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Drift transforms and Green function estimates for discontinuous processes

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Abstract

In this paper, we consider Girsanov transforms of pure jump type for discontinuous Markov processes. We show that, under some quite natural conditions, the Green functions of the Girsanov transformed process are comparable to those of the original process. As an application of the general results, the drift transform of symmetric stable processes is studied in detail. In particular, we show that the relativistic α-stable process in a bounded $C^{1,1}$-smooth open set D can be obtained from symmetric α-stable process in D through a combination of a pure jump Girsanov transform and a Feynman–Kac transform. From this, we deduce that the Green functions for these two processes in D are comparable.

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1. Introduction

Suppose that $\{X, \mathbf{P}_x, x \in E\}$ is a symmetric strong Markov process on a state space E and that K_t is a martingale additive functional of X. Let M_t be the Doleans–Dade exponential of K_t, that is, M_t is the unique solution of the following stochastic

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differential equation:

$$dM_t = M_{t-} dK_t \quad \text{with } M_0 = 1.$$

Then M_t is a nonnegative local martingale and thus is a supermartingale multiplicative functional of X. Let $\mathscr{F}_t$ be the σ-field generated by $\{X_s, s \leqslant t\}$. Then M_t uniquely determines a family of probability measures $\{\mathbf{Q}_x, x \in E\}$ on $\mathscr{F}_\infty$ by $d\mathbf{Q}_x = M_t \, d\mathbf{P}_x$ on $\mathscr{F}_t$, which is called a Girsanov transform by M_t. Let Y_t denote the process X_t under this family of new measures $\{\mathbf{Q}_x, x \in E\}$. It can be shown that Y is a strong Markov process. When X_t is a Brownian motion in $\mathbf{R}^n$ and $K_t = \int_0^t b(X_s)\, dX_s$ for some $\mathbf{R}^n$-valued function b, the corresponding Girsanov transform M_t is given by

$$M_t = \exp\left(\int_0^t b(X_s)\, dX_s - \frac{1}{2} \int_0^t |b(X_s)|^2 \, ds \right)$$

and the process Y obtained from X via M_t is a diffusion process in $\mathbf{R}^n$ with infinitesimal generator $\frac{1}{2}\Delta + b \cdot \nabla$. Thus, in the case of Brownian motion, the effect of a Girsanov transform is perturbing the generator of the process by a drift term. The effect of a Girsanov transform on the generator of X in the general case has been investigated in a recent paper [7] by Chen et al.

In [12], Cranston and Zhao derived sufficient conditions on the drift function b so that the Green functions and harmonic measures in a bounded Lipschitz domain for $\frac{1}{2}\Delta$ and $\frac{1}{2}\Delta + b \cdot \nabla$ are comparable. They proved the above result by showing that the conditional expectation of M_ζ is bounded between two positive numbers, where ζ is the lifetime of the conditional Brownian motion in D. In this paper, we are going to establish an analogue to the result of Cranston and Zhao for pure jump Girsanov transforms of discontinuous Markov processes.

Studying Girsanov transforms for discontinuous processes in depth is important both in theory and in application. It is well known (see, e.g., [20,28]) that many physical and economic systems should be and in fact have been successfully modeled by discontinuous processes, such as stable processes. A symmetric α-stable process in $\mathbf{R}^n$ with $0 < \alpha < 2$ is a Lévy process whose transition density $p(t, y - x)$ relative to the Lebesgue measure is uniquely determined by its Fourier transform $\int_{\mathbf{R}^n} e^{ix\cdot\xi} p(t,x)\, dx = e^{-t|\xi|^\alpha}$. Its infinitesimal generator is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. Stable processes are now widely used in physics, operations research, queuing theory, mathematical finance and risk estimation. In some physics literatures (e.g., [21]), these processes are called Lévy flights, and they have been applied to a wide range of very complex physics issues, such as turbulent diffusion, vortex dynamics, anomalous diffusion in rotating flows, and molecular spectral fluctuations. Another class of discontinues processes that arises from the study of relativistic Hamiltonian system in physics, and the references therein is the family of relativistic α-stable processes (cf. [4,24,25]). A relativistic α-stable process is a Lévy process in $\mathbf{R}^n$ with infinitesimal generator $m - (m^{2/\alpha} - \Delta)^{\alpha/2}$, where $m > 0$ is the mass of a relativistic particle. A relativistic α-stable process Y is intimately related to symmetric α-stable process X. We show in this paper that the part process Y^D of Y in a bounded $C^{1,1}$-smooth open set D (that is, D is a finite union of bounded $C^{1,1}$-domains that are separated from each other by a positive distance) can be obtained from the part process X^D of X in D through a combination of a pure jump Girsanov transform and a Feynman–Kac transform. From this, we can show that the Green function of Y^D is comparable to that of X^D.

In this paper, we will in fact investigate pure jump Girsanov transforms of a much larger class of discontinuous Markov processes which may not be symmetric and to show that, under some quite natural conditions, the Green functions of the Girsanov transformed process are comparable to those of the original process. Note that when X is a purely discontinuous process, for instance when X is a symmetric α-stable process or a relativistic α-stable process, it can be shown that there is no continuous martingale additive functional of X. Thus, any martingale additive functional K_t of X is a pure jump martingale. So the Girsanov transform M_t of X can only be of pure jump type. We call a pure jump Girsanov transform of a discontinuous process X as a drift transform of X. The effect of a drift transform on the process X is perturbing the jumping and killing measures of X. Now let us lay out the general setting of this paper.

Let E be a Lusin space (i.e., a space that is homeomorphic to a Borel subset of a compact metric space) and $\mathscr{B}(E)$ be the Borel σ-algebra on E, and let m be a σ-finite measure on $\mathscr{B}(E)$ with $\operatorname{supp}[m]=E$. Let $X=(\Omega,\mathscr{M},\mathscr{M}_t,X_t,\mathbf{P}_x,x\in E)$ be a Borel standard process on E which is transient in the sense of Getoor [16]. Here, a Borel standard process on the Lusin space E is a strong Markov process satisfying the following conditions: (i) it is right continuous, (ii) it has no branching points, (iii) its resolvents map Borel functions into Borel functions and (iv) it is quasi-left continuous on $(0,\zeta)$, where ζ is the lifetime of the process. The shift operators θ_t, $t\geqslant 0$, satisfy $X_s\circ\theta_t=X_{s+t}$ identically for $s,t\geqslant 0$. Adjoined to the state space E is an isolated point $\partial\notin E$; the process X retires to ∂ at its "lifetime" $\zeta:=\inf\{t\geqslant 0: X_t=\partial\}$. Denote $E\cup\{\partial\}$ by E_∂. The transition semigroup $\{P_t,t\geqslant 0\}$ of X_t is defined by

$$P_tf(x):=\mathbf{E}_x[f(X_t)]=\mathbf{E}_x[f(X_t);t<\zeta].$$

(Here and in the sequel, unless mentioned otherwise, we use the convention that a function defined on E takes the value 0 at the cemetery point ∂.)

Suppose we now have another transient Borel standard process $\hat{X}=(\hat{X}_t,\hat{\mathbf{P}}_x,x\in E)$ on E which is a strong dual of X with respect to the measure m. That is, the semigroup $\{\hat{P}_t\}_{t\geqslant 0}$ of $\hat{X}$ is the dual in $L^2(E,m)$ to the semigroup $\{P_t\}_{t\geqslant 0}$ of X:

$$\int_E f(x)P_tg(x)m(dx)=\int_E g(x)\hat{P}_tf(x)m(dx) \quad \text{for all } f,g\in L^2(E,m).$$

Furthermore for each $\alpha > 0$, the resolvent $\{U_\alpha\}$ and $\{\hat{U}_\alpha\}$ are related as follows: a potential density $G_\alpha(x,y)$ can be chosen to be $\mathscr{B}(E) \times \mathscr{B}(E)$ measurable and satisfy

(a) $U_\alpha(x,dy) = G_\alpha(x,y)m(dy),\ \ \hat{U}_\alpha(x,dy) = G_\alpha(y,x)m(dy)$;
(b) $x \to G_\alpha(x,y)$ is α-excessive for X, $y \to G_\alpha(x,y)$ is α-excessive for $\hat{X}$.

When $\alpha = 0$, we will drop the subscript and write G for G_0. In particular,

$$\mathbf{E}_x\left[\int_0^\zeta f(X_s)\,ds\right] = \int_E G(x,y)f(y)m(dy)$$

for all measurable $f \geqslant 0$ and $G(x,y)$ is called the Green function of X. Throughout this paper, the process X is assumed to be *m-irreducible* in the sense that if a measurable set A has positive m-measure then $\mathbf{P}_x[T_A < \infty] > 0$ for all $x \in E$, where $T_A = \inf\{t > 0, X_t \in A\}$ is the first hitting time of A. Note that the irreducibility condition holds if and only if $G(x,y) > 0$ for all x and y. The "if" part is obvious. For the "only if" part, fix y_0 and observe that $B := \{x : G(x,y_0) > 0\}$ is finely open and nonempty. Thus, $m(B) > 0$ because m is a reference measure. By the irreducibility assumption, $\mathbf{P}_x(T_B < \infty) > 0$ for all x. But, B^c, being the zero set of an excessive function, is absorbing; consequently, $\mathbf{P}_x(T_B < \infty) = 0$ for all $x \in B^c$. It follows that $B^c = \emptyset$.

We call a positive measure μ on E a smooth measure of X if there is a positive continuous additive functional (PCAF in abbreviation) A of X such that

$$\int_E f(x)\mu(dx) = \uparrow \lim_{t \downarrow 0} \mathbf{E}_m\left[\frac{1}{t}\int_0^t f(X_s)\,dA_s\right] \tag{1.1}$$

for any Borel $f \geqslant 0$. Here $\uparrow \lim_{t\downarrow 0}$ means the quantity is increasing as $t \downarrow 0$. The measure μ is called the Revuz measure of A. Throughout this paper, all additive functionals should be understood in the strict sense, that is, in the sense of Blumenthal and Getoor [3]. Since X is assumed to have a Green function, any m-polar set is polar. When X admits a strong dual, a PCAF having bounded 1-potential in the sense of [15] with an exceptional set can be uniquely refined into a PCAF in the strict sense. This can be shown by using the same argument as that of the proof of Theorem 5.1.6 of Fukushima et al. [15]. Therefore, Fitzsimmons and Getoor [14] gives a one-to-one correspondence between smooth measures with bounded 1-potentials and PCAFs in the strict sense with bounded 1-potentials. Note that by Revuz [26] for a PCAF A of X having bounded 1-potential with Revuz measure μ,

$$\mathbf{E}_x[A_\zeta] = \int_E G(x,y)\mu(dx).$$

Under the above assumptions on the process X, one can easily show that every excessive function of X is Borel measurable. By the assumed irreducibility we know that if an excessive function h of X is not identically zero then h is positive everywhere. For any excessive function h of X that is finite m-a.e. on E and not identically zero, let $E_h=\{x\in E: h(x)<\infty\}$ and

$$p^h(t,x,dy)=\frac{1}{h(x)}p(t,x,dy)h(y), \quad t>0,\ x,y\in E_h. \tag{1.2}$$

Then p^h is a transition probability and determines a Borel standard process X^h on E_h with lifetime ζ^h (see [13,22, Proposition 2.2]), which is called Doob's h-transformed process of X. The process X^h is standard because it is locally absolutely continuous with respect to the standard process X. By setting points of $E\backslash E_h$ to be absorbing states for X^h, it was shown in Proposition 5.4 of Getoor and Glover [17] that X^h and $\hat{X}$ are in strong duality on E with respect to the measure hm. For any $x\in E$, we are going to use $\mathbf{P}^h_x$ and $\mathbf{E}^h_x$, respectively, to denote the probability and expectation with respect to the h-conditioned process starting from x. Note that by (1.2), X^h has Green function

$$G^h(x,y):=\frac{G(x,y)}{h(x)} \tag{1.3}$$

with respect to the measure hm.

As is noted in Chen [6], a PCAF of X with Revuz measure μ (with respect to the reference measure m) is a PCAF of X^h with Revuz measure $h\mu$ (with respect to the reference measure hm).

When $h(\cdot)=G(\cdot,y)$ for some $y\in E$, we will use $\mathbf{P}^y_x$ and $\mathbf{E}^y_x$, respectively, to denote the probability and expectation for the h-conditioned process starting from x. In this case, the lifetime ζ^h will be denoted as ζ^y and X^h as X^y.

Let (N,H) be a Lévy system for X (cf. [2,29, Theorem 47.10]); that is, $N(x,dy)$ is a kernel on $(E,\mathscr{B}(E))$ and H_t is a PCAF of X with bounded 1-potential such that for any nonnegative Borel function f on $E\times E$ that vanishes on the diagonal and is extended to be zero off $E\times E$,

$$\begin{aligned}\mathbf{E}_x\left(\sum_{s\leqslant t} f(X_{s-},X_s)\right) &= \mathbf{E}_x\left(\sum_{s\leqslant t} f(X_{s-},X_s)1_{\{s<\zeta\}}\right)\\ &= \mathbf{E}_x\left(\int_0^t\int_E f(X_s,y)N(X_s,dy)\,dH_s\right)\end{aligned} \tag{1.4}$$

for every $x\in E$. The Revuz measure for H will be denoted as μ_H. If X is further assumed to be *special*, that is, if X_T is $\sigma(\cup_n\mathscr{M}_{T_n})$-measurable whenever $\{T_n\}$ is an increasing sequence of stopping times with limit T, then the above Lévy kernel $N(x,dy)$ can be extended to be a kernel on $(E_\partial,\mathscr{B}(E_\partial))$ such that for any nonnegative Borel function f on $E\times E_\partial$ vanishing on the diagonal and any $x\in E$,

$$\mathbf{E}_x\left(\sum_{s\leqslant t} f(X_{s-},X_s)\right)=\mathbf{E}_x\left(\int_0^t\int_{E_\partial} f(X_s,y)N(X_s,dy)\,dH_s\right). \tag{1.5}$$

We note that a pure jump Girsanov transform (see (2.7)) is a Feynman–Kac transform driven by a discontinuous additive functional. So the conditional gauge theorem proved in Chen [6] and Chen and Song [10] can be applied to study the relation between the Green functions of the original process and those of the process transformed by a pure jump exponential local martingale.

The content of this paper is organized as follows. In Section 2, we show that under some natural assumptions the conditional expectation of the pure jump exponential martingale is uniformly bounded between two positive numbers. This shows that the Green function of the transformed process is comparable to the Green function of the original process. In Section 3, we apply the results of Section 2 to the relativistic symmetric α-stable process. We show that the relativistic symmetric α-stable process in a bounded $C^{1,1}$-smooth open set D can be obtained from the symmetric α-stable process in D through a combination of a pure jump Girsanov transform and a Feynman–Kac transform. We further show that the Green functions of these two processes are comparable.

In this paper, we use "$:=$" as a way of definition, which is read as "is defined to be". For functions f and g, notation "$f \approx g$" means that there exist constants $c_2 > c_1 > 0$ such that $c_1 g \leqslant f \leqslant c_2 g$.

2. Pure jump Girsanov transforms

We first recall the following definitions from Chen [6], which extend those in Chen and Song [9,10]. For a signed measure μ, we use μ^+ and μ^- to denote its positive and negative parts, respectively. Let $d := \{(x,z) \in (E \times E) : G(x,z) = \infty\}$ and $E_z := \{x \in E : G(x,z) < \infty\}$ for $z \in E$.

Definition 2.1. Suppose that ν is a signed smooth measure. Let ν^+ and ν^- denotes its positive and negative part, and let $|\nu| = \nu^+ + \nu^-$.

(1) The measure ν is said to be in the class $\mathbf{K}_\infty(X)$ if for any $\varepsilon > 0$, there is a Borel set $K = K(\varepsilon)$ of finite $|\nu|$-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that for all measurable set $B \subset E$ with $|\nu|(B) < \delta$,

$$\sup_{x \in E} \int_{(E \setminus K) \cup B} G(x,y) |\nu|(dy) < \varepsilon.$$

(2) The measure ν is said to be in the class $\mathbf{K}_1(X)$ if there is a Borel set K of finite $|\nu|$-measure and a constant $\delta > 0$ such that

$$\beta_1(\nu) := \sup_{B \subset K : |\nu|(B) < \delta} \sup_{x \in E} \int_{(E \setminus K) \cup B} G(x,y) |\nu|(dy) < 1.$$

(3) The measure ν is said to be in the class $\mathbf{S}_\infty(X)$ if for any $\varepsilon > 0$ there is a Borel subset $K = K(\varepsilon)$ of finite $|\nu|$-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that for all

measurable sets $B \subset E$ with $|\nu|(B) < \delta$,

$$\sup_{(x,z)\in(E\times E)\setminus d} \int_{(E\setminus K)\cup B} \frac{G(x,y)G(y,z)}{G(x,z)} |\nu|(dy) \leqslant \varepsilon. \tag{2.1}$$

(4) The measure ν is said to be in the class $\mathbf{S}_1(X)$ if there is a Borel set K of finite $|\nu|$-measure and a constant $\delta > 0$ such that

$$\beta_2(\nu) := \sup_{B\subset K:|\nu|(B)<\delta} \ \sup_{(x,z)\in(E\times E)\setminus d} \int_{(E\setminus K)\cup B} \frac{G(x,y)G(y,z)}{G(x,z)} |\nu|(dy) < 1.$$

(5) A function q is said to be in the class $\mathbf{K}_\infty(X)$, $\mathbf{K}_1(X)$, $\mathbf{S}_\infty(X)$, or $\mathbf{S}_1(X)$, if $\nu(dx) := q(x)m(dx)$ is in the corresponding spaces.

Definition 2.2. Suppose F is a bounded function on $E \times E$ vanishing on the diagonal d and off $E \times E$. It is always extended to be zero off $E \times E$. Define $\mu_F(dx) := (\int_E F(x,y)N(x,dy))\mu_H(dx)$.

(1) We say that F belongs to the class $\mathbf{J}_\infty(X)$ if the measure $\mu_{|F|}$ belongs to $\mathbf{K}_\infty(X)$.

(2) F is said to be in the class $\mathbf{A}_\infty(X)$ if for any $\varepsilon > 0$ there is a Borel subset $K = K(\varepsilon)$ of finite $\mu_{|F|}$-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that for all measurable set $B \subset K$ with $\mu_{|F|}(B) < \delta$,

$$\sup_{(x,w)\in(E\times E)\setminus d} \int_{(K\setminus B)^c\times(K\setminus B)^c} G(x,y)\frac{|F(y,z)|G(z,w)}{G(x,w)} N(y,dz)\mu_H(dy) \leqslant \varepsilon. \tag{2.2}$$

(3) F is said to be in the class $\mathbf{A}_2(X)$ if $F \in \mathbf{A}_\infty(X)$ and if the measure $\mu_{|F|}$ is in $\mathbf{S}_\infty(X)$.

The main improvement of these new classes over those in Chen and Song [9,10], as was explained in Chen [6], is that they include measures that are not absolutely continuous with respect to the reference measure m and the gauge and conditional gauge theorems still hold. In particular, the Revuz measure μ_H of H in the Lévy system (N,H) for X needs not to be absolutely continuous with respect to m. From Chen [6], we know that $\mathbf{K}_\infty(X) \subset \mathbf{K}_1(X)$, $\mathbf{S}_\infty(X) \subset \mathbf{K}_\infty(X)$, $\mathbf{S}_1(X) \subset \mathbf{K}_1(X)$ and $\mathbf{A}_\infty(X) \subset \mathbf{J}_\infty(X)$. It follows from [6] that for any $\nu \in \mathbf{K}_1(X)$ we have $U|\mu|$ is bounded. Therefore, for any measure $\mu \in \mathbf{K}_1(X)$ we can uniquely associate an additive functional in the strict sense.

It is easy to see from the definition that if $F \in \mathbf{A}_\infty(X)$ satisfies

$$\inf_{x,y\in E} F(x,y) > -1,$$

then $\ln(1 + F(x,y))$ is also in $\mathbf{A}_\infty(X)$.

Although the above definitions for various Kato classes look complicated, they in fact arise quite naturally. For example, the Kato class $\mathbf{K}_\infty(X)$ is a genuine extension of the class of Green-tight measures introduced by Zhao [30] for Brownian motion in $\mathbf{R}^n$ (see [6]). We now give some concrete conditions for functions to be in classes $\mathbf{K}_\infty(X)$, $\mathbf{S}_\infty(X)$ and $\mathbf{A}_\infty(X)$, respectively, when X is a symmetric α-stable process in a bounded $C^{1,1}$-smooth open set.

Example 1. Suppose that X is a killed symmetric α-stable process on a bounded $C^{1,1}$ open set $D \subset \mathbf{R}^n$ for some $\alpha \in (0,2)$. The process X is a Hunt process, which is in particular a special standard process. So it has a Lévy system (N, H) in the sense of (1.5), where $H_t = t$, $N(x,dy) = \mathscr{A}(n,-\alpha)|x-y|^{-(n+\alpha)}\,dy$ on D and $N(x,\partial) = \mathscr{A}(n,-\alpha)\int_{D^c}|x-y|^{-(n+\alpha)}\,dy$. Here,

$$\mathscr{A}(n,-\alpha) = \frac{|\alpha|\Gamma(\frac{n+\alpha}{2})}{2^{1-\alpha}\pi^{n/2}\Gamma(1-\frac{\alpha}{2})}. \tag{2.3}$$

It follows from the $3G$-estimate in Chen and Song [8] as well as in Chen [5] and Corollary 4.2 in Chen and Song [9] that a Borel function q on D is in $\mathbf{S}_\infty(X)$ and hence in $\mathbf{K}_\infty(X)$ if

$$|q(x)| \leqslant f(x) + c\delta_D(x)^{-\beta} \quad \text{for almost every } x \in D,$$

where c and β are positive constants such that $\beta < \alpha$, and $f \geqslant 0$ is a function on D such that

$$\lim_{r\to 0} \sup_{x\in\mathbf{R}^n} \int_{B(x,r)\cap D} f(y)|x-y|^{\alpha-n}\,dy = 0. \tag{2.4}$$

Here $\delta_D(x)$ denotes the Euclidean distance between x and D^c. It is easy to see that condition (2.4) is satisfied if $f \in L^p(D)$ for some $p > d/\alpha$. It is illustrated in Example 2 of Chen and Song [10] that if F is a bounded function on $D \times D$ satisfying the inequality

$$|F(x,y)| \leqslant C|x-y|^\beta, \quad x,y \in D \tag{2.5}$$

for some constants $\beta > \alpha$ and $C > 0$, then $F \in \mathbf{A}_\infty(X)$. In fact, such a function F is also in the class $\mathbf{A}_2(X)$, as the function

$$\int_D |F(x,y)|N(x,dy) \leqslant \mathscr{A}(n,-\alpha)\int_D C|x-y|^{-n+(\beta-\alpha)}\,dy$$

is bounded on D and therefore is in the class $\mathbf{S}_\infty(X)$. $\square$

For a smooth measure μ associated with a continuous additive functional A^μ and a Borel function F on $E \times E$ that vanishes along the diagonal, define

$$e_{A^\mu+F}(t) = \exp\left(A_t^\mu + \sum_{0<s\leqslant t} F(X_{s-}, X_s)\right), \quad t \geqslant 0.$$

As mentioned previously, the continuous additive functional A_t^μ of X can be regarded as the continuous additive functional of X^w with Revuz measure $G(\cdot, w)\mu$. We recall the following results from Chen [6] established there as Theorems 2.13, 3.8 and 3.10, respectively.

Theorem 2.1. (1) (Gauge Theorem) *Assume that* $\mu \in \mathbf{K}_1(X)$ *and* $F \in \mathbf{J}_\infty(X)$. *Then the gauge function* $g(x) := \mathbf{E}_x[e_{A^\mu+F}(\zeta)]$ *is either bounded on* E *or identically infinite on* E.

(2) (Conditional Gauge Theorem) *Suppose that* $\mu \in \mathbf{S}_1(X)$ *and* $F \in \mathbf{A}_\infty(X)$. *Then the conditional gauge function* $(x, w) \mapsto \mathbf{E}_x^w[e_{A+F}(\zeta^w)]$ *is either bounded or identically infinite on* $(E \times E)\backslash d$.

(3) (Equivalence Theorem) *Suppose that* $\mu \in \mathbf{S}_1(X)$ *and* $F \in \mathbf{A}_\infty(X)$. *Then the following are equivalent.*

(3a) *The gauge function* $x \mapsto \mathbf{E}_x[e_{A^\mu+F}(\zeta)]$ *is bounded on* E;

(3b) *The conditional gauge function* $(x, w) \mapsto \mathbf{E}_x^w[e_{A^\mu+F}(\zeta^w)]$ *is bounded on* $(E \times E)\backslash d$;

(3c) *There is some* $x \in E$ *and a Borel set* $B \subset E$ *with* $0 < m(B) < \infty$ *such that*

$$\mathbf{E}_x\left[\int_0^\zeta e_{A^\mu+F}(t) 1_B(X_t)\, dt\right] < \infty.$$

Note that for $\mu \in \mathbf{S}_1(X)$ and $F \in \mathbf{A}_\infty(X)$,

$$\sup_{(x,w)\in(E\times E)\backslash d} \mathbf{E}_x^w\left[A_{\zeta^w}^{|\mu|} + \sum_{0<s<\zeta^w} |F(X_{s-}, X_s)|\right] < \infty$$

(see [6]) and therefore by Jensen's inequality

$$\inf_{(x,w)\in(E\times E)\backslash d} \mathbf{E}_x^w[e_{A^\mu+F}(\zeta^w)] > 0. \tag{2.6}$$

Suppose that F belongs to $\mathbf{A}_2(X)$ with

$$\inf_{x,y\in E} F(x, y) > -1.$$

Using the Lévy system, we see that

$$t \mapsto \sum_{0<s\leqslant t} F(X_{s-}, X_s) - \int_0^t \int_E F(X_s, y) N(X_s, dy)\, dH_s$$

is a local martingale. Its Doleans–Dade exponential is (cf. [18, Theorem 9.39])

$$\begin{aligned} M_t &= \exp\left(-\int_0^t \int_E F(X_s, y) N(X_s, dy)\, dH_s\right) \prod_{s\leqslant t}(1 + F(X_{s-}, X_s)) \\ &= \exp\left(\sum_{s\leqslant t} \ln(1 + F(X_{s-}, X_s)) - \int_0^t \int_E F(X_s, y) N(X_s, dy)\, dH_s\right). \end{aligned} \tag{2.7}$$

M_t is a nonnegative local martingale. M_t is called a pure jump Girsanov transform of X. In this paper, we are concerned with this pure jump Girsanov transform of X. It follows from (2.7) that M_t is a Feynman–Kac transform by a discontinuous additive functional. The latter is the subject of recent work of Chen and Song [10] and Chen [6].

The local martingale M_t is obviously a nonnegative supermartingale multiplicative functional of X. Let Y be the strong Markov process obtained from X through the pure jump Girsanov transform M, that is, the transition semigroup of Y is given by

$$Q_t f(x) := \mathbf{E}_x[M_t f(X_t)].$$

Theorem 2.4 of Chen and Song [10] *suggests* that the infinitesimal generator of the semigroup of Y takes the form of

$$f \mapsto \mathscr{L}^Y f(x) := \mathscr{L} f(x) + \left(\int_E (f(y) - f(x)) F(x,y) N(x,dy) \right) \mu_H(dx),$$

where $\mathscr{L}$ is the infinitesimal generator for the semigroup of X. We like to point out that if X is a symmetric Borel standard process and $F \in \mathbf{A}_2(X)$ is symmetric, then the Girsanov transformed process Y is symmetric with respect to the measure m as well.

We are going to use the recent results in Chen [6], some of which are summarized in Theorem 2.1, to show that the Green function G^Y of Y is comparable to that of X.

Theorem 2.2. *For* $F \in \mathbf{A}_2(X)$ *with*

$$\inf_{x,y \in E} F(x,y) > -1,$$

there is a constant $c > 1$ *such that*

$$c^{-1} \leqslant \mathbf{E}_x^y[M_{\zeta^y}] \leqslant c$$

for all $(x,y) \in (E \times E) \backslash d$. *Therefore, the Green function* $G^Y(x,y) = \mathbf{E}_x^y[M_{\zeta^y}] G(x,y)$ *of* Y *is comparable to* $G(x,y)$.

Proof. As $\ln(1+F) \in \mathbf{A}_\infty(X)$ and $\mu_{|F|} \in \mathbf{S}_\infty(X)$, we have by Theorem 2.1(2) that $u(x,y) := \mathbf{E}_x^y[M_{\zeta^y}]$ is either bounded or identically infinite on $(E \times E) \backslash d$. However, as $\mathbf{E}_x[M_\zeta] \leqslant 1$ for $x \in E$, we have by Theorem 2.1(3) $\mathbf{E}_x^y[M_{\zeta^y}]$ is bounded from above on $(E \times E) \backslash d$. By (2.6), this function is also bounded from below by a positive constant. That $G^Y(x,y) := \mathbf{E}_x^y[M_{\zeta^y}] G(x,y)$ is the Green function of Y is a consequence of Lemma 3.9 of Chen [6]. $\square$

The mixture of a pure jump Girsanov transform and a Feynman–Kac transform can also be handled similarly.

Theorem 2.3. *Suppose that* $F \in \mathbf{A}_2(X)$ *with*

$$\inf_{x,y\in E} F(x,y) > -1$$

and that a continuous additive functional A *of* X *is in* $\mathbf{S}_1(X)$. *Let* $K_t = M_t \exp(A_t)$, *where* M_t *is the exponential martingale in* (2.7) *determined by* F. *Then* $u(x,y) := \mathbf{E}_x^y[K_{\zeta^y}]$ *is either bounded or identically infinite. If* $\mathbf{E}_x[K_\zeta] < \infty$ *for some* $x \in E$, *then* u *is bounded.*

Proof. Note that by (2.7),

$$K_t = \exp\left(\sum_{0<s\leqslant t} \ln(1 + F(X_{s-}, X_s)) - \int_0^t \int_E F(X_s, y)N(X_s, dy)\, dH_s + A_t\right). \quad (2.8)$$

Let ν denote the signed Revuz measure of A_t. As $\ln(1+F) \in \mathbf{A}_\infty(X)$ and $|\nu| + \mu_{|F|}$ is in $\mathbf{S}_1(X)$, we have by Theorem 2.1(2) that u is either bounded or identically infinite on $(E \times E)\backslash d$. If $\mathbf{E}_x[K_\zeta] < \infty$ for some $x \in E$, then by Theorem 2.1(3), $u(x,y)$ is bounded on $(E \times E)\backslash d$. □

If the process X is special standard and has killing inside E, that is, $\mathbf{P}_x(X_{\zeta-} \in E, \zeta < \infty) > 0$ for any $x \in E$, the above two theorems can be proved by using Theorem 2.1(2) only (i.e., without using the equivalence in Theorem 2.1(3)). The above killing condition is equivalent to that the killing measure $\kappa(dx) := N(x, \partial)\mu_H(dx)$ is nontrivial. We will just present an alternative proof for Theorem 2.3. The alternative proof for Theorem 2.2 goes along the same lines. First, we give an interpretation of the y-conditioned process X^y, which may be of independent interest.

Theorem 2.4. *Assume further that* X *is a special standard process. Then for any* $\mathscr{M}_\infty$*-measurable function* $\Phi \geqslant 0$ *and* $x \in D$, *we have*

$$\mathbf{E}_x[\Phi | X_{\zeta-}] = \mathbf{E}_x^{X_{\zeta-}}[\Phi] \quad \text{on } \{X_{\zeta-} \in E, \zeta < \infty\}.$$

Proof. Using the monotone class theorem, we may assume that Φ is of the form $\Phi = \Phi_t 1_{\{t<\zeta\}}$, where Φ_t is $\mathscr{M}_t$-measurable and bounded. Clearly,

$$\mathbf{E}_x^y[\Phi] = \frac{1}{G(x,y)} \mathbf{E}_x[\Phi_t G(X_t, y); t < \zeta].$$

So $\mathbf{E}_x^y[\Phi]$ is Borel measurable in y, for every fixed x and Φ. We need to show that for any Borel subset A of E,

$$\mathbf{E}_x[\Phi; X_{\zeta-} \in A, \zeta < \infty] = \mathbf{E}_x[\mathbf{E}_x^{X_{\zeta-}}[\Phi]; X_{\zeta-} \in A, \zeta < \infty]. \quad (2.9)$$

The right-hand side of (2.9), by (1.5) and Fubini theorem, is equal to

$$\begin{aligned}
&\int_A G(x,y)\mathbf{E}_x^y[\Phi]\kappa(dy)\\
&\quad= \int_A \mathbf{E}_x[G(X_t,y)\Phi_t; t<\zeta]\kappa(dy)\\
&\quad= \mathbf{E}_x\left[\Phi_t\left(\int_A G(X_t,y)\kappa(dy)\right); t<\zeta\right]\\
&\quad= \mathbf{E}_x[\Phi_t\mathbf{P}_{X_t}(X_{\zeta-}\in A, \zeta<\infty); t<\zeta]\\
&\quad= \mathbf{E}_x[\Phi; X_{\zeta-}\in A, \zeta<\infty]
\end{aligned}$$

which is the left-hand side of (2.9). □

Second Proof for Theorem 2.3. (under the extra assumption that the process X is special standard and has killings inside E). If $\mathbf{E}_x[K_\zeta]<\infty$, then by (2.9),

$$\int_E G(x,y)\mathbf{E}_x^y[K_{\zeta^y}]\kappa(dy) = \mathbf{E}_x[K_\zeta; X_{\zeta-}\in E, \zeta<\infty]<\infty$$

and therefore there is some $(x,y)\in(E\times E)\backslash d$ such that $u(x,y)<\infty$. This implies that u has to be bounded by Theorem 2.1(2). □

3. Applications to relativistic stable processes

In this section, we will first apply the main results of the previous section to the case where the underlying process X is a symmetric α-stable process X^D in a bounded $C^{1,1}$ open set D, $\alpha\in(0,2)$. The process X^D is special standard, in fact a Hunt process, so all the results, including Theorem 2.4, of the previous section apply. We show that relativistic α-stable processes in D can be obtained from X^D through a Feynman–Kac transform of the form (2.8). We then use it to get sharp estimates for Green functions of relativistic α-stable processes in a bounded $C^{1,1}$ open set.

Let X be a symmetric α-stable process on $\mathbf{R}^n$ with $n\geqslant 2$ and $0<\alpha<2$. The process X is transient and it has Green function

$$G(x,y) = \mathscr{A}(n,\alpha)|x-y|^{\alpha-n},$$

where $\mathscr{A}(n,\alpha)$ is given by (2.3) with $-\alpha$ in place of α there, that is,

$$\mathscr{A}(n,\alpha) = \frac{\alpha\Gamma(\frac{n-\alpha}{2})}{2^{1+\alpha}\pi^{n/2}\Gamma(1+\frac{\alpha}{2})}.$$

It is well known that the Dirichlet form $(\mathscr{E}, \mathscr{F})$ associated with X is given by

$$\mathscr{E}(u,v) = \frac{1}{2}\mathscr{A}(n,-\alpha)\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\alpha}}\,dx\,dy,$$

$$\mathscr{F} = \left\{u\in L^2(\mathbf{R}^n) : \int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{(u(x)-u(y))^2}{|x-y|^{n+\alpha}}\,dx\,dy<\infty\right\}. \tag{3.1}$$

Here and in the rest of this section dx stands for the Lebesgue measure on $\mathbf{R}^n$.

Let Cap denote the 1-capacity associated with the process X (or equivalently, with the Riesz potential kernel $\mathscr{A}(n,\alpha)|x-y|^{\alpha-n}$). A function u is said to be quasi-continuous if there exists for any $\varepsilon>0$ an open set U such that $\mathrm{Cap}(U)<\varepsilon$ and $u|_{U^c}$ is continuous. It is known (cf. [15]) that every function u in $\mathscr{F}$ admits a quasi-continuous version. We will always use its quasi-continuous version in the sequel.

For any open set D, let X^D be the symmetric α-stable process X killed upon leaving D. The Dirichlet form associated with X^D is $(\mathscr{E}, \mathscr{F}^D)$, where

$$\mathscr{F}^D = \{u\in\mathscr{F} : u=0 \text{ on } D^c \text{ except for a set of zero capacity}\}. \tag{3.2}$$

Let $\mathscr{E}_1 := \mathscr{E} + (\cdot,\cdot)_{L^2(\mathbf{R}^n,dx)}$. Then $\mathscr{F}^D$ is in fact the $\mathscr{E}_1$-closure of $C_c^\infty(D)$, the space of smooth functions with compact support in D (cf. [15]). Note that for any $u, v\in\mathscr{F}^D$,

$$\mathscr{E}(u,v) = \int_D\int_D (u(x)-u(y))(v(x)-v(y))J(x,y)\,dx\,dy + \int_D u(x)v(x)\kappa_D(x)\,dx,$$

where

$$J(x,y) = \frac{1}{2}\mathscr{A}(n,-\alpha)\frac{1}{|x-y|^{n+\alpha}}$$

and

$$\kappa_D(x) = \mathscr{A}(n,-\alpha)\int_{D^c}\frac{1}{|x-y|^{n+\alpha}}\,dy.$$

The process X^D has Lévy system (N,H) where $N(x,dy) = \mathscr{A}(n,-\alpha)|x-y|^{-(n+\alpha)}\,dy$ on D and $N(x,\partial) = \kappa_D(x)$, and $H_t = t$.

It is well known that X^D has a continuous symmetric Green function $G_D(\cdot,\cdot)$, which is defined on $D\times D$ except along the diagonal such that for any Borel measurable function $f\geqslant 0$,

$$\mathbf{E}_x\left[\int_0^{\tau_D} f(X_s)\,ds\right] = \int_D G_D(x,y)f(y)\,dy.$$

When D is a bounded $C^{1,1}$ open set, sharp bounds on the Green function G_D have been established in Chen and Song [8], Kulczycki [23] and Chen [5]:

$$G_D(x,y) \approx \min\left\{\frac{1}{|x-y|^{n-\alpha}}, \frac{\delta_D^{\alpha/2}(x)\delta_D^{\alpha/2}(y)}{|x-y|^n}\right\} \quad \text{for } x,y \in D. \tag{3.3}$$

Recall that $\delta_D(x)$ denotes the Euclidean distance between x and D^c. Here for two functions f,g, $f \approx g$ means there are constants $c_2 > c_1 > 0$ such that $c_1 g \leqslant f \leqslant c_2 g$. Using the Lévy system, one easily deduces (cf. [19]) that

$$\mathbf{P}_x(X_{\tau_D -} \in A) = \int_A G_D(x,y)\kappa_D(y)\,dy \quad \text{for any measurable } A \subset D.$$

Suppose that $F \in \mathbf{A}_2(X)$ is symmetric with

$$\inf_{x,y \in E} F(x,y) > -1$$

and that μ is a signed measure in $\mathbf{S}_1(X)$. Let A^μ be the continuous additive functional of X^D with Revuz measure μ. As we already saw in Example 1, a bounded function F satisfying condition (2.5) is in $\mathbf{A}_2(X)$. Define

$$K_t = \exp\left(\sum_{0<s\leqslant t} \ln(1+F(X_{s-}^D, X_s^D)) - \int_0^t \int_D F(X_s^D, y)N(X_s^D, dy)\,ds + A_t^\mu\right). \tag{3.4}$$

Let

$$Q_t f(x) := \mathbf{E}_x[K_t f(X_t^D)], \quad t > 0,$$

with the convention that $f(\partial) = 0$. Then by Theorem 4.8 of Chen and Song [10], Q_t is a strongly continuous semigroup in $L^2(D,dx)$ whose associated quadratic form is $(\mathcal{Q}, \mathcal{F}^D)$, where

$$\begin{aligned}\mathcal{Q}(u,v) = &\int_D \int_D (u(x)-u(y))(v(x)-v(y))(1+F(x,y))J(x,y)\,dx\,dy \\ &+ \int_D u(x)v(x)\kappa_D(x)\,dx - \int_D u(x)v(x)\mu(dx) \quad \text{for } u,v \in \mathcal{F}^D.\end{aligned} \tag{3.5}$$

Let $\mathbf{P}_x^y$ and $\mathbf{E}_x^y$ denote the probability and expectation for the $G_D(\cdot,y)$-process of X^D starting from x. By Lemma 3.9 of Chen [6], for Borel function $f \geqslant 0$,

$$\mathbf{E}_x\left[\int_0^\infty K_t f(X_t^D)\,dt\right] = \int_D G_D(x,y)\mathbf{E}_x^y[K_{\zeta^y}]f(y)\,dy,$$

and so the semigroup Q_t has Green function $V(x,y) := G_D(x,y)\mathbf{E}_x^y[K_{\zeta^y}]$. If $\mathbf{E}_x[K_{\tau_D}] < \infty$ for some $x \in D$, then by Theorem 2.3 and (2.6),

$$V(x,y) \approx G_D(x,y). \tag{3.6}$$

Now we apply the above to relativistic α-stable processes in D. A relativistic α-stable process $Y = (Y_t, Q^x, x \in \mathbf{R}^n)$ in $\mathbf{R}^n$ is a Lévy process whose characteristic function is given by

$$e^{-t((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m)}, \quad \xi \in \mathbf{R}^n,$$

where $m > 0$ is a constant. In other words, Y has infinitesimal generator $m - (-\Delta + m^{2/\alpha})^{\alpha/2}$. It is clear that Y is symmetric with respect to the Lebesgue measure dx on $\mathbf{R}^n$ and that when the parameter m degenerates to 0, Y becomes a symmetric α-stable process on $\mathbf{R}^n$. Let $(\mathcal{Q}^Y, \mathcal{D}(\mathcal{Q}^Y))$ be the Dirichlet space of Y. Using Fourier transform $\hat{f}(x) := (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{ix\cdot y} f(y)\, dy$, it follows from Example 1.4.1 of Fukushima et al. [15] that

$$\mathcal{D}(\mathcal{Q}^Y) = \left\{ u \in L^2(\mathbf{R}^n, dx) : \int_{\mathbf{R}^n} |\hat{u}(\xi)|^2 ((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m)\, d\xi < \infty \right\},$$

$$\mathcal{Q}^Y(u,v) = \int_{\mathbf{R}^n} u(\xi) \bar{\hat{v}}(\xi) ((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m)\, d\xi.$$

Here $\bar{z}$ denotes the complex conjugate of a complex number z. The Dirichlet space $(\mathcal{E}, \mathcal{F})$ for the symmetric α-stable process X can also be characterized similarly by using Fourier transforms, only with $|\xi|^\alpha$ in place of $(|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$ above. Thus, there is a constant $c = c(m) > 1$ such that

$$c^{-1}\mathcal{E}_1(u,u) \leqslant \mathcal{Q}_1^Y(u,u) \leqslant c\, \mathcal{E}_1(u,u) \tag{3.7}$$

and so $\mathcal{D}(\mathcal{Q}^Y) = \mathcal{F}$. Here $\mathcal{Q}_1^Y(u,u) := \mathcal{Q}^Y(u,u) + \int_{\mathbf{R}^n} u(x)^2\, dx$. As noted in Bakry [1] and Ryznar [27], like symmetric stable processes, relativistic α-stable processes can be obtained from Brownian motions through subordinations. Using subordinations, it is easy to deduce the Lévy measure $v(x)\, dx$ for Y whose density is given by

$$v(x) = \frac{\alpha}{2\Gamma(1-\frac{\alpha}{2})} \int_0^\infty (4\pi u)^{-n/2} e^{-\frac{|x|^2}{4u}} e^{-m^{2/\alpha}u} u^{-(1+\frac{\alpha}{2})}\, du,$$

(see [27]). Using change of variables twice, first with $u = |x|^2 v$ then with $v = 1/s$, we have

$$\begin{aligned} v(x) &= \frac{\alpha}{2^{n+1}\pi^{n/2}\Gamma(1-\frac{\alpha}{2})} \int_0^\infty (|x|^2 v)^{-(1+\frac{n+\alpha}{2})} e^{-\frac{1}{4v} - m^{2/\alpha}|x|^2 v} |x|^2\, dv \\ &= \frac{\alpha}{2^{n+1}\pi^{n/2}\Gamma(1-\frac{\alpha}{2})} \frac{1}{|x|^{n+\alpha}} \int_0^\infty v^{-(1+\frac{n+\alpha}{2})} e^{-\frac{1}{4v} - m^{2/\alpha}|x|^2 v}\, dv \end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha}{2^{n+1}\pi^{n/2}\Gamma(1-\frac{\alpha}{2})}\frac{1}{|x|^{n+\alpha}}\int_0^\infty s^{\frac{n+\alpha}{2}-1}e^{-\frac{s}{4}-\frac{m^{2/\alpha}|x|^2}{s}}\,ds\\
&:= \frac{\alpha}{2^{n+1}\pi^{n/2}\Gamma(1-\frac{\alpha}{2})}\frac{1}{|x|^{n+\alpha}}I(m^{1/\alpha}|x|),
\end{aligned}
$$

where

$$I(r) = \int_0^\infty s^{\frac{n+\alpha}{2}-1}e^{-\frac{s}{4}-\frac{r^2}{s}}\,ds.$$

Using the change of variable $t = \frac{\sqrt{s}}{2} - \frac{r}{\sqrt{s}}$, we have

$$
\begin{aligned}
I(r) &= e^{-r}\int_0^\infty s^{\frac{n+\alpha}{2}-1}e^{-(\frac{\sqrt{s}}{2}-\frac{r}{\sqrt{s}})^2}\,ds\\
&= e^{-r}\int_{-\infty}^\infty \frac{2(t+\sqrt{t^2+2r})^{n+\alpha}}{\sqrt{t^2+2r}}e^{-t^2}\,dt
\end{aligned}
$$

and therefore

$$I(r) \approx e^{-r}\left(1 + r^{\frac{n+\alpha-1}{2}}\right) \quad \text{near } r = \infty.$$

Since $I(0) = 2^{n+\alpha}\Gamma(\frac{n+\alpha}{2})$, $\frac{\alpha}{2^{n+1}\pi^{n/2}\Gamma(1-\frac{\alpha}{2})}I(0) = \mathscr{A}(n,-\alpha)$. Defining $\psi(r) = I(r)/I(0)$, we have from the above that

$$v(x) = \mathscr{A}(n,-\alpha)|x|^{-n-\alpha}\,\psi(m^{1/\alpha}|x|) \tag{3.8}$$

with

$$\psi(r) \approx e^{-r}\left(1 + r^{\frac{n+\alpha-1}{2}}\right) \quad \text{near } r = \infty. \tag{3.9}$$

On the other hand,

$$\psi(r) = I(0)^{-1}\int_0^\infty s^{\frac{n+\alpha}{2}-1}e^{-\frac{s}{4}-\frac{r^2}{s}}\,ds$$

is a smooth function of r^2 and so

$$\psi(r) = 1 + \frac{\psi''(0)}{2}r^2 + o(r^4) \quad \text{near } r = 0. \tag{3.10}$$

The Lévy measure v determines the jumping measure J_1 of Y:

$$J_1(x,y) = \tfrac{1}{2}v(x-y) = \tfrac{1}{2}\mathscr{A}(n,-\alpha)|x-y|^{-(n+\alpha)}\,\psi(m^{1/\alpha}|x-y|).$$

As the Dirichlet space $(\mathscr{Q}^Y, \mathscr{F})$ for Y has no continuous part and killing part, so by Beurling–Deny decomposition it is uniquely determined by its jumping part (cf. [15])

and thus

$$\mathcal{Q}^Y(u,u) = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} (u(x)-u(y))^2 J_1(x,y)\,dx\,dy.$$

Suppose that D is a bounded $C^{1,1}$ open set in $\mathbf{R}^n$. Let Y^D be the relativistic α-stable process Y killed upon exiting D. Then (cf. [15]) the Dirichlet form corresponding to Y^D is $(\mathcal{Q}^Y, \mathcal{F}^D)$. For any $u \in \mathcal{F}^D$,

$$\mathcal{Q}^Y(u,v) = \int_D\int_D (u(x)-u(y))(v(x)-v(y))J_1(x,y)\,dx\,dy$$

$$+ \int_D u(x)v(x)\kappa_D^Y(x)\,dx \quad \text{for } u,v \in \mathcal{F}^D$$

with

$$\kappa_D^Y(x) = 2\int_{D^c} J_1(x,y)\,dy.$$

It follows from (3.8) and (3.9) that $J_1(x,y) \leqslant J(x,y)$ on $\mathbf{R}^n \times \mathbf{R}^n$. Moreover, we have from (3.8)–(3.10),

$$\frac{J_1(x,y)}{J(x,y)} \geqslant c > 0 \quad \text{and} \quad \left|\frac{J_1(x,y)}{J(x,y)} - 1\right| \leqslant C|x-y|^2 \quad \text{on } D\times D \tag{3.11}$$

for some positive constants $c = c(D,\alpha,m)$ and $C = C(D,\alpha,m)$. Therefore, the function $q(x) := \kappa_D(x) - \kappa_D^Y(x)$ satisfies the condition

$$\begin{aligned} 0 < q(x) &\leqslant C\int_{D^c} \frac{1}{|x-y|^{n+\alpha-2}}\,dy \\ &\leqslant C\int_{\{y\,:\,|y-x|>\delta_D(x)\}} \frac{1}{|x-y|^{n+\alpha-2}}\,dy \\ &= C\,\delta_D(x)^{2-\alpha}. \end{aligned}$$

Using the 3G theorem for Green function of X^D (cf. [8]), it is easy to see that $q \in \mathbf{S}_\infty(X^D)$. Let $F(x,y) = \frac{J_1(x,y)}{J(x,y)} - 1$. Then from (3.11) we see that on $(D\times D)\setminus d$, F is nonpositive and satisfies

$$\inf_{x,y\in D} F(x,y) \geqslant c-1 \quad \text{and} \quad |F(x,y)| \leqslant C|x-y|^2 \qquad \text{on } D\times D$$

for some constant $C>0$. So by Example 1 in the previous section we know $F \in \mathbf{A}_2(X^D)$ and $\ln(1+F) \in A_\infty(X^D)$. By (3.3) we also have $q \in \mathbf{S}_\infty(X^D)$ (see [9]).

Define

$$K_t = \exp\left(\sum_{0<s\leqslant t} \ln(1+F(X_{s-}^D, X_s^D)) - \int_0^t \int_E F(X_s^D, y) N(X_s^D, dy)\, ds \right.$$
$$\left. + \int_0^t q(X_s^D)\, ds \right).$$

We claim that Y^D can be obtained from X^D through the Feynman–Kac transform by the multiplicative functional K_t. For this, define a semigroup Q_t by

$$Q_t f(x) := \mathbf{E}_x[K_t f(X_t^D)], \quad x \in D.$$

It follows from (3.5) that the quadratic form associated with Q_t is exactly the Dirichlet space $(\mathcal{Q}^Y, \mathcal{F}^D)$ of Y^D and therefore Q_t is the semigroup of Y^D. Note that the killing intensity κ_D^Y of Y^D is bounded from below by a positive constant so it follows that

$$\inf\left\{ \mathcal{Q}^Y(u,u)\colon\ u \in \mathcal{F}^D \text{ with } \int_D u(x)^2\, dx = 1 \right\} > 0.$$

This implies that $\int_0^\infty Q_t\, dt$ is a bounded operator in $L^2(D, dx)$ and so for any Borel subset $B \subset E$ with $0 < m(B) < \infty$,

$$\int_0^\infty Q_t 1_B(x)\, dt = \mathbf{E}_x\left[\int_0^\infty K_t\, 1_B(X_t^D)\, dt \right] < \infty \quad \text{for } m\text{-a.e. } x \in E.$$

Thus, by Theorem 2.1(3) and (3.6), the Green function G_D^Y of Y^D is comparable to that of X^D. Therefore, we have established the following result.

Theorem 3.1. *For any bounded $C^{1,1}$ open set D in $\mathbf{R}^n$, there is a constant $C = C(D, \alpha, m) > 0$ such that*

$$C^{-1} G_D(x,y) \leqslant G_D^Y(x,y) \leqslant C G_D(x,y), \quad x, y \in D.$$

From (3.3) we immediately get the following sharp estimates on G_D^Y.

Theorem 3.2. *For any bounded $C^{1,1}$ open set D in $\mathbf{R}^n$, there is a constant $C = C(D, \alpha, m) > 0$ such that*

$$C^{-1} \min\left\{ \frac{1}{|x-y|^{n-\alpha}}, \frac{\delta_D^{\alpha/2}(x)\delta_D^{\alpha/2}(y)}{|x-y|^n} \right\} \leqslant G_D^Y(x,y) \leqslant C \min\left\{ \frac{1}{|x-y|^{n-\alpha}}, \frac{\delta_D^{\alpha/2}(x)\delta_D^{\alpha/2}(y)}{|x-y|^n} \right\}.$$

When D is a bounded $C^{1,1}$-domain, Theorem 3.2 is the main result of Ryznar [27], which was proved in a different way there.

By using an argument similar to that in Chung and Zhao [11], it can be shown that $G^Y(x,y)$ is continuous on $(D \times D)\backslash d$.

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