$

□

We now define an operator $(\Delta+\mathcal{A}, \mathfrak{D}(\Delta+\mathcal{A}))$ as follows:

$$\mathcal{A}f(x) := \lim_{\varepsilon\downarrow 0}\int_{B(x,\varepsilon)^c} (f(y)-f(x))\,j(|y-x|)\,dy,$$

$$\mathfrak{D}(\Delta+\mathcal{A}) := \left\{ f\in C^2(\mathbb{R}^d) : \lim_{\varepsilon\downarrow 0}\int_{B(x,\varepsilon)^c} (f(y)-f(x))\,j(|y-x|)\,dy \text{ exists and is finite} \right\}. \tag{3.6}$$

It is well known that $C_0^2(\mathbb{R}^d) \subset \mathfrak{D}(\Delta+\mathcal{A})$, where $C_0^2(\mathbb{R}^d)$ is the collection of C^2 functions in $\mathbb{R}^d$ vanishing at infinity, and that by the rotational symmetry of X, $\Delta+\mathcal{A}$ restricted to $C_0^2(\mathbb{R}^d)$ coincides with the infinitesimal generator of the process X (e.g. [24, Theorem 31.5]).

The proof of the next result is similar to that of [22, Theorem 4.3]. We give the proof here for completeness.

Theorem 3.4 *$(\Delta+\mathcal{A})w(x)$ is well defined and $(\Delta+\mathcal{A})w(x) = 0$ for all $x\in\mathbb{R}^d_+$.*

Proof. It follows from Proposition 3.3 and the fact that j is a Lévy density that for any $L>0$ and $\varepsilon\in(0,1/2)$

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d:\ 0<x_d<L}\left|\int_{B(x,\varepsilon)^c}(w(y)-w(x))j(|y-x|)dy\right| \\ &\le \sup_{x\in\mathbb{R}^d:\ 0<x_d<L}\int_{B(x,\varepsilon)^c} w(y)j(|y-x|)\,dy + V(L)\int_{B(x,\varepsilon)^c} j(|y|)dy < \infty\,. \end{aligned}\tag{3.7}$$

Hence, for every $\varepsilon \in (0, 1/2)$,

$$\mathcal{A}_\varepsilon w(x) := \int_{B(x,\varepsilon)^c} (w(y) - w(x)) j(|y - x|) dy$$

is well defined. Note that since $w(x) = V((x_d)^+)$ and V is smooth in $(0, \infty)$, it holds that w is smooth in $\mathbb{R}^d_+$. Hence,

$$\mathcal{A}_\varepsilon w(x) = \int_{B(x,\varepsilon)^c} \left(w(y) - w(x) - \mathbf{1}_{\{|y-x|<1\}}(y - x) \cdot \nabla w(x)\right) j(|y - x|) dy.$$

Moreover, by the smoothness of w,

$$x \mapsto \int_{B(x,\varepsilon)} (w(y) - w(x) - (y - x) \cdot \nabla w(x)) \, j(|y - x|) \, dy$$

converges to 0 locally uniformly in $\mathbb{R}^d_+$ as $\varepsilon \to 0$. Combining this with (3.7), we see that $\mathcal{A}w$ is well defined in $\mathbb{R}^d_+$ and $\mathcal{A}_\varepsilon w(x)$ converges to

$$\mathcal{A}w(x) = \int_{\mathbb{R}^d} \left(w(y) - w(x) - \mathbf{1}_{\{|y-x|<1\}}(y - x) \cdot \nabla w(x)\right) j(|y - x|) dy$$

locally uniformly in $\mathbb{R}^d_+$ as $\varepsilon \to 0$.

Moreover, for every $x \in \mathbb{R}^d_+$, $z \in B(x, (\varepsilon \wedge x_d)/2)$, and $y \in B(z, \varepsilon)^c$ it holds that $|y-z| \geq |y-x| - \frac{\varepsilon}{2}$ and $|y - z| \geq \frac{2}{3}|y - x|$. So, using (2.5) and (2.6), we have

$$\begin{aligned} & \mathbf{1}_{\{|y-z|>\varepsilon\}} \left|(w(y) - w(z) - \mathbf{1}_{\{|y-z|<1\}}(y - z) \cdot \nabla w(z))\right| j(|y - z|) \\ \leq \; & c \left(\sup_{\varepsilon/2 < s < x_d + 2} V''(s) \right) |y - x|^2 \mathbf{1}_{\{\varepsilon/2 < |y-x| < 2\}} \, j(|y - x|) \\ & + (w(y) + V(x_d + 1)) \mathbf{1}_{\{|y-x|>1/2\}} \, j(|y - x|). \end{aligned}$$

Using Proposition 3.3, the dominated convergence theorem and the fact that $j(|x|)dx$ is a Lévy measure, we can easily get that $x \to \mathcal{A}_\varepsilon w(x)$ is continuous for each ε. Therefore, by this and the local uniform convergence of $\mathcal{A}_\varepsilon w$, the function $\mathcal{A}w(x)$ is continuous in $\mathbb{R}^d_+$.

Suppose that U_1 and U_2 are relatively compact open subsets of $\mathbb{R}^d_+$ such that $\overline{U_1} \subset U_2 \subset \overline{U_2} \subset \mathbb{R}^d_+$. Let $r_0 := \text{dist}(U_1, U_2^c) > 0$. Then, by Proposition 3.3,

$$\begin{aligned} \int_{U_1} \int_{U_2^c} w(y) j(|x - y|) dy dx \;\; & \leq \;\; |U_1| \sup_{x \in U_1} \int_{U_2^c} w(y) j(|x - y|) dy \\ & \leq \;\; |U_1| \sup_{x \in U_1} \int_{B(x,r_0)^c} w(y) j(|x - y|) dy < \infty \, . \end{aligned} \tag{3.8}$$

By harmonicity of w, clearly $w(X_{\tau_{U_1}}) \in L^1(\mathbb{P}_x)$ and

$$\sup_{x \in U_1} \mathbb{E}_x \left[\mathbf{1}_{U_2^c}(X_{\tau_{U_1}}) w(X_{\tau_{U_1}})\right] \leq \sup_{x \in U_1} \mathbb{E}_x \left[w(X_{\tau_{U_1}})\right] = \sup_{x \in U_1} w(x) < \infty.$$

The last two displays show that the conditions [9, (2.4), (2.6)] are true. Thus, by [9, Lemma 2.3, Theorem 2.11(ii)], we have that for any $f \in C^2_c(\mathbb{R}^d_+)$,

$$0 = \int_{\mathbb{R}^d} \nabla w(x) \cdot \nabla f(x) \, dx + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (w(y) - w(x))(f(y) - f(x)) j(|y - x|) \, dx \, dy. \tag{3.9}$$

For $f \in C_c^2(\mathbb{R}^d_+)$ with $\mathrm{supp}(f) \subset \overline{U_1} \subset U_2 \subset \overline{U_2} \subset \mathbb{R}^d_+$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |w(y)-w(x)||f(y)-f(x)|j(|y-x|)dxdy \\
=&\int_{U_2}\int_{U_2} |w(y)-w(x)||f(y)-f(x)|j(|y-x|)dxdy + 2\int_{U_1}\int_{U_2^c} |w(y)-w(x)||f(x)|j(|y-x|)dxdy \\
\le& c_1 \int_{U_2\times U_2} |y-x|^2 j(|y-x|)dxdy + 2\|f\|_\infty |U_1| \left(\sup_{x\in U_1} w(x)\right)\int_{U_2^c} j(|y-x|)dy \\
&+ 2\|f\|_\infty \int_{U_1}\int_{U_2^c} w(y)j(|x-y|)dydx
\end{aligned}$$

is finite by (3.8) and the fact that $j(|x|)dx$ is a Lévy measure. Thus by (3.9), Fubini's theorem and the dominated convergence theorem, we have for any $f \in C_c^2(\mathbb{R}^d_+)$,

$$\begin{aligned}
0 =& \int_{\mathbb{R}^d} \nabla w(x)\cdot\nabla f(x)\,dx + \frac{1}{2}\lim_{\varepsilon\downarrow 0}\int_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d,\ |y-x|>\varepsilon\}} (w(y)-w(x))(f(y)-f(x))j(|y-x|)\,dx\,dy \\
=& -\int_{\mathbb{R}^d} \Delta w(x) f(x)\,dx - \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}^d_+} f(x)\left(\int_{B(x,\varepsilon)^c} (w(y)-w(x))j(|y-x|)dy\right)dx \\
=& -\int_{\mathbb{R}^d} \Delta w(x) f(x)\,dx - \int_{\mathbb{R}^d_+} f(x)\mathcal{A}w(x)\,dx \;=\; -\int_{\mathbb{R}^d} (\Delta+\mathcal{A})w(x)f(x)\,dx
\end{aligned}$$

where we have used the fact $\mathcal{A}_\varepsilon w \to \mathcal{A}w$ converges uniformly on the support of f. Hence, by the continuity of $(\Delta+\mathcal{A})w$, we have $(\Delta+\mathcal{A})w(x) = 0$ in $\mathbb{R}^d_+$. □

4 Analysis on $C^{1,1}$ open set

Recall that $\Lambda \ge 1$ and that D is a $C^{1,1}$ open set with characteristics (R,Λ) and D satisfies the uniform interior ball condition and the uniform exterior ball condition with radius $R \le 1$.

Lemma 4.1 *Fix $Q \in \partial D$ and define*

$$h(y) := V(\delta_D(y))\,\mathbf{1}_{D\cap B(Q,R)}(y).$$

There exists $C_1 = C_1(\Lambda, R) > 0$ independent of Q such that $(\Delta+\mathcal{A})h$ is well defined in $D \cap B(Q,R/4)$ a.e. and

$$|(\Delta+\mathcal{A})h(x)| \le C_1 \quad \textit{for a.e. } x \in D\cap B(Q,R/4)\,. \tag{4.1}$$

Proof. In this proof, we fix $x \in D\cap B(Q,R/4)$ and $x_0 \in \partial D$ satisfying $\delta_D(x) = |x-x_0|$. We also fix the $C^{1,1}$ function φ and the coordinate system $CS = CS_{x_0}$ in the definition of $C^{1,1}$ open set so that $x = (0, x_d)$ with $0 < x_d < R/4$ and $B(x_0,R)\cap D = \{y = (\widetilde{y}, y_d) \in B(0,R) \text{ in } CS : y_d > \varphi(\widetilde{y})\}$. Let

$$\varphi_1(\widetilde{y}) := R - \sqrt{R^2 - |\widetilde{y}|^2} \quad \text{and} \quad \varphi_2(\widetilde{y}) := -R + \sqrt{R^2 - |\widetilde{y}|^2}.$$

Due to the uniform interior ball condition and the uniform exterior ball condition with radius R, we have

$$\varphi_2(\widetilde{y}) \le \varphi(\widetilde{y}) \le \varphi_1(\widetilde{y}) \quad \text{for every } y \in D \cap B(x, R/4). \tag{4.2}$$

Define $H^+ := \{y = (\widetilde{y},\, y_d) \in CS : y_d > 0\}$ and let

$$A := \{y = (\widetilde{y}, y_d) \in (D \cup H^+) \cap B(x, R/4) : \varphi_2(\widetilde{y}) \le y_d \le \varphi_1(\widetilde{y})\},$$

$$E := \{y = (\widetilde{y}, y_d) \in B(x, R/4) : y_d > \varphi_1(\widetilde{y})\}.$$

Note that, since $|y - Q| \le |y - x| + |x - Q| \le R/2$ for $y \in B(x, R/4)$, we have $B(x, R/4) \cap D \subset B(Q, R/2) \cap D$.

Let

$$h_x(y) := V\left(\delta_{H^+}(y)\right).$$

Note that $h_x(x) = h(x)$. Moreover, since $\delta_{H^+}(y) = (y_d)^+$ in CS, it follows from Theorem 3.4 that $\mathcal{A}h_x$ is well defined in H^+ and

$$(\Delta + \mathcal{A})h_x(y) = 0, \quad \forall y \in H^+. \tag{4.3}$$

We show now that $\mathcal{A}(h - h_x)(x)$ is well defined. For each $\varepsilon > 0$ we have that

$$\begin{aligned} &\left| \int_{\{y \in D \cup H^+ :\, |y-x| > \varepsilon\}} (h(y) - h_x(y)) j(|y - x|)\, dy \right| \\ &\le \int_{B(x,R/4)^c} (h(y) + h_x(y)) j(|y - x|) dy + \int_A (h(y) + h_x(y)) j(|y - x|)\, dy \\ &\quad + \int_E |h(y) - h_x(y)| j(|y - x|) dy \;=:\; I_1 \,+\, I_2 \,+\, I_3. \end{aligned}$$

By the fact that $h(y) = 0$ for $y \in B(Q, R)^c$,

$$I_1 \le \sup_{z \in \mathbb{R}^d :\ 0 < z_d < R} \int_{B(z,R/4)^c \cap H^+} V(y_d) j(|z - y|) dy + c_1 \int_{B(0,R/4)^c} j(|y|) dy =: K_1 + K_2.$$

K_2 is clearly finite since J is the Lévy density of X while K_1 is finite by Proposition 3.3.

For $y \in A$, since V is increasing and $(R - \sqrt{R^2 - |\widetilde{y}|^2}) \le R^{-1}|\widetilde{y}|^2$, we see that

$$h_x(y) + h(y) \le 2V(\varphi_1(\widetilde{y}) - \varphi_2(\widetilde{y})) \le 2V(2R^{-1}|\widetilde{y}|^2) \le 2V(2R^{-1}|y - x|^2). \tag{4.4}$$

Using (4.4) and Lemma 3.1, we have

$$I_2 \le c_2 \int_A |y - x|^2 j(|y - x|) dy \le c_2 \int_{B(0,R/4)} |z|^2 j(|z|) dz < \infty. \tag{4.5}$$

For I_3, we consider two cases separately: If $0 < y_d = \delta_{H^+}(y) \le \delta_D(y)$, since v is decreasing,

$$h(y) - h_x(y) \le V(y_d + R^{-1}|\widetilde{y}|^2) - V(y_d) = \int_{y_d}^{y_d + R^{-1}|\widetilde{y}|^2} v(z) dz \le R^{-1}|\widetilde{y}|^2 v(y_d). \tag{4.6}$$

If $y_d = \delta_{_{H^+}}(y) > \delta_D(y)$ and $y \in E$, using the fact that $\delta_D(y)$ is greater than or equal to the distance between y and the graph of φ_1 and

$$y_d - R + \sqrt{|\widetilde{y}|^2 + (R-y_d)^2} = \frac{|\widetilde{y}|^2}{\sqrt{|\widetilde{y}|^2 + (R-y_d)^2} + (R-y_d)} \le \frac{|y-x|^2}{2(R-y_d)} \le \frac{|y-x|^2}{R},$$

we have

$$h_x(y) - h(y) \le \int_{R-\sqrt{|\widetilde{y}|^2+(R-y_d)^2}}^{y_d} v(z)dz \le R^{-1}|y-x|^2\, v(R - \sqrt{|\widetilde{y}|^2 + (R-y_d)^2}). \tag{4.7}$$

Thus, by (4.6)-(4.7) and Lemma 3.1,

$$I_3 \le c_3 \int_E |y-x|^2 j(|y-x|)dy \le c_3 \int_{B(0,R/4)} |z|^2 j(|z|)dz < \infty.$$

We have proved

$$|\mathcal{A}(h-h_x)(x)| \le I_1 + I_2 + I_3 \le c_4 \tag{4.8}$$

for some constant $c_4 = c_4(R, \Lambda) > 0$.

The estimate (4.8) shows in particular that the limit

$$\lim_{\varepsilon \downarrow 0} \int_{\{y \in D \cup H^+ : |y-x| > \varepsilon\}} (h(y) - h_x(y)) j(|y-x|)\, dy$$

exists and hence $\mathcal{A}(h-h_x)(x)$ is well defined.

Note that for a.e. $x \in D \cap B(Q, R/4)$, the second order partial derivatives of the function $y \to \delta_D(y)$ exist at x. Without loss of generality we assume that x has been chosen so that the second order partial derivatives of the function $y \to \delta_D(y)$ exist at x. In the coordinate system CS, since $h_x(y) = V\left((y_d)^+\right)$, we have $\Delta h_x(x) = v'(x_d)$. Moreover, since $\delta_D(y) = y_d$ for $y = (\widetilde{0}, x_d + \varepsilon)$ when $|\varepsilon|$ is small, $\partial^2_{x_d} h(x) = v'(x_d)$. Thus

$$\begin{aligned}\Delta(h-h_x)(x) &= \sum_{i=1}^{d-1} \frac{\partial^2 V\left(\delta_D(y)\right)}{\partial y_i^2}|_{y=x} = \sum_{i=1}^{d-1} \frac{\partial}{\partial y_i}\left(v(\delta_D(y)) \frac{\partial \delta_D(y)}{\partial y_i}\right)|_{y=x} \\ &= \sum_{i=1}^{d-1} v'(\delta_D(x)) \left(\frac{\partial \delta_D(y)}{\partial y_i}|_{y=x}\right)^2 + v(\delta_D(x)) \frac{\partial^2 \delta_D(y)}{\partial y_i^2}|_{y=x}\ .\end{aligned} \tag{4.9}$$

In the coordinate system CS,

$$\frac{\partial \delta_D(y)}{\partial y_i}|_{y=x} = 0 \quad \text{and} \quad \left| \frac{\partial^2 \delta_D(y)}{\partial y_i^2}|_{y=x} \right| \le \frac{4}{3R}, \quad i = 1, \dots, d-1. \tag{4.10}$$

Indeed, let $\epsilon \in \mathbb{R}$ with $|\epsilon|$ small, and $x_{\epsilon,i} := (0, \dots, \epsilon, \dots 0, x_d)$, $i = 1, \dots, d-1$. Due to the uniform interior ball condition and the uniform exterior ball condition with radius R, we have

$$R - \sqrt{\epsilon^2 + (R-x_d)^2} - x_d \le \delta_D(x_{\epsilon,i}) - \delta_D(x) \le \sqrt{\epsilon^2 + (R+x_d)^2} - R - x_d\,,$$

so

$$\begin{aligned}\frac{1}{\epsilon}|\delta_D(x_{\epsilon,i}) - \delta_D(x)| \le& \frac{1}{\epsilon}\left(\sqrt{\epsilon^2 + (R-x_d)^2} - (R-x_d)\right) \vee \frac{1}{\epsilon}\left(\sqrt{\epsilon^2 + (R+x_d)^2} - (R+x_d)\right) \\ =& \left(\frac{\epsilon}{\sqrt{\epsilon^2 + (R-x_d)^2} + (R-x_d)} \right) \vee \left(\frac{\epsilon}{\sqrt{\epsilon^2 + (R+x_d)^2} + (R+x_d)} \right),\end{aligned}$$

which goes to zero as $\epsilon \to 0$. The bound involving the second partial derivatives can be proved in a similar way using the elementary fact that $\frac{\partial^2 \delta_D(y)}{\partial x_i^2}|_{y=x} = \lim_{\epsilon \to 0} \frac{1}{h^2}(\delta_D(x_{\epsilon,i}) + \delta_D(x_{-\epsilon,i}) - 2\delta_D(x))$. Therefore, combining (4.9), (4.10) and Lemma 3.1, we have

$$|\Delta(h - h_x)(x)| \le c_5 \sum_{i=1}^{d-1} \left| \frac{\partial^2 \delta_D(y)}{\partial x_i^2}|_{y=x} \right| \le c_6.$$

Using this, (4.3), (4.8), and linearity we get that $(\Delta + \mathcal{A})h(x)$ is well defined and $|(\Delta + \mathcal{A})h(x)| \le c_7$. □

Recall that we use $C_c^\infty(\mathbb{R}^d)$ to denote the space of infinitely differentiable functions with compact support. Using the fact that $\Delta + \mathcal{A}$ restricted to $C_c^\infty(\mathbb{R}^d)$ coincides with the infinitesimal generator of the process X, we see that the following Dynkin's formula is true: for $f \in C_c^\infty(\mathbb{R}^d)$ and any bounded open subset U of $\mathbb{R}^d$,

$$\mathbb{E}_x \int_0^{\tau_U} (\Delta + \mathcal{A}) f(X_t) dt = \mathbb{E}_x[f(X_{\tau_U})] - f(x). \tag{4.11}$$

Lemma 4.2 *For every $r_1 > 0$ and every $a \in (0,1)$, there exists a positive constant $c = c(r_1, d, a)$ such that for any $r \in (0, r_1]$ and any open sets U and D with $B(0, ar) \cap D \subset U \subset D$, we have*

$$\mathbb{P}_x\left(X_{\tau_U} \in D\right) \le c\, r^{-2}\, \mathbb{E}_x[\tau_U], \qquad x \in D \cap B(0, ar/2).$$

Proof. For fixed $a \in (0,1)$, take a sequence of radial functions ϕ_m in $C_c^\infty(\mathbb{R}^d)$ such that $0 \le \phi_m \le 1$,

$$\phi_m(y) = \begin{cases} 0, & |y| < a/2 \\ 1, & a \le |y| \le m+1 \\ 0, & |y| > m+2, \end{cases}$$

and that $\| \sum_{i,j} |\frac{\partial^2}{\partial y_i \partial y_j} \phi_m| \|_\infty < c_1 = c_1(a) < \infty$. Define $\phi_{m,r}(y) = \phi_m(\frac{y}{r})$ so that $0 \le \phi_{m,r} \le 1$,

$$\phi_{m,r}(y) = \begin{cases} 0, & |y| < ar/2 \\ 1, & ar \le |y| \le r(m+1) \\ 0, & |y| > r(m+2), \end{cases} \quad \text{and} \quad \sup_{y \in \mathbb{R}^d} \sum_{i,j} \left| \frac{\partial^2}{\partial y_i \partial y_j} \phi_{m,r}(y) \right| < c_1\, r^{-2}. \tag{4.12}$$

Using (4.12), we see that

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} (\phi_{m,r}(x+y) - \phi_{m,r}(x) - (\nabla \phi_{m,r}(x) \cdot y) 1_{B(0,1)}(y)) J(y) dy \right| \\ &\le \left| \int_{\{|y| \le 1\}} (\phi_{m,r}(x+y) - \phi_{m,r}(x) - (\nabla \phi_{m,r}(x) \cdot y) 1_{B(0,1)}(y)) J(y) dy \right| + 2 \int_{\{|y|>1\}} J(y) dy \\ &\le \frac{c_2}{r^2} \int_{\{|y| \le 1\}} |y|^2 J(y) dy + 2 \int_{\{|y|>1\}} J(y) dy \le \frac{c_3}{r^2}. \end{aligned} \tag{4.13}$$

Now, by combining (4.11), (4.12) and (4.13), we get that for any $x \in D \cap B(0, ar/2)$,

$$\mathbb{P}_x\left(X_{\tau_U} \in \{y \in D : ar \le |y| < (m+1)r\}\right) = \mathbb{E}_x\left[\phi_{m,r}\left(X_{\tau_U}\right) : X_{\tau_U} \in \{y \in D : ar \le |y| < (m+1)r\}\right]$$

$$\le \mathbb{E}_x\left[\phi_{m,r}\left(X_{\tau_U}\right)\right] = \mathbb{E}_x\left[\int_0^{\tau_U} (\Delta + \mathcal{A}) \phi_{m,r}(X_t) dt\right] \le \|(\Delta + \mathcal{A})\phi_{m,r}\|_\infty\, \mathbb{E}_x[\tau_U] \le c_4 r^{-2} \mathbb{E}_x[\tau_U].$$

Therefore, since $B(0,ar)\cap D\subset U$,

$$\mathbb{P}_x\left(X_{\tau_U}\in D\right)=\mathbb{P}_x\left(X_{\tau_U}\in\{y\in D: ar\le |y|\}\right) \\ =\lim_{m\to\infty}\mathbb{P}_x\left(X_{\tau_U}\in\{y\in D: ar\le |y|<(m+1)r\}\right)\ \le\ c_5\, r^{-2}\mathbb{E}_x[\tau_U].$$

□

Define $\rho_Q(x):=x_d-\varphi_Q(\widetilde{x})$, where $(\widetilde{x},x_d)$ are the coordinates of x in CS_Q. Note that for every $Q\in\partial D$ and $x\in B(Q,R)\cap D$ we have

$$(1+\Lambda^2)^{-1/2}\,\rho_Q(x)\ \le\ \delta_D(x)\ \le\ \rho_Q(x). \tag{4.14}$$

We define for $r_1,r_2>0$

$$D_Q(r_1,r_2):=\{y\in D: r_1>\rho_Q(y)>0,\ |\widetilde{y}|<r_2\}\,.$$

Let $R_1:=R/(4\sqrt{1+(1+\Lambda)^2})$.

Lemma 4.3 *There are constants $\lambda_0>2R_1^{-1}$, $\kappa_0\in(0,1)$ and $c=c(R,\Lambda)>0$ such that for every $\lambda\ge\lambda_0$, $Q\in\partial D$ and $x\in D_Q(2^{-1}(1+\Lambda)^{-1}\kappa_0\lambda^{-1},\kappa_0\lambda^{-1})$ with $\widetilde{x}=0$,*

$$\mathbb{P}_x\left(X_{\tau_{D_Q(\kappa_0\lambda^{-1},\lambda^{-1})}}\in D\right)\ \le\ c\,\lambda\,V(\delta_D(x)) \tag{4.15}$$

and

$$\mathbb{E}_x\left[\tau_{D_Q(\kappa_0\lambda^{-1},\lambda^{-1})}\right]\ \le\ c\,\lambda^{-1}\,V(\delta_D(x)). \tag{4.16}$$

Proof. Without loss of generality, we assume $Q=0$. Let $\varphi=\varphi_0:\mathbb{R}^{d-1}\to\mathbb{R}$ be the $C^{1,1}$ function and CS_0 be the coordinate system in the definition of $C^{1,1}$ open set so that $B(0,R)\cap D=\big\{(\widetilde{y},\,y_d)\in B(0,R)$ in $CS_0: y_d>\varphi(\widetilde{y})\big\}$. Let $\rho(y):=y_d-\varphi(\widetilde{y})$ and $D(a,b):=D_0(a,b)$.

Note that

$$|y|^2=|\widetilde{y}|^2+|y_d|^2<r^2+(|y_d-\varphi(\widetilde{y})|+|\varphi(\widetilde{y})|)^2<(1+(1+\Lambda)^2)r^2\quad\text{for every } y\in D(r,r)\,. \tag{4.17}$$

By this and the definition of R_1, we have $D(r,s)\subset D(R_1,R_1)\subset B(0,R/4)\cap D\subset B(0,R)\cap D$ for every $r,s\le R_1$.

Using Lemma 3.1, we can and will choose $\delta_0\in(0,R_1)$ small such that

$$2r^2\le V((1+\Lambda^2)^{-1/2}r)\qquad\text{for all } r\le 4\delta_0.$$

Then, by (4.14), the subadditivity and monotonicity of V, for every $\lambda\ge1$ and every $y\in B(0,R)\cap D$ with $\rho(y)\le 4\lambda^{-1}\delta_0$, we have

$$2\lambda^2\rho(y)^2\le V(\lambda\delta_D(y))\le(\lambda+1)V(\delta_D(y))\le 2\lambda V(\delta_D(y)). \tag{4.18}$$

Since $\Delta\varphi(\tilde{y})$ is well defined for a.e. $\tilde{y}$ with respect to the $(d-1)$-dimensional Lebesgue measure, it follows that $\Delta\rho(y)$ exists for a.e. y with respect to the d-dimensional Lebesgue measure. Using the fact that the derivative of a Lipschitz function is essentially bounded by its Lipschitz constant, we have for a.e. $y \in B(0,R) \cap D$ that

$$\Delta\rho(y)^2 = \Delta(y_d - \varphi(\widetilde{y}))^2 = 2(1 + |\nabla\varphi(\widetilde{y})|^2) - 2\,\rho(y)\Delta\varphi(\widetilde{y}) \geq 2(1 - \rho(y)\|\Delta\varphi\|_\infty)\,.$$

Choosing $\delta_0 \in (0, R_1)$ smaller if necessary we can get that

$$\Delta\rho(y)^2 \geq 1 \quad \text{ for a.e. } y \in B(0,R) \cap D \text{ with } \rho(y) \leq 2\delta_0. \tag{4.19}$$

Let $g(y) = g(\widetilde{y}, y_d)$ be a smooth function on $\mathbb{R}^d$ with $0 \leq g(\widetilde{y}, y_d) \leq 2$, $g(\widetilde{y}, y_d) \leq y_d^2$,

$$\sum_{i,j=1}^d |\frac{\partial^2 g}{\partial y_i \partial y_j}| + \sum_{i=1}^d |\frac{\partial g}{\partial y_i}| \leq c_1, \tag{4.20}$$

and

$$g(y) = \begin{cases} 0, & \text{if } -\infty < y_d < 0, \text{ or } y_d \geq 4 \text{ or } |\widetilde{y}| > 2 \\ y_d^2, & \text{if } 0 \leq y_d < 1 \text{ and } |\widetilde{y}| < 1 \\ -(y_d - 2)^2 + 2, & \text{if } 1 \leq y_d \leq 3 \text{ and } |\widetilde{y}| < 1 \\ (y_d - 4)^2, & \text{if } 3 \leq y_d \leq 4 \text{ and } |\widetilde{y}| < 1. \end{cases}$$

Thus $\text{supp}(g) \subset \{|\widetilde{y}| \leq 2, 0 \leq y_d \leq 4\}$.

For $\lambda > 1$, let $g_\lambda(y) := g_\lambda(\widetilde{y}, y_d) := g(\lambda\delta_0^{-1}\widetilde{y}, \lambda\delta_0^{-1}\rho(y))$ so that

$$\text{supp}(g_\lambda) \subset \{|\widetilde{y}| \leq 2\lambda^{-1}\delta_0,\ \ 0 \leq \rho(y) \leq 4\lambda^{-1}\delta_0\}. \tag{4.21}$$

Then, since $\sum_{i,j=1}^d |\frac{\partial^2}{\partial y_i \partial y_j} g(y)|$ is essentially bounded, using (4.20), we have

$$\sum_{i,j=1}^d |\frac{\partial^2}{\partial y_i \partial y_j}\, g_\lambda(y)| \leq c_2\lambda^2 \quad \text{ a.e. } y. \tag{4.22}$$

Note that, by the definition of g, $g_\lambda(y) = \lambda^2\delta_0^{-2}\rho(y)^2$ on $D(\lambda^{-1}\delta_0, \lambda^{-1}\delta_0)$. Thus, from (4.19) we get

$$\Delta g_\lambda(y) \geq \lambda^2\delta_0^{-2} \quad \text{ for a.e. } y \in D(\lambda^{-1}\delta_0, \lambda^{-1}\delta_0). \tag{4.23}$$

On the other hand, by (4.22) we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} (g_\lambda(y+z) - g_\lambda(y) - (\nabla g_\lambda(y)\cdot z)1_{B(0,\lambda^{-1})}(z))J(z)\,dz\right| \\
&\leq \left|\int_{\{|z|\leq\lambda^{-1}\}} (g_\lambda(y+z) - g_\lambda(y) - (\nabla g_\lambda(y)\cdot z)1_{B(0,\lambda^{-1})}(z))J(z)\,dz\right| \\
&\quad + \int_{\{\lambda^{-1}<|z|\leq 1\}} J(z)g_\lambda(y+z)dz + \left(\int_{\{\lambda^{-1}<|z|\leq 1\}} J(z)dz\right) g_\lambda(y) + 2\int_{\{1<|z|\}} J(z)\,dz \\
&\leq c_3\lambda^2 \int_{\{|z|\leq\lambda^{-1}\}} |z|^2 J(z)\,dz + 2\int_{\{1<|z|\}} J(z)\,dz \\
&\quad + \int_{\{\lambda^{-1}<|z|\leq 1\}} J(z)g_\lambda(y+z)dz + \left(\int_{\{\lambda^{-1}<|z|\leq 1\}} J(z)dz\right) g_\lambda(y).
\end{aligned}$$

Thus

$$
\begin{aligned}
&\lambda^{-2}\left|\int_{\mathbb{R}^d}(g_\lambda(y+z)-g_\lambda(y)-(\nabla g_\lambda(y)\cdot z)1_{B(0,\lambda^{-1})}(z))J(z)dz\right|\\
\le\ & c_3\int_{\{|z|\le\lambda^{-1}\}}|z|^2J(z)\,dz+2\lambda^{-2}\int_{\{1<|z|\}}J(z)\,dz\\
&+\lambda^{-2}\int_{\{\lambda^{-1}<|z|\le 1\}}J(z)g_\lambda(y+z)dz+\lambda^{-2}\left(\int_{\{\lambda^{-1}<|z|\le 1\}}J(z)dz\right)g_\lambda(y)\\
\le\ & c_3\int_{\{|z|\le\lambda^{-1}\}}|z|^2J(z)\,dz+2\lambda^{-2}\int_{\{1<|z|\}}J(z)\,dz\\
&+\int_{\{\lambda^{-1}<|z|\le 1\}}J(z)|z|^2g_\lambda(y+z)dz+\left(\int_{\{0<|z|\le 1\}}|z|^2J(z)dz\right)g_\lambda(y).
\end{aligned}
\tag{4.24}
$$

We claim that for every $\lambda>1$ and $y\in D(\lambda^{-1}\delta_0,\lambda^{-1}\delta_0)$, the function $z\to g_\lambda(y+z)$ is supported in $B(0,3\lambda^{-1}\delta_0\sqrt{(4\Lambda)^2+1})$.

Fix $\lambda>1$ and $y\in D(\lambda^{-1}\delta_0,\lambda^{-1}\delta_0)$ and suppose that $z\in B(0,3\lambda^{-1}\delta_0\sqrt{(4\Lambda)^2+1})^c$. Then either $|\widetilde{z}|\ge 3\lambda^{-1}\delta_0$, or $|\widetilde{z}|<3\lambda^{-1}\delta_0$ and $|z_d|\ge 12\lambda^{-1}\delta_0\Lambda$. If $|\widetilde{z}|\ge 3\lambda^{-1}\delta_0$, then clearly $|\widetilde{y}+\widetilde{z}|\ge|\widetilde{z}|-|\widetilde{y}|\ge 3\lambda^{-1}\delta_0-\lambda^{-1}\delta_0=2\lambda^{-1}\delta_0$. Thus by (4.21), $g_\lambda(y+z)=0$. Now assume $|\widetilde{z}|<3\lambda^{-1}\delta_0$ and $|z_d|\ge 12\lambda^{-1}\delta_0\Lambda$. If $z_d\le -12\lambda^{-1}\delta_0\Lambda$, then $g_\lambda(y+z)=0$. If $z_d\ge 12\lambda^{-1}\delta_0\Lambda$, we have

$$
\rho(y+z)\ge z_d-|\psi(\widetilde{y}+\widetilde{z})|\ge 12\lambda^{-1}\delta_0\Lambda-\Lambda(|\widetilde{z}|+|\widetilde{y}|)\ge\lambda^{-1}\Lambda(12\delta_0-3\delta_0-\delta_0)=7\lambda^{-1}\Lambda\delta_0.
$$

Thus by (4.21), $g_\lambda(y+z)=0$. The claim is proved.

Using the above claim and the fact that $g_\lambda(y)=\lambda^2\delta_0^{-2}\rho(y)^2$ on $D(\lambda^{-1}\delta_0,\lambda^{-1}\delta_0)$, we have from (4.24), that for $y\in D(\lambda^{-1}\delta_0,\lambda^{-1}\delta_0)$

$$
\begin{aligned}
&\lambda^{-2}\left|\int_{\mathbb{R}^d}(g_\lambda(y+z)-g_\lambda(y)-(\nabla g_\lambda(y)\cdot z)1_{B(0,\lambda^{-1})}(z))J(z)dz\right|\\
\le& c_3\int_{\{|z|\le\lambda^{-1}\}}|z|^2J(z)\,dz+2\lambda^{-2}\int_{\{1<|z|\}}J(z)\,dz\\
&+\int_{\{\lambda^{-1}<|z|\le 1\wedge 3\lambda^{-1}\delta_0\sqrt{(4\Lambda)^2+1}\}}J(z)|z|^2dz+c_4\lambda^2\delta_0^{-2}\rho(y)^2\\
\le&(c_3+1)\int_{\{|z|\le 3\lambda^{-1}\delta_0\sqrt{(4\Lambda)^2+1}\}}(1\wedge|z|^2)J(z)\,dz+2\lambda^{-2}\int_{\{1<|z|\}}J(z)\,dz+c_4\lambda^2\delta_0^{-2}\rho(y)^2,
\end{aligned}
\tag{4.25}
$$

where $c_4:=2^{-1}\vee\int_{\{0<|z|\le 1\}}|z|^2J(z)dz$. Define

$$
h(y):=V(\delta_D(y))\mathbf{1}_{B(0,R)\cap D}(y)\quad\text{ and }\quad h_\lambda(y):=\lambda h(y)-g_\lambda(y).
$$

Choose $\lambda_*\ge 2$ large such that for every $\lambda\ge\lambda_*$,

$$
(c_3+1)\int_{\{|z|\le 2\lambda^{-1}\delta_0\sqrt{(4\Lambda)^2+1}\}}(1\wedge|z|^2)J(z)\,dz+2\lambda^{-2}\int_{\{1<|z|\}}J(z)\,dz\le 4^{-1}\delta_0^{-2}\quad\text{ and }\quad\frac14\lambda\delta_0^{-2}\ge C_1,
$$

where C_1 is the constant from Lemma 4.1. Then by (4.23) and (4.25), for every $\lambda\ge\lambda_*$ and a.e. $y\in D(\lambda^{-1}2^{-1}c_4^{-1/2}\delta_0,\lambda^{-1}\delta_0)$,

$$
(\Delta+\mathcal{A})g_\lambda(y)\ge\Delta g_\lambda(y)-|\mathcal{A}g_\lambda(y)|\ge\lambda^2\delta_0^{-2}-4^{-1}\lambda^2\delta_0^{-2}-c_4\lambda^4\delta_0^{-2}\rho(y)^2\ge\frac12\lambda^2\delta_0^{-2}
\tag{4.26}
$$

and

$$(\Delta+\mathcal{A})h_\lambda(y) \le \lambda|(\Delta+\mathcal{A})h(y)| - (\Delta+\mathcal{A})g_\lambda(y) \le \lambda(C_1 - \frac{1}{2}\lambda\delta_0^{-2}) \le -\frac{1}{4}\lambda^2\delta_0^{-2}\,. \tag{4.27}$$

Let $\delta_* := 2^{-1}c_4^{-1/2}\delta_0$ and f be a non-negative smooth radial function with compact support such that $f(x)=0$ for $|x|>1$ and $\int_{\mathbb{R}^d} f(x)dx = 1$. For $k\ge 1$, define $f_k(x) = 2^{kd}f(2^k x)$ and

$$h_\lambda^{(k)}(z) := (f_k * h_\lambda)(z) := \int_{\mathbb{R}^d} f_k(y)h_\lambda(z-y)dy\,.$$

Let

$$B_k^\lambda := \left\{ y\in D(\lambda^{-1}\delta_*, \lambda^{-1}\delta_0) : \delta_{D(\lambda^{-1}\delta_*,\lambda^{-1}\delta_0)}(y) \ge 2^{-k} \right\}$$

and consider large k's such that B_k^λ's are non-empty open sets. Since $h_\lambda^{(k)}$ is in C_c^∞, $\mathcal{A}h_\lambda^{(k)}$ is well defined everywhere. We claim that for every $\lambda\ge\lambda_*$ and k large enough,

$$(\Delta+\mathcal{A})h_\lambda^{(k)} \le -\frac{1}{4}\lambda^2\delta_0^{-2} \quad \text{on } B_k^\lambda\,. \tag{4.28}$$

Indeed, for any $x\in B_k^\lambda$ and $z\in B(0,2^{-k})$, when k is large enough, it holds that $x-z\in D(\lambda^{-1}\delta_*, \lambda^{-1}\delta_0)$. By the proof of Lemma 4.1 the following limit exists:

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{B(x,\varepsilon)^c} (h_\lambda(y-z) - h_\lambda(x-z))\; j(|x-y|)\,dy \\
&= \lim_{\varepsilon\to 0}\int_{B(x-z,\varepsilon)^c} \big(h_\lambda(y') - h_\lambda(x-z)\big)\; j(|(x-z)-y'|)\,dy' \;=\; \mathcal{A}h_\lambda(x-z)\,.
\end{aligned}$$

Moreover, by (4.27) it holds that for every $\lambda\ge\lambda_*$, $(\Delta+\mathcal{A})h_\lambda \le -\frac{1}{4}\lambda^2\delta_0^{-2}$ a.e. on $D(\lambda^{-1}\delta_*,\lambda^{-1}\delta_0)$. Next,

$$\begin{aligned}
&\int_{B(x,\varepsilon)^c} (h_\lambda^{(k)}(y) - h_\lambda^{(k)}(x))\, j(|x-y|)\,dy \\
&= \int_{|x-y|>\varepsilon}\left(\int_{\mathbb{R}^d} f_k(z)(h_\lambda(y-z) - h_\lambda(x-z))\,dz\right) j(|x-y|)\,dy \\
&= \int_{B(0,2^{-k})} f_k(z)\left(\int_{B(x,\varepsilon)^c} (h_\lambda(y-z) - h_\lambda(x-z))\; j(|x-y|)\,dy\right)\,dz.
\end{aligned}$$

By letting $\varepsilon\to 0$ and using the dominated convergence theorem, we get that for every $\lambda\ge\lambda_*$ and k large enough,

$$(\Delta+\mathcal{A})h_\lambda^{(k)}(x) = \int_{|z|<2^{-k}} f_k(z)(\Delta+\mathcal{A})h_\lambda(x-z)\,dz \le -\frac{1}{4}\lambda^2\delta_0^{-2}\int_{|z|<2^{-k}} f_k(z)\,dz = -\frac{1}{4}\lambda^2\delta_0^{-2}\,.$$

By using Dynkin's formula (4.11), the estimates (4.28) and the fact that $h_\lambda^{(k)}$ are in $C_c^\infty(\mathbb{R}^d)$, and by letting $k\to\infty$ we get for every $\lambda\ge\lambda_*$ and $x\in D(\lambda^{-1}\delta_*,\lambda^{-1}\delta_0)$ with $\widetilde{x}=0$,

$$\begin{aligned}
\mathbb{E}_x[h_\lambda(X_{\tau_{D(\lambda^{-1}\delta_*,\lambda^{-1}\delta_0)}})] - \lambda V(\delta_D(x)) &\le \mathbb{E}_x[h_\lambda(X_{\tau_{D(\lambda^{-1}\delta_*,\lambda^{-1}\delta_0)}})] - h_\lambda(x) \\
&\le -\frac{1}{4}\lambda^2\delta_0^{-2}\mathbb{E}_x[\tau_{D(\lambda^{-1}\delta_*,\lambda^{-1}\delta_0)}]\,. \qquad (4.29)
\end{aligned}$$

It is easy to see that $h_\lambda \geq 0$. In fact, if $y \in (B(0,R) \cap D)^c$, then both $h(y)$ and $g_\lambda(y)$ are zero. If $y \in B(0,R) \cap D$ and $\rho(y) \geq 4\lambda^{-1}\delta_0$, then $g_\lambda(y) = 0$. Finally, if $y \in B(0,R) \cap D$ and $\rho(y) \leq 4\lambda^{-1}\delta_0$, then, since $g(y) \leq y_d^2$ by (4.20), we have from (4.18),

$$h_\lambda(y) = \lambda V(\delta_D(y)) - g(\lambda\delta_0^{-1}\widetilde{y}, \lambda\delta_0^{-1}\rho(y)) \geq \lambda V(\delta_D(y)) - \lambda^2\rho(y)^2 \geq 0.$$

Therefore, from (4.29),

$$V(\delta_D(x)) \geq \frac{1}{4}\lambda\,\delta_0^{-2}\,\mathbb{E}_x[\tau_{D(\lambda^{-1}\delta_*,\lambda^{-1}\delta_0)}]. \tag{4.30}$$

Since $B(0,(1+\Lambda)^{-1}\delta_*\lambda^{-1}) \cap D \subset D(\lambda^{-1}\delta_*, \lambda^{-1}\delta_0)$, using Lemma 4.2 and (4.30), we have that for every $\lambda \geq \lambda_*$ and $x \in B(0, 2^{-1}(1+\Lambda)^{-1}\delta_*\lambda^{-1})$ with $\widetilde{x} = 0$,

$$\mathbb{P}_x\left(X_{\tau_{D_Q(\lambda^{-1}\delta_*,\lambda^{-1}\delta_0)}} \in D\right) \leq c_7\,\lambda^2\,\mathbb{E}_x[\tau_{D(\lambda^{-1}\delta_*,\lambda^{-1}\delta_0)}] \leq c_8\,\lambda\,V(\delta_D(x)).$$

We have proved the lemma with $\lambda_0 := \lambda_*\delta_0^{-1}$. □

Lemma 4.4 *There is a constant $c = c(R,\Lambda) > 0$ such that for every $\lambda \geq \lambda_0$, $\kappa \in (0,1]$, $Q \in \partial D$ and $x \in D_Q(\kappa\lambda^{-1}, \lambda^{-1})$ with $\widetilde{x} = 0$,*

$$\mathbb{P}_x\left(X_{\tau_{D_Q(\kappa\lambda^{-1},\lambda^{-1})}} \in D_Q(2\kappa\lambda^{-1}, \lambda^{-1})\right) \geq c\lambda V(\delta_D(x)). \tag{4.31}$$

Proof. Fix $\lambda \geq \lambda_0$ and $\kappa \in (0,1]$. For simplicity we denote $D_Q(\kappa\lambda^{-1}, \lambda^{-1})$ by $\widehat{D}$. Further, let

$$B = \left\{y \in D : \rho_Q(y) = \kappa\lambda^{-1} \text{ and } |\widetilde{y}| < \lambda^{-1}\right\}$$

be the upper boundary of $\widehat{D}$.

Let $\tau^W_{\widehat{D}}$ be the first time the Brownian motion W exits $\widehat{D}$ and $W^{\widehat{D}}$ the killed Brownian motion in $\widehat{D}$. Let $Y = (Y_t : t \geq 0)$ be the subordinate killed Brownian motion defined by $Y_t = W^{\widehat{D}}_{S_t}$. Let ζ denote the lifetime of Y. Recall that u is the potential density of the subordinator S. It follows from [29, Corollary 4.4] that

$$\mathbb{P}_x(X_{\tau_{\widehat{D}}} \in B) \geq \mathbb{P}_x(Y_{\zeta-} \in B) = \mathbb{E}_x\left[u(\tau^W_{\widehat{D}}); W_{\tau^W_{\widehat{D}}} \in B\right].$$

Thus, since u is deceasing, for any $t > 0$,

$$\begin{aligned}\mathbb{P}_x(X_{\tau_{\widehat{D}}} \in B) &\geq \mathbb{E}_x\left[u(\tau^W_{\widehat{D}}); W_{\tau^W_{\widehat{D}}} \in B, \tau^W_{\widehat{D}} \leq t\right] \geq u(t)\mathbb{P}_x\big(W_{\tau^W_{\widehat{D}}} \in B, \tau^W_{\widehat{D}} \leq t\big)\\ &= u(t)\left[\mathbb{P}_x\big(W_{\tau^W_{\widehat{D}}} \in B\big) - \mathbb{P}_x\big(\tau^W_{\widehat{D}} > t\big)\right] \geq u(t)\left[\mathbb{P}_x\big(W_{\tau^W_{\widehat{D}}} \in B\big) - t^{-1}\mathbb{E}_x\big[\tau^W_{\widehat{D}}\big]\right].\end{aligned}$$

Now we use the following two estimates which are valid for Brownian motions (for example, see [10, Lemma 3.4] with $a = 0$). There exist constants $c_1 > 0$ and $c_2 > 0$ (independent of $\lambda \geq \lambda_0$) such that $\mathbb{P}_x\big(W_{\tau^W_{\widehat{D}}} \in B\big) \geq c_1\lambda\delta_D(x)$ and $\mathbb{E}_x\big[\tau^W_{\widehat{D}}\big] \leq c_2\lambda^{-1}\delta_D(x)$. Then, by choosing $t_0 > 0$ so that $c_1 - t_0^{-1}c_2\lambda^{-2} \geq c_1 - t_0^{-1}c_2\lambda_0 \geq c_1/2 =: c_3$, we get

$$\mathbb{P}_x\big(X_{\tau_{\widehat{D}}} \in B\big) \geq u(t)(c_1 - c_2t^{-1}\lambda^{-2})\lambda\delta_D(x) \geq c_3u(t_0)\lambda\delta_D(x).$$

□

5 Carleson estimate and Boundary Harnack principle

Recall that for any open set $U \subset \mathbb{R}^d$, $\tau_U = \inf\{t > 0 : X_t \notin U\}$ is the first exit time from U by X.

Lemma 5.1 *For every $\varrho > 0$, there exists $c = c(\varrho) > 0$ such that for every $x_0 \in \mathbb{R}^d$ and $r \in (0, \varrho]$,*

$$c^{-1} r^2 \le \mathbb{E}_{x_0}\left[\tau_{B(x_0,r)}\right] \le c\, r^2. \tag{5.1}$$

Proof. See [23] for a proof. □

In this section, we give the proof of the boundary Harnack principle for X. We first prove the Carleson estimate for X on Lipschitz open sets.

We recall that an open set D in $\mathbb{R}^d$ is said to be a Lipschitz open set if there exist a localization radius $R_2 > 0$ and a constant $\Lambda_1 > 0$ such that for every $Q \in \partial D$, there exist a Lipschitz function $\psi = \psi_Q : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\psi(0) = 0$, $|\psi(x) - \psi(y)| \le \Lambda_1 |x - y|$, and an orthonormal coordinate system CS_Q: $y = (y_1, \ldots, y_{d-1}, y_d) =: (\widetilde{y}, y_d)$ with its origin at Q such that

$$B(Q, R_2) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R_2) \text{ in } CS_Q : y_d > \psi(\widetilde{y})\}.$$

The pair (R_2, Λ_1) is called the characteristics of the Lipschitz open set D. Without loss of generality, we will assume throughout this section that $R_1 < 1$. Note that a Lipschitz open set can be unbounded and disconnected. For Lipschitz open set D and every $Q \in \partial D$ and $x \in B(Q, R_2) \cap D$, we define

$$\rho_Q(x) := x_d - \psi_Q(\tilde{x}),$$

where $(\tilde{x}, x_d)$ are the coordinates of x in CS_Q.

The proof of the next lemma is similar to that of [10, Lemma 4.1].

Lemma 5.2 *Let $D \subset \mathbb{R}^d$ be a Lipschitz open set with characteristics (R_2, Λ_1). There exists a constant $\delta = \delta(R_2, \Lambda_1) > 0$ such that for all $Q \in \partial D$ and $x \in D$ with $\rho_Q(x) < R_2/2$,*

$$\mathbb{P}_x(X_{\tau(x)} \in D^c) \ge \delta,$$

where $\tau(x) := \tau_{D \cap B(x, 2\rho_Q(x))} = \inf\{t > 0 : X_t \notin D \cap B(x, 2\rho_Q(x))\}$.

Proof. Let $D_x := D \cap B(x, 2\rho_Q(x))$ and W^{D_x} be the killed Brownian motion in D_x. As in the proof of Lemma 4.4, we define the subordinate killed Brownian motion $Y = (Y_t : t \ge 0)$ in D_x by $Y_t := W^{D_x}(S_t)$. We will use ζ to denote the lifetime of Y and let $C_x := \partial D \cap B(x, 2\rho_Q(x))$ and $\tau^W_U := \inf\{t > 0 : W_t \notin U\}$.

Since, see [29], $\mathbb{P}_x\left(X_{\tau(x)} \in C_x\right) \ge \mathbb{P}_x\left(Y_{\zeta-} \in C_x\right) = \mathbb{E}_x\left[u(\tau^W_{D_x}); W_{\tau^W_{D_x}} \in C_x\right]$, we have

$$\begin{aligned}
&\mathbb{P}_x\left(X_{\tau(x)} \in D^c\right) \ge \mathbb{P}_x\left(X_{\tau(x)} \in C_x\right) \ge \mathbb{E}_x\left[u(\tau^W_{D_x}); W_{\tau^W_{D_x}} \in C_x, \tau^W_{D_x} \le t\right] \\
&\ge u(t)\mathbb{P}_x\left[W_{\tau^W_{D_x}} \in C_x, \tau^W_{D_x} \le t\right] \ge u(t)\left(\mathbb{P}_x(W_{\tau^W_{D_x}} \in C_x) - \mathbb{P}_x(\tau^W_{D_x} > t)\right), \quad t > 0.
\end{aligned} \tag{5.2}$$

By the fact that D is a Lipschitz open set, there exists $c_1 = c_1(R_2, \Lambda_1) > 0$ such that

$$\mathbb{P}_x(W_{\tau^W_{D_x}} \in C_x) \ge c_1. \tag{5.3}$$

(See the proof of [10, Lemma 4.1].) Since

$$\mathbb{P}_x(\tau^W_{D_x} > t) \le \frac{\mathbb{E}_x[\tau^W_{D_x}]}{t} \le \frac{\mathbb{E}_x[\tau^W_{B(x,2\rho_Q(x))}]}{t} \le c_2 \frac{(\rho_Q(x))^2}{t} \le c_2 \frac{R_2^2}{t},$$

by using (5.3) and (5.2), we obtain that

$$\mathbb{P}_x\left(X_{\tau(x)} \in D^c\right) \ge u(t)\left(\mathbb{P}_x(W_{\tau^W_{D_x}} \in C_x) - \mathbb{P}_x(\tau^W_{D_x} > t)\right) \ge u(t)\left(c_1 - c_2\frac{R_1^2}{t}\right) \ge c_1 u(t_0)/2 > 0,$$

where $t_0 = t_0(R_2, \Lambda_1) > 0$ is chosen so that $c_1 - c_2 R_2^2/t \ge c_1/2$. The lemma is thus proved. □

Suppose that D is an open set and that U and V are bounded open sets with $V \subset \overline{V} \subset U$ and $D \cap V \ne \emptyset$. If f vanishes continuously on $D^c \cap U$, then by a finite covering argument, it is easy to see that f is bounded in an open neighborhood of $\partial D \cap V$. The proof of the next result is the same as that of [10, Lemma 4.2]. So we omit the proof.

Lemma 5.3 *Let D be an open set and U and V be bounded open sets with $V \subset \overline{V} \subset U$ and $D \cap V \ne \emptyset$. Suppose f is a nonnegative function in $\mathbb{R}^d$ that is harmonic in $D \cap U$ with respect to X and vanishes continuously on $D^c \cap U$. Then f is regular harmonic in $D \cap V$ with respect to X, i.e.,*

$$f(x) = \mathbb{E}_x\left[f(X_{\tau_{D\cap V}})\right] \qquad \text{for all } x \in D \cap V\,. \tag{5.4}$$

Theorem 5.4 (Carleson estimate) *Let $D \subset \mathbb{R}^d$ be a Lipschitz open set with the characteristics (R_2, Λ_1). Then there exists a positive constant $A = A(R_2, \Lambda_1)$ such that for every $Q \in \partial D$, $0 < r < R_2/2$, and any nonnegative function f in $\mathbb{R}^d$ that is harmonic in $D \cap B(Q, r)$ with respect to X and vanishes continuously on $D^c \cap B(Q, r)$, we have*

$$f(x) \le A f(x_0) \qquad \text{for } x \in D \cap B(Q, r/2), \tag{5.5}$$

where $x_0 \in D \cap B(Q, r)$ with $\rho_Q(x_0) = r/2$.

Proof. Since D is Lipschitz and $r < R_2/2$, by the Harnack inequality and a standard chain argument, it suffices to prove (5.5) for $x \in D \cap B(Q, r/12)$ and $\widetilde{x}_0 = \widetilde{Q}$. Without loss of generality, we may assume that $f(x_0) = 1$. In this proof, the constants δ, β, η and c_i's are always independent of r.

Let $\nu = \nu(3)$ be the constant in (2.7) with $K = 3$, choose $0 < \gamma < (\nu^{-1} \wedge (1 - \nu^{-1}))$ and let

$$B_0(x) = D \cap B(x, 2\rho_Q(x))\,, \qquad B_1(x) = B(x, r^{1-\gamma}\rho_Q(x)^\gamma)$$

and

$$B_2 = B(x_0, \rho_Q(x_0)/3)\,, \qquad B_3 = B(x_0, 2\rho_Q(x_0)/3).$$

By Lemma 5.2, there exists $\delta = \delta(R_2, \Lambda_1) > 0$ such that

$$\mathbb{P}_x(X_{\tau_{B_0(x)}} \in D^c) \ge \delta\,, \quad x \in B(Q, r/4)\,. \tag{5.6}$$

By the Harnack inequality and a chain argument, there exists $\beta > 0$ such that

$$f(x) < (\rho_Q(x)/r)^{-\beta} f(x_0)\,, \quad x \in D \cap B(Q, r/4)\,. \tag{5.7}$$

In view of Lemma 5.3, f is regular harmonic in $B_0(x)$ with respect to X. So

$$f(x) = \mathbb{E}_x\big[f\big(X_{\tau_{B_0(x)}}\big); X_{\tau_{B_0(x)}} \in B_1(x)\big] + \mathbb{E}_x\big[f\big(X_{\tau_{B_0(x)}}\big); X_{\tau_{B_0(x)}} \notin B_1(x)\big] \qquad \text{for } x \in B(Q, r/4). \tag{5.8}$$

We first show that there exists $\eta > 0$ such that

$$\mathbb{E}_x\big[f\big(X_{\tau_{B_0(x)}}\big); X_{\tau_{B_0(x)}} \notin B_1(x)\big] \le f(x_0) \quad \text{if } x \in D \cap B(Q, r/12) \text{ with } \rho_Q(x) < \eta r\,. \tag{5.9}$$

Let $\eta_0 := 2^{-2\nu}$, then, since $\gamma < 1 - \nu^{-1}$, for $\rho_Q(x) < \eta_0 r$,

$$2\rho_Q(x) \le r^{1-\gamma}\rho_Q(x)^\gamma - 2\rho_Q(x).$$

Thus if $x \in D \cap B(Q, r/12)$ with $\rho_Q(x) < \eta_0 r$, then $|x - y| \le 2|z - y|$ for $z \in B_0(x)$, $y \notin B_1(x)$. Moreover, by the triangle inequality, $|x-y| \le |x-z| + |z-y| \le 1 + |z-y|$. Thus we have by (2.5), (2.6), (2.3) and Lemma 5.1

$$\begin{aligned}
&\mathbb{E}_x\big[f\big(X_{\tau_{B_0(x)}}\big); X_{\tau_{B_0(x)}} \notin B_1(x)\big] \\
=&\mathbb{E}_x \int_0^{\tau_{B_0(x)}} \int_{2>|y-x|>r^{1-\gamma}\rho_Q(x)^\gamma} j(|X_t - y|) f(y)\, dy\, dt + \mathbb{E}_x \int_0^{\tau_{B_0(x)}} \int_{|y-x|>2} j(|X_t - y|) f(y)\, dy\, dt \\
\le& c_1 \mathbb{E}_x[\tau_{B_0(x)}] \left(\int_{2>|y-x|>r^{1-\gamma}\rho_Q(x)^\gamma} j(|x-y|) f(y)\, dy + \int_{|y-x|>2} j(|x-y|) f(y)\, dy \right) \\
\le& c_1 c_2 \rho_Q(x)^2 \left(\int_{|y-x|>r^{1-\gamma}\rho_Q(x)^\gamma, |y-x_0|>2\rho_Q(x_0)/3} j(|x-y|) f(y)\, dy \right. \\
&\qquad \left. + \int_{|y-x_0|\le 2\rho_Q(x_0)/3} j(|x-y|) f(y)\, dy \right) =: c_3 \rho_Q(x)^2 (I_1 + I_2)\,.
\end{aligned} \tag{5.10}$$

On the other hand, for $z \in B_2$ and $y \notin B_3$, we have $|z-y| \le |z-x_0| + |x_0-y| \le \rho_Q(x_0)/3 + |x_0-y| \le 2|x_0-y|$ and $|z-y| \le |z-x_0| + |x_0-y| \le 1 + |x_0-y|$. We have again by (2.3), (2.5), (2.6) and Lemma 5.1

$$\begin{aligned}
f(x_0) \;\ge\; & \mathbb{E}_{x_0}\left[f(X_{\tau_{B_2}}), X_{\tau_{B_2}} \notin B_3\right] \\
\ge\; & \mathbb{E}_{x_0} \int_0^{\tau_{B_2}} \left(\int_{2>|y-x_0|>2\rho_Q(x_0)/3} j(|X_t - y|) f(y)\, dy + \int_{|y-x_0|\ge 2} j(|X_t-y|) f(y)\, dy \right) dt \\
\ge\; & c_4 \mathbb{E}_{x_0}[\tau_{B_2}] \left(\int_{2>|y-x_0|>2\rho_Q(x_0)/3} j(|x_0-y|) f(y)\, dy + \int_{|y-x_0|\ge 2} j(|x_0-y|) f(y)\, dy \right) \\
=\; & c_5 \rho_Q(x_0)^2 \int_{|y-x_0|>2\rho_Q(x_0)/3} j(|x_0-y|) f(y)\, dy\,.
\end{aligned} \tag{5.11}$$

Suppose now that $|y-x| \ge r^{1-\gamma}\rho_Q(x)^\gamma$ and $x \in B(Q, r/4)$. Then

$$|y - x_0| \le |y-x| + r \le |y-x| + r^\gamma \rho_Q(x)^{-\gamma}|y-x| \le 2r^\gamma \rho_Q(x)^{-\gamma}|y-x|.$$

Thus, using (2.7), we get for $|x-y| \le 2$,

$$j(|y-x|) \le c_7 (\rho_Q(x)/r)^{-\nu\gamma} j(|y-x_0|). \tag{5.12}$$

Now, using (2.5), (2.6) and (5.12),

$$
\begin{aligned}
I_1 &\le c_7 \int_{R_0/2>|y-x|>r^{1-\gamma}\rho_Q(x)^\gamma, |y-x_0|>2\rho_Q(x_0)/3} (\rho_Q(x)/r)^{-\nu\gamma} j(|y-x_0|) f(y)\, dy \\
&\quad + c_8 \int_{|y-x|\ge R_0/2, |y-x_0|>2\rho_Q(x_0)/3} j(|x_0-y|) f(y)\, dy \\
&\le c_9 \left((\rho_Q(x)/r)^{-\nu\gamma} + 1 \right) \int_{|y-x_0|>2\rho_Q(x_0)/3} j(|x_0-y|)\, f(y)\, dy \\
&\le c_5^{-1} c_9 \rho_Q(x_0)^{-2} \left((\rho_Q(x)/r)^{-\nu\gamma} + 1 \right) f(x_0) \\
&\le 2 c_5^{-1} c_9 (\rho_Q(x)/r)^{-\nu\gamma} \rho_Q(x_0)^{-2} f(x_0)\,, \qquad (5.13)
\end{aligned}
$$

where the second to last inequality is due to (5.11).

If $|y-x_0| < 2\rho_Q(x_0)/3$, then $|y-x| \ge |x_0-Q| - |x-Q| - |y-x_0| > \rho_Q(x_0)/6$. This together with the Harnack inequality implies that

$$
\begin{aligned}
I_2 &\le c_{10} \int_{|y-x_0|\le 2\rho_Q(x_0)/3} j(|x-y|) f(x_0)\, dy \le c_{10} f(x_0) \int_{|y-x|>\rho_Q(x_0)/6} j(|x-y|)\, dy \\
&= c_{10} f(x_0) \left(\int_{R_0>|z|>\rho_Q(x_0)/6} j(|z|)\, dz + \int_{R_0\le |z|} j(|z|)\, dz \right) \\
&\le c_{10} f(x_0) \left(\int_{R_0>|z|>\rho_Q(x_0)/6} j(|z|)\, dz + c_{11} \right). \qquad (5.14)
\end{aligned}
$$

Combining (5.10), (5.13) and (5.14) we obtain

$$
\begin{aligned}
&\mathbb{E}_x[f(X_{\tau_{B_0(x)}});\, X_{\tau_{B_0(x)}} \notin B_1(x)] \\
\le\ & c_{12} f(x_0) \Big(\rho_Q(x)^2 (\rho_Q(x)/r)^{-\gamma\nu} \rho_Q(x_0)^{-2} \\
&\quad (\rho_Q(x)/r)^2 (\rho_Q(x_0)/6)^2 \int_{R_0>|z|>\rho_Q(x_0)/6} j(|z|)\, dz + (\rho_Q(x)/r)^2 r^2 \Big) \\
\le\ & c_{13} f(x_0) \left((\rho_Q(x)/r)^{2-\gamma\nu} + (\rho_Q(x)/r)^2 \Big(\int_{R_0>|z|>\rho_Q(x_0)/6} |z|^2 j(|z|)\, dz + 1 \Big) \right) \\
\le\ & c_{14} f(x_0) \left((\rho_Q(x)/r)^{2-\gamma\nu} + (\rho_Q(x)/r)^2 \right), \qquad (5.15)
\end{aligned}
$$

where we used the fact that $\rho_Q(x_0) = r/2$. Since $2 - \gamma\nu > 0$, choose now $\eta \in (0, \eta_0)$ so that

$$
c_{14}\ \left(\eta^{2-\gamma\nu} + \eta^2 \right)\ \le\ 1\,.
$$

Then for $x \in D \cap B(Q, r/12)$ with $\rho_Q(x) < \eta r$, we have by (5.15),

$$
\mathbb{E}_x \left[f(X_{\tau_{B_0(x)}});\, X_{\tau_{B_0(x)}} \notin B_1(x) \right] \ \le\ c_{14}\, f(x_0) \left(\eta^{2-\gamma\nu} + \eta^2 \right) \le f(x_0)\,.
$$

We now prove the Carleson estimate (5.5) for $x \in D \cap B(Q, r/12)$ by a method of contradiction. Recall that $f(x_0) = 1$. Suppose that there exists $x_1 \in D \cap B(Q, r/12)$ such that $f(x_1) \ge K > \eta^{-\beta} \vee (1+\delta^{-1})$, where K is a constant to be specified later. By (5.7) and the assumption $f(x_1) \ge K > \eta^{-\beta}$, we have $(\rho_Q(x_1)/r)^{-\beta} > f(x_1) \ge K > \eta^{-\beta}$, and hence $\rho_Q(x_1) < \eta r$. By (5.8) and (5.9),

$$
K \le f(x_1) \le \mathbb{E}_{x_1} \left[f(X_{\tau_{B_0(x_1)}}); X_{\tau_{B_0(x_1)}} \in B_1(x_1) \right] + 1\,,
$$

and hence

$$\mathbb{E}_{x_1}\left[f(X_{\tau_{B_0(x_1)}}); X_{\tau_{B_0(x_1)}} \in B_1(x_1)\right] \ge f(x_1) - 1 > \frac{1}{1+\delta} f(x_1)\,.$$

In the last inequality of the display above we used the assumption that $f(x_1) \ge K > 1 + \delta^{-1}$. If $K \ge 2^{\beta/\gamma}$, then $D^c \cap B_1(x_1) \subset D^c \cap B(Q, r)$. By using the assumption that $f = 0$ on $D^c \cap B(Q, r)$, we get from (5.6)

$$\begin{aligned}\mathbb{E}_{x_1}[f(X_{\tau_{B_0(x_1)}}), X_{\tau_{B_0(x_1)}} \in B_1(x_1)] &= \mathbb{E}_{x_1}[f(X_{\tau_{B_0(x_1)}}), X_{\tau_{B_0(x_1)}} \in B_1(x_1) \cap D] \\ &\le \mathbb{P}_x(X_{\tau_{B_0(x_1)}} \in D) \sup_{B_1(x_1)} f \le (1-\delta) \sup_{B_1(x_1)} f\,.\end{aligned}$$

Therefore, $\sup_{B_1(x_1)} f > f(x_1)/((1+\delta)(1-\delta))$, i.e., there exists a point $x_2 \in D$ such that

$$|x_1 - x_2| \le r^{1-\gamma}\rho_Q(x_1)^\gamma \quad \text{and} \quad f(x_2) > \frac{1}{1-\delta^2} f(x_1) \ge \frac{1}{1-\delta^2} K\,.$$

By induction, if $x_k \in D \cap B(Q, r/12)$ with $f(x_k) \ge K/(1-\delta^2)^{k-1}$ for $k \ge 2$, then there exists $x_{k+1} \in D$ such that

$$|x_k - x_{k+1}| \le r^{1-\gamma}\rho_Q(x_k)^\gamma \quad \text{and} \quad f(x_{k+1}) > \frac{1}{1-\delta^2} f(x_k) > \frac{1}{(1-\delta^2)^k} K\,. \tag{5.16}$$

From (5.7) and (5.16) it follows that $\rho_Q(x_k)/r \le (1-\delta^2)^{(k-1)/\beta} K^{-1/\beta}$, for every $k \ge 1$. Therefore,

$$\begin{aligned}|x_k - Q| &\le |x_1 - Q| + \sum_{j=1}^{k-1} |x_{j+1} - x_j| \le \frac{r}{12} + \sum_{j=1}^{\infty} r^{1-\gamma}\rho_Q(x_j)^\gamma \\ &\le \frac{r}{12} + r^{1-\gamma}\sum_{j=1}^{\infty}(1-\delta^2)^{(j-1)\gamma/\beta} K^{-\gamma/\beta} r^\gamma = \frac{r}{12} + r^{1-\gamma} r^\gamma K^{-\gamma/\beta}\sum_{j=0}^{\infty}(1-\delta^2)^{j\gamma/\beta} \\ &= \frac{r}{12} + rK^{-\gamma/\beta}\frac{1}{1-(1-\delta^2)^{\gamma/\beta}}.\end{aligned}$$

Choose

$$K = \eta \vee (1+\delta^{-1}) \vee 12^{\beta/\gamma}(1-(1-\delta^2)^{\gamma/\beta})^{-\beta/\gamma}.$$

Then $K^{-\gamma/\beta}(1-(1-\delta^2)^{\gamma/\beta})^{-1} \le 1/12$, and hence $x_k \in D \cap B(Q, r/6)$ for every $k \ge 1$. Since $\lim_{k\to\infty} f(x_k) = +\infty$, this contradicts the fact that f is bounded on $B(Q, r/2)$. This contradiction shows that $f(x) < K$ for every $x \in D \cap B(Q, r/12)$. This completes the proof of the theorem. □

Proof of Theorem 1.2 . We recall that $R_1 = R/(4\sqrt{1+(1+\Lambda)^2})$ and $\lambda_0 > 2R_1^{-1}$ and $\kappa_0 \in (0,1)$ are the constants in the statement of Lemma 4.3.

Since D is a $C^{1,1}$ open set and $r < R$, by the Harnack inequality and a standard chain argument, it suffices to prove (1.4) for $x, y \in D \cap B(Q, 2^{-1}r\kappa_0\lambda_0^{-1})$. In this proof, the constants η and c_i's are always independent of r.

For any $r \in (0, R]$ and $x \in D \cap B(Q, 2^{-1}r\kappa_0\lambda_0^{-1})$, let Q_x be the point $Q_x \in \partial D$ so that $|x - Q_x| = \delta_D(x)$ and let $x_0 := Q_x + \frac{r}{8}(x - Q_x)/|x - Q_x|$. We choose a $C^{1,1}$-function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\varphi(0) = 0$, $\nabla\varphi(0) = (0, \dots, 0)$, $\|\nabla\varphi\|_\infty \le \Lambda$, $|\nabla\varphi(y) - \nabla\varphi(z)| \le \Lambda|y - z|$, and an orthonormal coordinate system CS with its origin at Q_x such that

$$B(Q_x, R) \cap D = \{y = (\widetilde{y}, y_d) \in B(0, R) \text{ in } CS : y_d > \varphi(\widetilde{y})\}.$$

In the coordinate system CS we have $\widetilde{x} = \widetilde{0}$ and $x_0 = (\widetilde{0}, r/8)$. For any $b_1, b_2 > 0$, we define

$$D(b_1, b_2) := \left\{ y = (\widetilde{y}, y_d) \text{ in } CS : 0 < y_d - \varphi(\widetilde{y}) < b_1 r \kappa_0 \lambda_0^{-1},\ |\widetilde{y}| < b_2 r \lambda_0^{-1} \right\}.$$

It is easy to see that $D(2,2) \subset D \cap B(Q, r/2)$. In fact, since $\Lambda \geq 1$ and $R \leq 1$, for every $z \in D(2,2)$,

$$\begin{aligned}
&|z - Q| \leq |Q - x| + |x - Q_x| + |Q_x - z| \leq |Q - x| + |x - Q_x| + |z_d - \varphi(\widetilde{z})| + |\varphi(\widetilde{z})| \\
&< r\lambda_0^{-1}((1+\Lambda) + 4) < 2^{-1} r R((1+\Lambda)+4)/(4\sqrt{1+(1+\Lambda)^2}) \leq \frac{r}{2}.
\end{aligned}$$

Thus if f is a nonnegative function on $\mathbb{R}^d$ that is harmonic in $D \cap B(Q, r)$ with respect to X and vanishes continuously in $D^c \cap B(Q, r)$, then, by Lemma 5.3, f is regular harmonic in $D \cap B(Q, r/2)$ with respect to X, hence also in $D(2,2)$. Thus by the Harnack inequality, we have

$$\begin{aligned}
f(x) &= \mathbb{E}_x\left[f\big(X_{\tau_{D(1,1)}}\big)\right] \geq \mathbb{E}_x\left[f\big(X_{\tau_{D(1,1)}}\big); X_{\tau_{D(1,1)}} \in D(2,1)\right] \\
&\geq c_1 f(x_0) \mathbb{P}_x\Big(X_{\tau_{D(1,1)}} \in D(2,1)\Big) \geq c_2 f(x_0) \delta_D(x)/r.
\end{aligned} \tag{5.17}$$

In the last inequality above we have used (4.31).

Let $w = (\widetilde{0}, r\lambda_0^{-1}\kappa_0 4)$. Then it is easy to see that there exists a constant $\eta = \eta(\Lambda, \delta_0) \in (0, 1/4)$ such that $B(w, \eta r \lambda_0^{-1}\kappa_0) \in D(1,1)$. By (2.5), (2.6), (2.3) and Lemma 5.1,

$$\begin{aligned}
&f(w) \geq \mathbb{E}_w\left[f\big(X_{\tau_{D(1,1)}}\big); X_{\tau_{D(1,1)}} \notin D(2,2)\right] = \mathbb{E}_w \int_0^{\tau_{D(1,1)}} \int_{\mathbb{R}^d \setminus D(2,2)} f(y) j(|X_t - y|) dy dt \\
&\geq c_3\, \mathbb{E}_w\left[\tau_{B(w, \eta r \lambda_0^{-1}\kappa_0)}\right] \int_{\mathbb{R}^d \setminus D(2,2)} f(y) j(|w-y|) dy \geq c_4\, r^2 \int_{\mathbb{R}^d \setminus D(2,2)} f(y) j(|w-y|) dy.
\end{aligned}$$

Hence by (2.5), (2.6), (4.16),

$$\begin{aligned}
&\mathbb{E}_x\left[f\left(X_{\tau_{D(1,1)}}\right);\, X_{\tau_{D(1,1)}} \notin D(2,2)\right] = \mathbb{E}_x \int_0^{\tau_{D(1,1)}} \int_{\mathbb{R}^d \setminus D(2,2)} f(y) j(|X_t - y|) dy dt \\
&\leq c_5\, \mathbb{E}_x[\tau_{D(1,1)}] \int_{\mathbb{R}^d \setminus D(2,2)} f(y) j(|w-y|) dy \\
&\leq c_6\, \delta_D(x) r \int_{\mathbb{R}^d \setminus D(2,2)} f(y) j(|w-y|) dy \leq \frac{c_6\, \delta_D(x)}{c_4\, r} f(w).
\end{aligned}$$

On the other hand, by the Harnack inequality and the Carleson estimate, we have

$$\mathbb{E}_x\left[f\left(X_{\tau_{D(1,1)}}\right);\, X_{\tau_{D(1,1)}} \in D(2,2)\right] \leq c_7\, f(x_0) \mathbb{P}_x\left(X_{\tau_{D(1,1)}} \in D(2,2)\right) \leq c_8\, f(x_0) \delta_D(x)/r.$$

In the last inequality above we have used (4.15). Combining the two inequalities above, we get

$$\begin{aligned}
&f(x) = \mathbb{E}_x\left[f\left(X_{\tau_{D(1,1)}}\right);\, X_{\tau_{D(1,1)}} \in D(2,2)\right] + \mathbb{E}_x\left[f\left(X_{\tau_{D(1,1)}}\right);\, X_{\tau_{D(1,1)}} \notin D(2,2)\right] \\
&\leq \frac{c_8}{r} \delta_D(x) f(x_0) + \frac{c_6\, \delta_D(x)}{c_4\, r} f(w) \leq \frac{c_9}{r}\, \delta_D(x)(f(x_0) + f(w)) \leq \frac{c_{10}}{r}\, \delta_D(x) f(x_0).
\end{aligned} \tag{5.18}$$

In the last inequality above we have used the Harnack inequality.

From (5.17)–(5.18), we have that for every $x, y \in D \cap B(Q, 2^{-1} r \kappa_0 \lambda_0^{-1})$,

$$\frac{f(x)}{f(y)} \leq \frac{c_{10}}{c_2} \frac{\delta_D(x)}{\delta_D(y)},$$

which proves the theorem. □

6 Counterexample

In this section, we present an example of a (bounded) $C^{1,1}$ domain (open and connected) D on which the boundary Harnack principle for the independent sum of a Brownian motion and a finite range rotationally invariant Lévy process fails, even for regular harmonic function vanishing on D^c. A similar example appears in [18, Section 6] for the case of truncated stable process.

Suppose that Z is a rotationally invariant Lévy process whose Lévy measure has a density $J(x) = j(|x|)$ with $j(r) = 0$ for all $r \geq 1$. Suppose that Z is independent of the Brownian motion W. We will consider the process $Y = W + Z$. For any Borel sets U and V in $\mathbb{R}^d$ with $V \subset \overline{U}^c$, we have

$$\mathbb{P}_x(Y_{\tau_U^Y} \in V) = \mathbb{E}_x \int_0^{\tau_U^Y} \int_V j(|Y_t - z|) \mathbf{1}_{\{|Y_t - z| < 1\}}(|Y_t - z|)\, dz\, dt \quad x \in U, \tag{6.1}$$

where $\tau_U^Y := \inf\{t > 0 : Y_t \notin U\}$.

We consider the bounded domain in $\mathbb{R}^d$

$$D := (-100, 100)^d \setminus \Big((-100, 49]^{d-1} \times [-1/2, 0]\Big).$$

Suppose that the (not necessarily scale invariant) boundary Harnack principle is true for Y on D at the origin for regular harmonic function vanishing on D^c, i.e., there exist constants $R_1 > 0$ and $M_1 > 1$ such that for any $r < R_1$ and any nonnegative functions u, v on $\mathbb{R}^d$ which are regular harmonic with respect to Y in $D \cap B(0, M_1 r)$ and vanish in D^c, we have

$$\frac{u(x)}{v(x)} \leq c_r \frac{u(y)}{v(y)} \quad \text{for any } x, y \in D \cap B(0, r), \tag{6.2}$$

where $c_r = c_r(D) > 0$ is independent of the harmonic functions u and v. Choose an $r_1 < R_1$ with $M_1 r_1 < 1/2$ and let $A := (\widetilde{0}, \frac{1}{2} r_1)$. We define a function v by

$$v(x) := \mathbb{P}_x\left(Y_{\tau^Y_{D \cap B(0, M_1 r_1)}} \in \{y \in D; y_d > 0\}\right).$$

By definition v is regular harmonic in $D \cap B(0, M_1 r_1)$ with respect to Y and vanishes in D^c. Applying the function v above to (6.2), we get a Carleson type estimate at 0, i.e., for any nonnegative function u which is regular harmonic with respect to Y in $D \cap B(0, M_1 r_1)$ and vanishes in D^c we have

$$u(A) \geq c_{r_1}^{-1} \frac{v(A)}{v(x)} u(x) \geq c_{r_1}^{-1} v(A) u(x) = c_1\, u(x), \quad x \in D \cap B(0, r_1), \tag{6.3}$$

where $c_1 = c_{r_1}^{-1} v(A) > 0$. We will construct a bounded positive function u on $\mathbb{R}^d$ which is regular harmonic with respect to Y in $D \cap B(0, M_1 r)$ and vanishes in D^c for which (6.3) fails.

For $n \geq 1$, we put

$$\begin{aligned} C_n &:= \left\{(\widetilde{x}, x_d) \in D : \quad |\widetilde{x}| \leq 2^{-n-3} r_1, \quad x_2 \leq -1 + 2^{-n} r_1^2\right\}, \\ D_n &:= \{(\widetilde{y}, y_d) \in D : \quad y_d > 0, \quad |x - y| < 1 \quad \text{for some } x \in C_n\}. \end{aligned}$$

It is easy to see that

$$\overline{D_n} \subset \{(\widetilde{y}, y_d) : |\widetilde{y}| \leq (2^{-n-3} + 2^{-(n-1)/2}) r_1, 0 \leq y_d \leq 2^{-n} r_1^2\} \subset B(0, r_1) \cap D, \quad \text{for } n \geq 2. \tag{6.4}$$

In fact, for any $y \in \overline{D_n}$, we have $y_d \in [0, 2^{-n} r_1^2]$ and $|y - x| \le 1$ for some $x \in C_n$. If $|\widetilde{y}| > (2^{-n-3} + 2^{-(n-1)/2}) r_1$, $y_d \ge 0$ and $x \in C_n$, then

$$|x - y|^2 \ge x_d^2 + (|\widetilde{y}| - |\widetilde{x}|)^2 \ge (1 - 2^{-n} r_1^2)^2 + 2^{-(n-1)} r_1^2 > 1.$$

Thus, in this case $y \notin \overline{D_n}$.

For any n, let $T^Y_{D_n}$ be the first hitting time of D_n by the process Y. By (6.4)

$$\mathbb{P}_A \left(\tau^Y_{D \cap B(0, M_1 r_1)} > T^Y_{D_n} \right) \to \mathbb{P}_A \left(\tau^Y_{D \cap B(0, M_1 r_1)} > T^Y_{\{0\}} \right) = 0, \qquad \text{as } n \to \infty.$$

Fix $n_0 \ge 2$ large so that

$$\mathbb{P}_A \left(\tau^Y_{D \cap B(0, M_1 r_1)} > T^Y_{D_{n_0}} \right) < \frac{c_1}{2} \tag{6.5}$$

and define

$$u(x) := \mathbb{P}_x \left(Y_{\tau^Y_{D \cap B(0, M_1 r_1)}} \in C_{n_0} \right).$$

u is a nonnegative bounded function which is regular harmonic in $D \cap B(0, M_1 r_1)$ with respect to Y and vanishes in D^c. It also vanishes continuously on $\partial D \cap B(0, M_1 r_1)$. Note that by (6.1),

$$\mathbb{P}_A \left(Y_{\tau^Y_{D \cap B(0, M_1 r_1)}} \in C_{n_0}, \, \tau^Y_{D \cap B(0, M_1 r_1)} \le T^Y_{D_{n_0}} \right) = \mathbb{P}_A \left(Y_{\tau^Y_{D \cap B(0, M_1 r_1) \setminus D_{n_0}}} \in C_{n_0} \right) = 0.$$

Thus by the strong Markov property,

$$\begin{aligned} u(A) &= \mathbb{P}_A \left(Y_{\tau^Y_{D \cap B(0, M_1 r_1)}} \in C_{n_0}, \, \tau^Y_{D \cap B(0, M_1 r_1)} > T^Y_{D_{n_0}} \right) \\ &= \mathbb{E}_A \left[\mathbb{P}_{Y_{T^Y_{D_{n_0}}}} \left(Y_{\tau^Y_{D \cap B(0, M_1 r_1)}} \in C_{n_0} \right); \, \tau^Y_{D \cap B(0, M_1 r_1)} > T^Y_{D_{n_0}} \right] \\ &\le \mathbb{P}_A \left(\tau^Y_{D \cap B(0, M_1 r_1)} > T^Y_{D_{n_0}} \right) \left(\sup_{x \in D_{n_0}} u(x) \right) < \frac{c_1}{2} \left(\sup_{x \in D \cap B(0, r_1)} u(x) \right). \end{aligned}$$

In the last inequality above, we have used (6.4)–(6.5). But by (6.3), $u(A) \ge c_1 \sup_{x \in D \cap B(0, r_1)} u(x)$, which gives a contradiction. Thus the boundary Harnack principle is not true for D at the origin.

By smoothing off the corners of D, we can easily construct a bounded $C^{1,1}$ domain on which the boundary Harnack principle for Y fails at 0.

7 Remarks on proofs of Theorems 1.4 and 1.5

As already said in the introduction, once the boundary Harnack principle has been established, the proofs of Theorems 1.4 and 1.5 are similar to the corresponding proofs in [11] for the operator $\Delta + a^\alpha \Delta^{\alpha/2}$. In fact, the proof are even simpler, because [11] strives for uniformity in the weight a.

The proof of Theorem 1.4 in the case $d \ge 3$ is by now quite standard. Once the interior estimates are established, the full estimates in connected $C^{1,1}$ open sets follow from the boundary Harnack principle by the method developed by Bogdan [2] and Hansen [17]. For the operator $\Delta + a\Delta^{\alpha/2}$ this is accomplished in [11, Section 3]. In the present setting the proof from [11] carries over almost verbatim. In several places in [11] on refers to the form of the Lévy density, but in fact, only the form of the constant is what matters there in order to establish uniformity in the weight a.

When $d=2$, the above method ceases to work due to the nature of the logarithmic potential associated with the Laplacian. The proof in [11, Section 4] for the operator $\Delta + a\Delta^{\alpha/2}$ uses a capacitary argument to derive the interior upper bound estimate for the Green function. By a scaling consideration and applying the boundary Harnack principle, one gets sharp Green function upper bound estimates. For the lower bound estimates, [11] compares the process with the subordinate killed Brownian motion when D is connected, and then extend it to general bounded $C^{1,1}$ by using the jumping structure of the process. In the present setting, the proof of the lower bound is exactly the same as in [11] (see proofs of Theorems 4.2 and 4.4). The proof of the upper bound is essentially the same as the one in [11], except that one has to make several minor modifications. Lemma 4.5 in [11] should be replaced by the following statement: There exists $c>0$ such that for any $L>0$,

$$\mathrm{Cap}^0_{B(0,L)}(\overline{B(0,r)}) \geq \frac{c}{\log(L/r)} \quad \text{for every } r \in (0, 3L/4).$$

This is proved in the same way as [11, Lemma 4.5] by using the explicit formula for the Green function of the ball $B(0,L) \subset \mathbb{R}^2$:

$$G^0_{B(0,L)}(x,y) = \frac{1}{2\pi} \log\left(1 + \frac{(L^2-|x|^2)(L^2-|y|^2)}{L^2|x-y|^2}\right).$$

The statement of Lemma 4.6 in [11] should be changed to: There exists $c>0$ such that for any $L>0$ and bounded open set D in $\mathbb{R}^2$ containing $B(0,L)$ and any $x \in \overline{B(0,\frac{3L}{4})}$

$$G_D(x,0) \leq \frac{c}{\mathrm{Cap}^0_D\left(\overline{B(0,|x|/2)}\right)} \mathbb{P}_x\left(\sigma_{\overline{B(0,|x|/2)}} < \tau_D\right),$$

(we refer to [11] for all unexplained notation). Next, Corollary 4.7 in [11] should be replaced by the statement: There exists $c>0$ such that for any $L>0$ and any $x \in \overline{B(0,3L/4)}$

$$G_{B(0,L)}(x,0) \leq c\log\left(L/|x|\right).$$

Finally, the last change is in the proof of Lemma 4.8 in [11] which uses a scaling argument. This in our setting can be circumvented by using the modified statement of [11, Lemma 4.6]. The rest of the proof remains exactly the same.

The proof of Theorem 1.5 is also quite standard. In the current setting we follow step-by-step the proof of the corresponding result in [11] (see Section 6). The main difference is that [11] uses the explicit form of the Lévy density j^a for the operator $\Delta + a\Delta^{\alpha/2}$ which is $c(\alpha,d,a)r^{-d-\alpha}$. This Lévy density is now replaced by j, and it suffice to use properties (2.5) and (2.6) to carry over all arguments. The reader can also compare with [19, Section 6] where the Martin boundary was identified with the Euclidean boundary for purely discontinuous processes whose jumping kernel satisfies (2.5) and (2.6).

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Panki Kim

Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, San56-1 Shinrim-dong Kwanak-gu, Seoul 151-747, Republic of Korea

E-mail: `pkim@snu.ac.kr`

Renming Song

Department of Mathematics, University of Illinois, Urbana, IL 61801, USA

E-mail: `rsong@math.uiuc.edu`

Zoran Vondraček

Department of Mathematics, University of Zagreb, Bijenička c. 30, Zagreb, Croatia

Email: `vondra@math.hr`