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Continuity of eigenvalues of subordinate processes in domains

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Abstract. Let $X = \{X_t, t \geq 0\}$ be a symmetric Markov process in a state space E and D an open set of E. Let $S^{(n)} = \{S_t^{(n)}, t \geq 0\}$ be a subordinator with Laplace exponent ϕ_n and $S = \{S_t, t \geq 0\}$ a subordinator with Laplace exponent ϕ. Suppose that X is independent of S and $S^{(n)}$. In this paper we consider the subordinate processes $X^{\phi_n} := \{X_{S_t^{(n)}},\ t \geq 0\}$ and $X^{\phi} := \{X_{S_t},\ t \geq 0\}$, and their subprocesses $X^{\phi_n, D}$ and $X^{\phi, D}$ killed upon leaving D. Suppose that the spectra of the semigroups of $X^{\phi_n, D}$ and $X^{\phi, D}$ are all discrete, with $\{-\lambda_k^{\phi_n, D}; k \geq 1\}$ being the eigenvalues of the generator of $X^{\phi_n, D}$ and $\{-\lambda_k^{\phi, D}; k \geq 1\}$ being the eigenvalues of the generator of $X^{\phi, D}$. We show that, if $\lim_{n\to\infty} \phi_n(\lambda) = \phi(\lambda)$ for every $\lambda > 0$, then

$$\lim_{n\to\infty} \lambda_k^{\phi_n, D} = \lambda_k^{\phi, D} \qquad \text{for all } k \geq 1.$$

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1. Introduction

Let $Y^n = \{Y_t^n,\ t \geq 0\}$ and $Y = \{Y_t,\ t \geq 0\}$ be symmetric right processes on a Lusin space E with respect to a σ-finite measure m on E with full support. Let D be an open subset of E. In this paper, we will consider conditions under which the spectrum of the generator of the subprocess $Y^{n,D}$ of Y^n killed upon leaving D

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converges to that of the generator of Y^D as $n \to \infty$. For a non-positive definite selfadjoint operator $\mathcal{A}$, we often need to assume that it has discrete spectrum. We often use $\{-\lambda_k;\ k \geq 1\}$ to denote the eigenvalues of $\mathcal{A}$, arranged in decreasing order in k and repeated according to multiplicity. When there are only finitely many, say N, eigenvalues, we put $\lambda_{N+1} = \lambda_{N+2} = \cdots = \infty$. We are in particular interested in the case when Y^n and Y are subordinate processes of a symmetric right process X on E. When D is an open subset of $\mathbf{R}^d$, we will use $L^2(D)$ to denote the space of all square-integrable functions with respect to the Lebesgue measure in D. The following is one of the main results of this paper.

Theorem 1.1. *Let $\{Y^n;\ n \geq 1\}$ be a sequence of symmetric Lévy processes in $\mathbf{R}^d$ with Lévy exponents $\{\Phi_n;\ n \geq 1\}$, and Y a symmetric Lévy process in $\mathbf{R}^d$ with Lévy exponent Φ. Let $D \subset \mathbf{R}^d$ be an open set. Assume that*

$$\lim_{n\to\infty} \Phi_n(\xi) = \Phi(\xi) \quad \text{for every } \xi \in \mathbf{R}^d. \tag{1.1}$$

Then

(i) *For every $t > 0$, the semigroup $P_t^{n,D}$ of $Y^{n,D}$ converges strongly to the semigroup P_t^D of Y^D in $L^2(D)$, that is, for every $f \in L^2(D)$, $P_t^{n,D} f$ converges to $P_t^D f$ in $L^2(D)$. Consequently, the spectral family for the generator of $Y^{n,D}$ converges to that of the generator of Y^D.*

(ii) *Assume that for each $n \geq 1$, the generators of $Y^{n,D}$ and Y^D have discrete spectra $\{-\lambda_k^{(n)};\ k \geq 1\}$ and $\{-\lambda_k;\ k \geq 1\}$, respectively, arranged in decreasing order in k and repeated according to multiplicity. Then*

$$\lim_{n\to\infty} \lambda_k^{(n)} = \lambda_k \quad \text{for every } k \geq 1.$$

Moreover, if, for every $k \geq 1$ and $n \geq 1$, $\psi_k^{(n)}$ is an eigenfunction for the generator of $Y^{n,D}$ corresponding to the eigenvalue $-\lambda_k^{(n)}$ with $\|\psi_k^{(n)}\|_{L^2(D)} = 1$, then any limit point of $\{\psi_k^{(n)};\ n \geq 1\}$ in $L^2(D)$ is a unit eigenfunction for the generator of Y^D with eigenvalue $-\lambda_k$.

See the sentence following the statement of Proposition 2.2 for the definition of the convergence of spectral family. The second main result of paper is concerned with subordinate processes of a symmetric Markov process. Let $X = \{X_t, t \geq 0\}$ be a symmetric right process in a Lusin space E with respect to a σ-finite measure m on E with full support. Using quasi-homeomorphism [3], without loss of generality, we may and do assume that X is an m-symmetric Hunt process associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a locally compact separable metric space E with m being a Radon measure with full support on E. Let $S^{(n)} = \{S_t^{(n)}, t \geq 0\}$ be a subordinator with Laplace exponent ϕ_n and $S = \{S_t, t \geq 0\}$ a subordinator with Laplace exponent ϕ. Suppose that X is independent of S and $S^{(n)}$. Consider the subordinate processes $X^{\phi_n} := \left\{X_{S_t^{(n)}},\ t \geq 0\right\}$ and $X^{\phi} := \{X_{S_t},\ t \geq 0\}$, and their subprocesses $X^{\phi_n, D}$ and $X^{\phi, D}$ killed upon leaving D, where D is an open subset of E.

Theorem 1.2. *Assume that*

$$\lim_{n\to\infty} \phi_n(\lambda) = \phi(\lambda) \quad \textit{for every } \lambda \geq 0. \tag{1.2}$$

Then

(i) *For every $t > 0$, the semigroup $P_t^{\phi_n,D}$ of $X^{\phi_n,D}$ converges strongly to the semigroup $P_t^{\phi,D}$ of $X^{\phi,D}$ in $L^2(D;m)$. Consequently the spectral family for the generator of $X^{\phi_n,D}$ converges to that of the generator of $X^{\phi,D}$.*

(ii) *Assume that for each $n \geq 1$, the generators of $X^{\phi_n,D}$ and $X^{\phi,D}$ have discrete spectra $\{-\lambda_k^{\phi_n,D};\ k \geq 1\}$ and $\{-\lambda_k^{\phi,D};\ k \geq 1\}$, respectively, arranged in decreasing order in k and repeated according to multiplicity. Then*

$$\lim_{n\to\infty} \lambda_k^{\phi_n,D} = \lambda_k^{\phi,D} \quad \textit{for every } k \geq 1. \tag{1.3}$$

Moreover, if, for every $k \geq 1$ and $n \geq 1$, $\psi_k^{(n)}$ is an eigenfunction for the generator of $X^{\phi_n,D}$ corresponding to the eigenvalue $-\lambda_k^{\phi_n,D}$ with $\|\psi_k^{(n)}\|_{L^2(D;m)} = 1$, then any limit point of $\{\psi_k^{(n)};\ n \geq 1\}$ in $L^2(D;m)$ is a unit eigenfunction for the generator of $X^{\phi,D}$ with eigenvalue $-\lambda_k^{\phi,D}$.

In many concrete examples such as in the fractional Laplacian case, using the Sobolev embedding theorem, it is possible to show that $\{\psi_k^{(n)};\ n \geq 1\}$ in Theorems 1.1 and 1.2 is precompact in $L^2(D;m)$. See Examples 5.1 and 5.3 in Section 5 below. For two-sided eigenvalue estimates for subordinate processes in D, see Chen and Song [4].

To rephrase Theorem 1.2 analytically, let $\mathcal{A}$ be the L^2-infinitesimal generator of X, which is a non-positive definite self-adjoint operator in $L^2(E;m)$. It admits a spectral representation

$$\mathcal{A} = -\int_0^\infty \lambda \, dE_\lambda.$$

For any Bernstein function ϕ with $\phi(0) = 0$ (or, equivalently, a function ϕ that is the Laplace exponent of a subordinator), define

$$\phi(-\mathcal{A}) := \int_0^\infty \phi(\lambda)\, dE_\lambda,$$

which is a non-negative definite self-adjoint operator in $L^2(E;m)$. Let $\phi(-\mathcal{A})_D$ be the operator $\phi(-\mathcal{A})$ in D satisfying the zero "exterior" condition on D^c. Then the main results of this paper can be restated as follows.

(a) the spectral family of $\phi_n(-\mathcal{A})_D$ converges to that of $\phi(-\mathcal{A})_D$ if condition (1.2) holds;

(b) If the spectra of $\phi_n(-\mathcal{A})_D$ and $\phi(-\mathcal{A})_D$ are all discrete, with $\{\lambda_k^{\phi_n,D}; k \geq 1\}$ being the eigenvalues of $\phi_n(-\mathcal{A})_D$ and $\{\lambda_k^{\phi,D}; k \geq 1\}$ being the eigenvalues of $\phi(-\mathcal{A})_D$, then (1.3) holds under condition (1.2).

We remark here that it is easy to see that, under condition (1.2), the spectral family of $\phi_n(-\mathcal{A})$ converges to that of $\phi(-\mathcal{A})$ (see Lemma 4.2 and Proposition 2.2 below). But here we are concerned with the spectral convergence for the operators $\phi_n(-\mathcal{A})$ and $\phi(-\mathcal{A})$ in D satisfying zero "exterior" condition on D^c. Note that typically, $\phi_n(-\mathcal{A})$ and $\phi(-\mathcal{A})$ are non-local operators even if $\mathcal{A}$ is a differential operator.

In [6], DeBlassie and Méndez-Hernández considered the case of X being a Brownian motion in $\mathbf{R}^d$ running twice fast as the standard Brownian motion, $\phi_\alpha(\lambda) = \lambda^{\alpha/2}$ for $\alpha \in (0, 2)$ and D is a connected open subset of $\mathbf{R}^d$ having finite Lebesgue measure. In this case, X^{ϕ_α} is a spherically symmetric α-stable process in $\mathbf{R}^d$, whose generator is the fractional Laplacian $-(-\Delta)^{\alpha/2}$. The process $X^{\phi_\alpha, D}$ is a spherically symmetric α-stable process killed upon leaving D with generator $-(-\Delta)^{\alpha/2}$ in D satisfying zero Dirichlet exterior condition on D^c. Since D has finite Lebesgue measure, the generator of $X^{\phi_\alpha, D}$ has discrete spectrum with eigenvalues denoted as $\{-\lambda_k^{(\alpha)};\ k \geq 1\}$. It is shown in [6] that for every $k \geq 1$, the eigenvalue $\lambda_k^{(\alpha)}$ as a function of $\alpha \in (0, 2)$ is right continuous, and it is continuous if D is a bounded Lipschitz domain in $\mathbf{R}^d$. However as a special case of the main results of this paper, we can show (see Example 5.1 below) that in fact for any *open subset* $D \subset \mathbf{R}^d$ having finite Lebesgue measure, the eigenvalue $\lambda_k^{(\alpha)}$ is continuous in $\alpha \in (0, 2]$ for every $k \geq 1$. (Note that the right endpoint 2 is included and D need not be connected.)

The rest of the paper is organized as follows. In section 2, we collect some results concerning the convergence of spectral family. It is well-known that for self-adjoint operators, the strong L^2-convergence of resolvents implies the spectral convergence, while the strong L^2-convergence of resolvents is equivalent to the strong L^2-convergence of semigroups, which is also equivalent to the Mosco convergence of the associated quadratic forms. Theorem 1.1 is proved in section 3 by showing that the Dirichlet form of $Y^{n,D}$ converges to that of Y^D in the sense of Mosco. This result is in particular applicable to subordinate processes of a symmetric Lévy process. In section 4, we give the proof of Theorem 1.2 by showing that the Dirichlet form of $X^{\phi_n, D}$ converges to that of $X^{\phi, D}$ in the sense of Mosco. Some concrete examples are given in section 5 to illustrate the main results of this paper.

In this paper, we use ":=" as a way of definition, which is read as "is defined to be". For two real numbers a and b, $a \wedge b := \min\{a, b\}$. For a locally compact separable metric space E, we use $C_c(E)$ to denote the space of continuous functions with compact support in E. The space of smooth functions with compact support in an open subset $D \subset \mathbf{R}^d$ will be denoted as $C_c^\infty(D)$. For any $u \in L^1(\mathbf{R}^d)$, its Fourier transform $\widehat{u}$ is defined as

$$\widehat{u}(\xi) := \int_{\mathbf{R}^d} e^{i\xi\cdot x} u(x)\, dx \quad \text{for } \xi \in \mathbf{R}^d.$$

It is well-known that $\widehat{u}$ is well-defined for $u \in L^2(\mathbf{R}^d)$ and $\|\widehat{u}\|_{L^2(\mathbf{R}^d)} = \|u\|_{L^2(\mathbf{R}^d)}$.

2. Eigenvalue convergence

Let E be a locally compact separable metric space and m a Radon measure on E with full support. Given a densely defined quadratic form $(\mathcal{E}, \mathcal{F})$ in $L^2(E; m)$, we can extend its domain of definition to $L^2(E; m)$ by setting $\mathcal{E}(u, u) = \infty$ for $u \in L^2(E; m) \setminus \mathcal{F}$. Throughout this paper, we will use this extension and, unless otherwise specified, all the quadratic forms encountered will be assumed to be densely defined in $L^2(E; m)$.

Definition 2.1. *A sequence of closed quadratic forms* $\{(\mathcal{E}^{(n)}, \mathcal{F}^{(n)}),\ n \geq 1\}$ *on* $L^2(E; m)$ *is said to be convergent to a closed quadratic form* $(\mathcal{E}, \mathcal{F})$ *on* $L^2(E; m)$ *in the sense of Mosco (cf. [9]) if*

(i) *For every sequence* $\{u_n, n \geq 1\}$ *in* $L^2(E; m)$ *that converges weakly to* u *in* $L^2(E; m)$,

$$\liminf_{n\to\infty} \mathcal{E}^{(n)}(u_n, u_n) \geq \mathcal{E}(u, u),$$

(ii) *For every* $u \in L^2(E; m)$, *there is a sequence* $\{u_n, n \geq 1\}$ *in* $L^2(E; m)$ *converging strongly to* u *in* $L^2(E; m)$ *such that*

$$\limsup_{n\to\infty} \mathcal{E}^{(n)}(u_n, u_n) \leq \mathcal{E}(u, u).$$

Let $\{P_t,\ t \geq 0\}$ and $\{P_t^{(n)},\ t \geq 0\}$ be the semigroups of $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$, respectively, and $\{G_\alpha, \alpha > 0\}$ and $\{G_\alpha^{(n)}, \alpha > 0\}$ their corresponding resolvents, respectively. The following result is known (see Theorem 2.4.1 and Corollary 2.6.1 of [9]).

Proposition 2.1. *The following are equivalent for closed quadratic forms* $(\mathcal{E}, \mathcal{F})$ *and* $\{(\mathcal{E}^{(n)}, \mathcal{F}^{(n)}),\ n \geq 1\}$ *on* $L^2(E; m)$.

(i) $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ *converges to* $(\mathcal{E}, \mathcal{F})$ *in the sense of Mosco;*
(ii) *For every* $\alpha > 0$ *and* $f \in L^2(E; m)$, $G_\alpha^{(n)} f$ *converges to* $G_\alpha f$ *in* $L^2(E; m)$;
(iii) *For every* $t > 0$ *and* $f \in L^2(E; m)$, $P_t^{(n)} f$ *converges to* $P_t f$ *in* $L^2(E; m)$.

Let $\mathcal{A}$ be the L^2-infinitesimal generator of $(\mathcal{E}, \mathcal{F})$, which is non-positive definite and self-adjoint. So it has a spectral representation

$$\mathcal{A} = -\int_0^\infty \lambda dE_\lambda \quad \text{with}$$
$$\mathrm{Dom}(\mathcal{A}) = \left\{ u \in L^2(E; m) : \int_0^\infty \lambda^2\, d(E_\lambda u, u) < \infty \right\}.$$

Here $\{E_\lambda, \lambda \geq 0\}$ is a right continuous increasing sequence of orthogonal projections in $L^2(E; m)$ with $E_0 = 0$ and $E_\infty = I$ the identity operator. The family $\{E_\lambda, \lambda \geq 0\}$ is called the spectral family of $-\mathcal{A}$ or, equivalently, of $(\mathcal{E}, \mathcal{F})$. For $n \geq 1$, let $\{E_\lambda^{(n)}, \lambda \geq 0\}$ be the spectral family of $\left(\mathcal{E}^{(n)}, \mathcal{F}^{(n)}\right)$.

The following result that resolvent convergence implies spectral convergence is well-known (see, e.g., Theorem VIII.24 in [11] or Corollary 2.7.1 of [9]).

Proposition 2.2. *Suppose that* $\left(\mathcal{E}^{(n)}, \mathcal{F}^{(n)}\right)$ *is a sequence of closed quadratic forms in* $L^2(E; m)$ *that converges to a closed quadratic form* $(\mathcal{E}, \mathcal{F})$ *in* $L^2(E; m)$ *in the sense of Mosco. Then for any pair* $\lambda > \mu$ *that are points of continuity for the spectral family* $\{E_\lambda, \lambda \geq 0\}$ *of* $(\mathcal{E}, \mathcal{F})$*,*

$$\text{the orthogonal projection } E_\lambda^{(n)} - E_\mu^{(n)} \text{ converges strongly to } E_\lambda - E_\mu \text{ as } n \to \infty.$$

Whenever the conclusion of Proposition 2.2 holds, we say the spectral family of $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ converges to that of $(\mathcal{E}, \mathcal{F})$. We will say a closed quadratic form $(\mathcal{E}, \mathcal{F})$ has discrete spectrum if its L^2-generator $\mathcal{A}$ has discrete spectrum, and an eigenvalue (respectively, an eigenfunction) of $-\mathcal{A}$ will be called an eigenvalue (respectively, an eigenfunction) of $(\mathcal{E}, \mathcal{F})$. Note that when $(\mathcal{E}, \mathcal{F})$ has discrete spectrum, the linear subspace $\left(E_\lambda - E_\mu\right) L^2(E; m)$ is the linear span of eigenfunctions corresponding to all eigenvalues of $(\mathcal{E}, \mathcal{F})$ that are in the interval $(\mu, \lambda]$. In the theorem below, the quadratic forms are only assumed to be closed and do not need to be Markovian (i.e. they do not have to be Dirichlet forms).

Theorem 2.3. *Let* $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ *be a sequence of closed quadratic forms in* $L^2(E; m)$ *that converges to a closed quadratic form* $(\mathcal{E}, \mathcal{F})$ *in* $L^2(E; m)$ *in the sense of Mosco. Assume that for each* $n \geq 1$, $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ *and* $(\mathcal{E}, \mathcal{F})$ *have discrete spectra* $\{\lambda_k^{(n)};\ k \geq 1\}$ *and* $\{\lambda_k;\ k \geq 1\}$*, respectively, arranged in increasing order in k and repeated according to multiplicity. Then*

$$\lim_{n \to \infty} \lambda_k^{(n)} = \lambda_k \quad \text{for } k \geq 1. \tag{2.1}$$

Moreover, if, for every $k \geq 1$ *and* $n \geq 1$, $\psi_k^{(n)}$ *is an eigenfunction of* $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ *corresponding to the eigenvalue* $\lambda_k^{(n)}$ *with* $\|\psi_k^{(n)}\|_{L^2(E;m)} = 1$*, then any limit point of* $\{\psi_k^{(n)};\ n \geq 1\}$ *in* $L^2(E; m)$ *is a unit eigenfunction of* $(\mathcal{E}, \mathcal{F})$ *with eigenvalue* λ_k*.*

Proof. We only deal with the case when $(\mathcal{E}, \mathcal{F})$ has infinitely many eigenvalues. The case when $(\mathcal{E}, \mathcal{F})$ has finitely many eigenvalues is similar. Since $(\mathcal{E}, \mathcal{F})$ is a non-negative definite closed quadratic form having discrete spectrum, $\{\lambda_k; k \geq 1\}$ has no finite accumulation point. In other words, $\lim_{k\to\infty} \lambda_k = \infty$. For any $k \geq 1$, there is $\varepsilon_0 = \varepsilon_0(k) > 0$ such that $(\lambda_k - \varepsilon_0,\ \lambda_k + \varepsilon_0)$ contains no other eigenvalues except λ_k (counted with multiplicity). It follows from Proposition 2.2 that for every $\varepsilon \in (0,\ \varepsilon_0)$, as $n \to \infty$,

$$\text{the orthogonal projection } E^{(n)}_{\lambda_k+\varepsilon} - E^{(n)}_{\lambda_k-\varepsilon} \text{ converges strongly to } E_{\lambda_k+\varepsilon} - E_{\lambda_k-\varepsilon}. \tag{2.2}$$

Consequently,

$$\lim_{n\to\infty} \dim\left(\left(E^{(n)}_{\lambda_k+\varepsilon} - E^{(n)}_{\lambda_k-\varepsilon}\right) L^2(E; m)\right) = \dim\left(\left(E_{\lambda_k+\varepsilon} - E_{\lambda_k-\varepsilon}\right) L^2(E; m)\right), \tag{2.3}$$

where the right hand side equals to the multiplicity of the eigenvalue λ_k. Since (2.3) holds for any $k \geq 1$ and for any $\varepsilon < \varepsilon_0(k)$, it is easy to see now that (2.1) holds.

Suppose now that for $k \geq 1$ and $n \geq 1$, $\psi_k^{(n)}$ is an eigenfunction of $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ corresponding to the eigenvalue $\lambda_k^{(n)}$ and $\|\psi_k^{(n)}\|_{L^2(E;m)} = 1$ and ψ is a limit point of $\{\psi_k^{(n)};\ n \geq 1\}$ in $L^2(E;m)$. There is a subsequence $\{n_j;\ j \geq 1\}$ so that $\psi_k^{(n_j)}$ converges to ψ in $L^2(E;m)$. Consequently,

$$\lim_{j\to\infty} \|(E^{(n_j)}_{\lambda_k+\varepsilon} - E^{(n_j)}_{\lambda_k-\varepsilon})(\psi_k^{(n_j)} - \psi)\|_{L^2(E;m)} \leq \lim_{j\to\infty} \|(\psi_k^{(n_j)} - \psi)\|_{L^2(E;m)} = 0.$$

On the other hand, $(E^{(n_j)}_{\lambda_k+\varepsilon} - E^{(n_j)}_{\lambda_k-\varepsilon})\psi_k^{(n_j)} = \psi_k^{(n_j)}$ and, by (2.2), $(E^{(n_j)}_{\lambda_k+\varepsilon} - E^{(n_j)}_{\lambda_k-\varepsilon})\psi$ converges in $L^2(E;m)$ to $(E_{\lambda_k+\varepsilon} - E_{\lambda_k-\varepsilon})\psi$. It follows that $\psi = (E_{\lambda_k+\varepsilon} - E_{\lambda_k-\varepsilon})\psi$ and so ψ is a unit eigenfunction of $(\mathcal{E}, \mathcal{F})$ with eigenvalue λ_k. □

Remark 2.4. In many concrete examples such as in the fractional Laplacian case, using the Sobolev embedding theorem, it is possible to show that $\{\psi_k^{(n)};\ n \geq 1\}$ in Theorem 2.3 is precompact in $L^2(E;m)$. See Examples 5.1 and 5.3 in Section 5 below.

By Propositions 2.1-2.2 and Theorem 2.3, in order to show spectral convergence, one can check one of the three equivalent conditions in Proposition 2.1 holds. In Sections 3 and 4 we will establish respectively the spectral convergences of symmetric Lévy processes in domains and of subordinate processes in domains by proving the Mosco convergence of the corresponding Dirichlet forms.

3. Spectral convergence of symmetric Lévy processes in domains

We first recall some basic facts about symmetric Lévy processes and then establish a lemma that will be used later in this section.

Suppose that Y is a Lévy process in $\mathbf{R}^d$ with Lévy exponent Φ; that is,

$$\mathbf{E}\left[e^{i\xi\cdot(Y_t-Y_0)}\right] = e^{-t\Phi(\xi)} \quad \text{for every } t > 0 \text{ and } \xi \in \mathbf{R}^d.$$

By the Lévy-Khintchine formula, there is a constant vector $\vec{b} \in \mathbf{R}^d$, a non-negative definite symmetric constant matrix $A = (a_{ij})$ and a measure Π on $\mathbf{R}^d \setminus \{0\}$ with $\int (1 \wedge |y|^2)\Pi(dy) < \infty$ such that

$$\Phi(\xi) = i\,\vec{b}\cdot\xi + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j + \int_{\mathbf{R}^d}\left(1 - e^{i\xi\cdot y} + i\xi\cdot y 1_{\{|y|<1\}}\right)\Pi(dy) \quad (3.1)$$

for every $\xi = (\xi_1, \cdots, \xi_d) \in \mathbf{R}^d$.

Now assume that Y is a symmetric Lévy process in $\mathbf{R}^d$ with Lévy exponent Φ. Here, *symmetric* means the semigroup of Y is a symmetric operator in $L^2(\mathbf{R}^d)$; or equivalently, $Y - Y_0$ and $-(Y - Y_0)$ have the same law. This amounts to say that the Lévy exponent Φ is real-valued. Hence by (3.1), we have

$$\Phi(\xi) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j + \int_{\mathbf{R}^d}(1 - \cos(\xi\cdot y))\,\Pi(dy) \quad (3.2)$$

Thus

$$0 \le \Phi(\xi) \le \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j + \int_{\mathbf{R}^d} \left(1 \wedge (\xi \cdot y)^2\right) \Pi(dy)$$
$$\le \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j + (1 + |\xi|^2) \int_{\mathbf{R}^d} \left(1 \wedge |y|^2\right) \Pi(dy) \tag{3.3}$$

Lemma 3.1. *Let $\{Y^n;\ n \ge 1\}$ be a sequence of symmetric Lévy processes in $\mathbf{R}^d$ with Lévy exponents $\{\Phi_n;\ n \ge 1\}$. If $\Phi_n(\xi)$ converges as $n \to \infty$ for every $\xi \in \mathbf{R}^d$ to the Lévy exponent of a Lévy process in $\mathbf{R}^d$, then there is a constant $C > 0$ such that*

$$\Phi_n(\xi) \le C(1 + |\xi|^2) \quad \textit{for every } n \ge 1 \textit{ and every } \xi \in \mathbf{R}^d. \tag{3.4}$$

Proof. In view of (3.2), for every $n \ge 1$, there is a non-negative definite symmetric constant matrix $A^n = (a^n_{ij})$ and a measure Π_n on $\mathbf{R}^d \setminus \{0\}$ with $\int (1 \wedge |y|^2)\Pi_n(dy) < \infty$ such that

$$\Phi_n(\xi) = \frac{1}{2} \sum_{i,j=1}^{d} a^n_{ij}\xi_i\xi_j + \int_{\mathbf{R}^d} (1 - \cos(\xi \cdot y))\, \Pi_n(dy).$$

Since Φ_n converges pointwise to the Lévy exponent of a Lévy process, it follows from [12, (2.8.12)] that $\left\{\int_{R^d} (1 \wedge |y|^2)\Pi_n(dy);\ n \ge 1\right\}$ is bounded. On the other hand, since

$$\Phi_n(\xi) \ge \frac{1}{2} \sum_{i,j=1}^{d} a^n_{ij}\xi_i\xi_j \quad \text{for every } \xi \in \mathbf{R}^d,$$

the pointwise convergence of Φ_n implies that a^n_{ij} is uniformly bounded in $i, j \in \{1, \cdots, d\}$ and in $n \ge 1$. Thus the conclusion of the lemma follows from (3.3). □

Let Y be a symmetric Lévy process with Lévy exponent Φ. Its L^2-generator is denoted by $\mathcal{A}$. By definition, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of Y is given by $\mathcal{F} = \text{Dom}\sqrt{-\mathcal{A}}$ and

$$\mathcal{E}(u, v) = (\sqrt{-\mathcal{A}}u,\ \sqrt{-\mathcal{A}}v)_{L^2(\mathbf{R}^d)} \quad \text{for } u, v \in \mathcal{F}.$$

Thus

$$\mathcal{F} = \left\{u \in L^2(\mathbf{R}^d) :\ \int_{\mathbf{R}^d} |\widehat{u}(\xi)|^2\, \Phi(\xi)\, d\xi < \infty\right\} \tag{3.5}$$
$$\mathcal{E}(u, v) = \int_{\mathbf{R}^d} \widehat{u}(\xi)\, \overline{\widehat{v}(\xi)}\, \Phi(\xi)\, d\xi \quad \text{for } u, v \in \mathcal{F}. \tag{3.6}$$

It is known (cf. [7, Example 1.4.1]) that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(\mathbf{R}^d)$ with a core $C_c^\infty(\mathbf{R}^d)$. We are going to use P_t to denote the transition semigroup of Y.

Let $\{Y^n;\ n \geq 1\}$ be a sequence of symmetric Lévy processes in $\mathbf{R}^d$ with Lévy exponents $\{\Phi_n;\ n \geq 1\}$. We are going to use P_t^n to denote the transition semigroup of Y^n and $(\mathcal{E}^n, \mathcal{F}^n)$ to denote the Dirichlet form of Y^n.

Let $D \subset \mathbf{R}^d$ be an open set. Denote by $Y^{n,D}$ and Y^D the subprocesses of Y^n and Y, respectively, killed upon leaving D. We will use $P_t^{n,D}$ and P_t^D to denote the transition semigroups of $Y^{n,D}$ and Y^D respectively. The Dirichlet forms of $Y^{n,D}$ and Y^D will be denoted by $(\mathcal{E}^n, \mathcal{F}^{n,D})$ and $(\mathcal{E}, \mathcal{F}^D)$ respectively.

Lemma 3.2. *If* $\lim_{n\to\infty} \Phi_n(\xi) = \Phi(\xi)$ *for every* $\xi \in \mathbf{R}^d$*, then, for every* $t > 0$ *and every* $f \in L^2(\mathbf{R}^d)$*,* $P_t^n f$ *converges to* $P_t f$ *in* $L^2(\mathbf{R}^d)$*. Consequently, the Dirichlet forms* $(\mathcal{E}^n, \mathcal{F}^n)$ *converge to* $(\mathcal{E}, \mathcal{F})$ *as* $n \to \infty$ *in the sense of Mosco.*

Proof. It is well-known (see, for instance, Example 1.4.1 of [7]) that for any $t > 0$, $\xi \in \mathbf{R}^d$ and $f \in L^2(\mathbf{R}^d)$,

$$\widehat{P_t^n f}(\xi) = e^{-t\Phi_n(\xi)}\widehat{f}(\xi), \quad \widehat{P_t f}(\xi) = e^{-t\Phi(\xi)}\widehat{f}(\xi).$$

So by the dominated convergence theorem, $P_t^n f$ converges to $P_t f$ in $L^2(\mathbf{R}^d)$. The last assertion follows from Proposition 2.1. □

Theorem 3.3. *If* $\lim_{n\to\infty} \Phi_n(\xi) = \Phi(\xi)$ *for every* $\xi \in \mathbf{R}^d$*, then the Dirichlet forms* $(\mathcal{E}^n, \mathcal{F}^{n,D})$ *converge to* $(\mathcal{E}, \mathcal{F}^D)$ *as* $n \to \infty$ *in the sense of Mosco.*

Proof. Since by Lemma 3.2, $(\mathcal{E}^n, \mathcal{F}^n)$ converges to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco, we have in particular that for any v_n that converges weakly to v in $L^2(D)$,

$$\liminf_{n\to\infty} \mathcal{E}^n(v_n, v_n) \geq \mathcal{E}(v, v).$$

So to show that $(\mathcal{E}^n, \mathcal{F}^{n,D})$ converges to $(\mathcal{E}, \mathcal{F}^D)$ in the sense of Mosco, we only need to show that condition (ii) in Definition 2.1 is satisfied.

Since for any $u \in L^2(\mathbf{R}^d)$ we have

$$\mathcal{E}(u, u) = \int_{\mathbf{R}^d} |\widehat{u}(\xi)|^2\, \Phi(\xi)\, d\xi$$

and

$$\mathcal{E}^n(u, u) = \int_{\mathbf{R}^d} |\widehat{u}(\xi)|^2\, \Phi_n(\xi)\, d\xi,$$

we can apply Lemma 3.1 and the dominated convergence theorem to get that

$$\lim_{n\to\infty} \mathcal{E}^n(u, u) = \mathcal{E}(u, u) \quad \text{for } u \in C_c^\infty(\mathbf{R}^d). \tag{3.7}$$

It is well-known that $C_c^\infty(D)$ is a core for $(\mathcal{E}, \mathcal{F}^D)$ and $(\mathcal{E}^n, \mathcal{F}^{n,D})$. Using these facts and (3.7) we can show that, for any $u \in \mathcal{F}^D$ there exists a sequence $\{u_n\} \subset C_c^\infty(D)$ converging strongly to u in $L^2(D)$ such that

$$\lim_{n\to\infty} \mathcal{E}^n(u_n, u_n) = \mathcal{E}(u, u).$$

In fact, for any $u \in \mathcal{F}^D$, there exists a sequence $\{v_j\} \subset C_c^\infty(D)$ converging strongly to u in $L^2(D)$ such that

$$\lim_{j\to\infty} \mathcal{E}(v_j, v_j) = \mathcal{E}(u, u).$$

Using (3.7) and induction we can find a subsequence $\{n_j\}$ such that

$$|\mathcal{E}^n(v_j, v_j) - \mathcal{E}(v_j, v_j)| \le \frac{1}{j} \quad \text{for } n \ge n_j.$$

Now we put $u_1 = \cdots = u_{n_1-1} = 0$ and $u_{n_j} = \cdots = u_{n_{j+1}-1} = v_j$ for $j \ge 1$. We can easily see that $\{u_n\}$ is a sequence in $C_c^\infty(D)$ converging strongly to u in $L^2(D)$ such that

$$\lim_{n\to\infty} \mathcal{E}^n(u_n, u_n) = \mathcal{E}(u, u).$$

For $u \in L^2(D) \setminus \mathcal{F}^D$, since $\mathcal{E}(u, u) = \infty$, it trivially holds that

$$\limsup_{n\to\infty} \mathcal{E}^n(u, u) \le \mathcal{E}(u, u).$$

This shows that the condition (ii) in Definition 2.1 is satisfied. The theorem is now proved. □

In particular, the above theorem yields the semigroup continuity for subordinate Lévy processes in domains. Assume that X is a symmetric Lévy process in $\mathbf{R}^d$ with Lévy exponent Ψ. Suppose that S is a subordinator with Laplace exponent ϕ and that, for any $n \ge 1$, $S^{(n)}$ is a subordinator with Laplace exponent ϕ_n. Let $S^{(n)}$ and S be independent of X. Let $Y_t^n := X_{S_t^n}$ and $Y_t := X_{S_t}$ be the subordinate processes of X. Let $D \subset \mathbf{R}^d$ be an open set. Denote by $Y^{n,D}$ and Y^D the subprocesses of Y^n and Y, respectively, killed upon leaving D. We will use $P_t^{n,D}$ and P_t^D to denote the the transition semigroups of $Y^{n,D}$ and Y^D, respectively.

Corollary 3.4. *Assume that* $\lim_{n\to\infty} \phi_n(\lambda) = \phi(\lambda)$ *for every* $\lambda \ge 0$. *Then for every* $t > 0$, $P_t^{n,D}$ *converges strongly to* P_t^D *in* $L^2(D)$.

Proof. Note that $Y_t^n = X_{S_t^n}$ and $Y_t = X_{S_t}$ are symmetric Lévy processes with Lévy exponents $\Phi_n(\xi) := \phi_n(\Psi(\xi))$ and $\Phi(\xi) := \phi(\Psi(\xi))$, respectively. Thus

$$\lim_{n\to\infty} \Phi_n(\xi) = \Phi(\xi) \quad \text{for every } \xi \in \mathbf{R}^d.$$

The Corollary now follows immediately from Theorem 3.3. □

Proof of Theorem 1.1. Combining Theorem 3.3 with Propositions 2.1-2.2 and Theorem 2.3 establishes Theorem 1.1. □

4. Spectral convergence of subordinate processes in domains

Let X be an m-symmetric right process on a Lusin space E. Using quasi-homeomorphism (see [3]), without loss of generality, we may and do assume that X is an m-symmetric Hunt process associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a locally compact separable metric space E, where m is a Radon measure with full support on E. Let $\mathcal{A}$ be its L^2-infinitesimal generator, which is a non-positive definite self-adjoint operator. So it has a spectral representation

$$-\mathcal{A} = \int_0^\infty \lambda dE_\lambda.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X is defined as $\mathcal{F} = \mathrm{Dom}(\sqrt{-\mathcal{A}})$ and

$$\mathcal{E}(u, v) = (\sqrt{-\mathcal{A}}u, \sqrt{-\mathcal{A}}v)_{L^2(E;m)} \quad \text{for } u, v \in \mathcal{F}.$$

Using spectral representation,

$$\mathcal{F} = \left\{ u \in L^2(E; m) : \int_0^\infty \lambda\, d(E_\lambda u, u) < \infty \right\},$$
$$\mathcal{E}(u, v) = \int_0^\infty \lambda\, d(E_\lambda u, v) \quad \text{for } u, v \in \mathcal{F}.$$

We now recall some basic facts about subordinators and then establish a lemma that will be used later in the section.

Suppose that $S = \{S_t,\ t \geq 0\}$ is a subordinator; that is, S is a one-dimensional Lévy process taking values in $[0, \infty)$ with $S_0 = 0$. Let ϕ be its Laplace exponent:

$$\mathbf{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)} \quad \text{for every } t > 0 \text{ and } \lambda > 0.$$

It is known (see, e.g., [2, page 72]) that the Laplace exponent ϕ can be expressed as

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\pi(dx), \quad \lambda > 0 \tag{4.1}$$

for some $b \geq 0$ and Lévy measure π on $(0, \infty)$ with $\int_0^\infty (1 \wedge x)\pi(dx) < \infty$. Clearly the Laplace exponent ϕ is a Bernstein function with $\phi(0) = 0$. Here a C^∞ function $f : (0, \infty) \to \mathbf{R}_+$ is called a Bernstein function if $(-1)^n \frac{d^n f(x)}{dx^n} \leq 0$ on $(0, \infty)$ for every $n \geq 1$. Conversely, any Bernstein function f with $f(0) = 0$ is the Laplace exponent of a subordinator. Let $\overline{\pi}(x) := \pi(x, \infty)$ be the tail of the Lévy measure π. Then

$$\frac{\phi(\lambda)}{\lambda} = b + \int_0^\infty e^{-\lambda t}\overline{\pi}(t)\, dt. \tag{4.2}$$

Note that $\frac{\phi(\lambda)}{\lambda}$ is a decreasing function in $\lambda \in (0, \infty)$.

Lemma 4.1. *Let $\{\phi_n;\ n \geq 1\}$ be the Laplace exponents of a sequence of subordinators $\{S^n;\ n \geq 1\}$. If $\{\phi_n(\lambda_0);\ n \geq 1\}$ is bounded at some $\lambda_0 > 0$, then there is a constant $C > 0$ such that*

$$\phi_n(\lambda) \leq C\ \max\{1,\ \lambda\}. \quad \textit{for every } n \geq 1 \textit{ and } \lambda \geq 0. \tag{4.3}$$

Proof. Note that any Laplace exponent ϕ can be represented by (4.1). So there exist $b_n \geq 0$ and Lévy measure π_n such that (4.1) holds for ϕ_n with b_n and π_n in place of b and π. With $\overline{\pi}_n(x) := \pi_n(x, \infty)$, we have

$$\frac{\phi_n(\lambda)}{\lambda} = b_n + \int_0^\infty e^{-\lambda t}\, \overline{\pi}_n(t)\, dt. \tag{4.4}$$

Since $\{\phi_n(\lambda_0);\ n \geq 1\}$ is bounded, the sequence $\{b_n + \int_0^\infty e^{-\lambda_0 t}\, \overline{\pi}_n(t) dt;\ n \geq 1\}$ is bounded. Hence the sequence $\{b_n + \int_0^\infty e^{-\lambda t}\, \overline{\pi}_n(t)\, dt;\ n \geq 1\}$ is uniformly bounded in $\lambda \geq \lambda_0$. Thus, by (4.2)-(4.4), there is $c > 0$ such that

$$\phi_n(\lambda) \leq c\, \lambda \quad \text{for every } n \geq 1 \text{ and } \lambda \geq \lambda_0. \tag{4.5}$$

On the other hand, it follows from the representation (4.1), for every $n \geq 1$, $\phi_n(\lambda)$ is an increasing function in $\lambda > 0$. So (4.3) follows from (4.5) □

Suppose that $S := \{S_t,\ t \geq 0\}$ is a subordinator with Laplace exponent ϕ and that, for any $n \geq 1$, $S^{(n)} := \{S_t^{(n)}, t \geq 0\}$ is a subordinator with Laplace exponent ϕ_n. We suppose that $S^{(n)}$ and S are independent of X. Let X^{ϕ_n} be the subordinate process $\{X_{S_t^{(n)}},\ t \geq 0\}$ and X^ϕ the subordinate process $\{X_{S_t},\ t \geq 0\}$. The Dirichlet form associated with X^{ϕ_n} will be denoted as $(\mathcal{E}^{\phi_n}, \mathcal{F}^{\phi_n})$ and the Dirichlet form associated with X^ϕ will be denoted as $(\mathcal{E}^\phi, \mathcal{F}^\phi)$. It follows from spectral representation and Theorem 2.1 of [10] that

$$\mathcal{F}^\phi = \left\{u \in L^2(E; m) :\ \int_0^\infty \phi(\lambda) d(E_\lambda u, u) < \infty\right\},$$
$$\mathcal{F}^{\phi_n} = \left\{u \in L^2(E; m) :\ \int_0^\infty \phi_n(\lambda) d(E_\lambda u, u) < \infty\right\},$$

and

$$\mathcal{E}^\phi(u, v) = \int_0^\infty \phi(\lambda) d(E_\lambda u, v) \quad \text{for } u, v \in \mathcal{F}^\phi,$$
$$\mathcal{E}^{\phi_n}(u, v) = \int_0^\infty \phi_n(\lambda) d(E_\lambda u, v) \quad \text{for } u, v \in \mathcal{F}^{\phi_n}.$$

Thus the generator of X^{ϕ_n} is $\mathcal{L}_n$ with

$$-\mathcal{L}_n = \int_0^\infty \phi_n(\lambda) dE_\lambda \quad \text{and}$$
$$\text{Dom}(\mathcal{L}_n) = \left\{u \in L^2(E; m) :\ \int_0^\infty \phi_n(\lambda)^2 d(E_\lambda u, u) < \infty\right\},$$

and the generator of X^ϕ is $\mathcal{L}$ with

$$-\mathcal{L} = \int_0^\infty \phi(\lambda) dE_\lambda \quad \text{and}$$
$$\mathrm{Dom}(\mathcal{L}_n) = \left\{ u \in L^2(E; m) : \int_0^\infty \phi(\lambda)^2 d(E_\lambda u, u) < \infty \right\}.$$

We are going to use $\{G_\beta, \beta > 0\}$ to denote the resolvent of X^ϕ and $\{G_\beta^{(n)}, \beta > 0\}$ to denote the resolvent of X^{ϕ_n}. We shall assume that

$$\lim_{n\to\infty} \phi_n(\lambda) = \phi(\lambda), \quad \lambda \geq 0. \tag{4.6}$$

Lemma 4.2. *Under the condition (4.6), we have that $G_\beta^{(n)} f$ converges to $G_\beta f$ in $L^2(E, m)$ for every $\beta > 0$ and $f \in L^2(E, m)$. Consequently, the Dirichlet form $(\mathcal{E}^{\phi_n}, \mathcal{F}^{\phi_n})$ converges to $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ as $n \to \infty$ in the sense of Mosco.*

Proof. From the above expressions for the generators $\mathcal{L}_n$ and $\mathcal{L}$, we can easily see that

$$\beta G_\beta^{(n)} = \int_0^\infty \frac{\beta}{\beta + \phi_n(\lambda)} dE_\lambda$$

and

$$\beta G_\beta = \int_0^\infty \frac{\beta}{\beta + \phi(\lambda)} dE_\lambda.$$

So for any $\beta > 0$ and $f \in L^2(E; m)$, by the dominated convergence theorem,

$$\lim_{n\to\infty} \|G_\beta^{(n)} f - G_\beta f\|^2_{L^2(E;m)}$$
$$= \lim_{n\to\infty} \int_0^\infty \left(\frac{\beta}{\beta + \phi_n(\lambda)} - \frac{\beta}{\beta + \phi(\lambda)} \right)^2 d(E_\lambda f, f) = 0.$$

The Mosco convergence of the corresponding Dirichlet forms follows immediately from Proposition 2.1 □

Let D be an open subset of E. We will use $X^{\phi_n, D}$ to denote the part process of X^{ϕ_n} on D and $X^{\phi, D}$ to denote the part process of X^ϕ on D. Let $(\mathcal{E}^{\phi_n}, \mathcal{F}^{\phi_n, D})$ be the Dirichlet form associated with $X^{\phi_n, D}$ and $(\mathcal{E}^\phi, \mathcal{F}^{\phi, D})$ the Dirichlet form associated with $X^{\phi, D}$.

Theorem 4.3. *Under the condition (4.6), the Dirichlet form $(\mathcal{E}^{\phi_n}, \mathcal{F}^{\phi_n, D})$ converges to $(\mathcal{E}^\phi, \mathcal{F}^{\phi, D})$ in the sense of Mosco.*

Proof. The proof of this theorem is similar to that of Theorem 3.3. For the reader's convenience, we spell out the details below. Since by Lemma 4.2, $(\mathcal{E}^{\phi_n}, \mathcal{F}^{\phi_n})$ converges to $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ in the sense of Mosco, we have in particular that for any v_n that converges weakly to v in $L^2(D; m)$,

$$\liminf_{n\to\infty} \mathcal{E}^{\phi_n}(v_n, v_n) \geq \mathcal{E}^\phi(v, v).$$

So to show that $(\mathcal{E}^{\phi_n}, \mathcal{F}^{\phi_n,D})$ converges to $(\mathcal{E}^{\phi}, \mathcal{F}^{\phi,D})$ in the sense of Mosco, we only need to show that condition (ii) in Definition 2.1 is satisfied.

We know that for any $u \in \mathcal{F}$ we have

$$\int_0^\infty \lambda\, d(E_\lambda u, u) < \infty.$$

Since for any $u \in L^2(D; m)$ we have

$$\mathcal{E}^\phi(u, u) = \int_0^\infty \phi(\lambda) d(E_\lambda u, u)$$

and

$$\mathcal{E}^{\phi_n}(u, u) = \int_0^\infty \phi_n(\lambda) d(E_\lambda u, u),$$

we can apply Lemma 4.1 and the dominated convergence theorem to get that

$$\lim_{n\to\infty} \mathcal{E}^{\phi_n}(u, u) = \mathcal{E}^\phi(u, u) \quad \text{for } u \in \mathcal{F}.$$

In particular we have

$$\lim_{n\to\infty} \mathcal{E}^{\phi_n}(u, u) = \mathcal{E}^\phi(u, u) \quad \text{for } u \in \mathcal{F}^D. \tag{4.7}$$

It follows from Section 1.4 of [7] that $\mathcal{C} := \mathcal{F} \cap C_c(E)$ is a special standard core of $\mathcal{E}$. Therefore using Theorem 2.1 of [10] we can see that $\mathcal{C}$ is also a special standard core of $\mathcal{E}^\phi$ and $\mathcal{E}^{\phi_n}$ for every $n \geq 1$. Now we can conclude from Theorem 4.4.3 of [7] that

$$\mathcal{C}_D := \{u \in \mathcal{C} : \text{Supp}[u] \subset D\}$$

is a core of $\mathcal{E}^{\phi,D}$ and $\mathcal{E}^{\phi_n,D}$ for every $n \geq 1$. Of course $\mathcal{C}_D$ is also a core of $\mathcal{E}^D$. Using these facts and (4.7) we can show that, for any $u \in \mathcal{E}^{\phi,D}$ there exists a sequence $\{u_n\} \subset \mathcal{C}_D$ converging strongly to u in $L^2(D; m)$ such that

$$\lim_{n\to\infty} \mathcal{E}^{\phi_n}(u_n, u_n) = \mathcal{E}^\phi(u, u).$$

In fact, for any $u \in \mathcal{F}^{\phi,D}$, there exists a sequence $\{v_j\} \subset \mathcal{C}_D$ converging strongly to u in $L^2(D; m)$ such that

$$\lim_{j\to\infty} \mathcal{E}^\phi(v_j, v_j) = \mathcal{E}^\phi(u, u).$$

Using (4.7) and induction we can find an increasing subsequence $\{n_j\}$ such that

$$|\mathcal{E}^{\phi_n}(v_j, v_j) - \mathcal{E}^\phi(v_j, v_j)| \leq \frac{1}{j} \quad \text{for } n \geq n_j.$$

Now we put $u_1 = \cdots = u_{n_1-1} = 0$ and $u_{n_j} = \cdots = u_{n_{j+1}-1} = v_j$ for $j \geq 1$. We can easily see that $\{u_n\}$ is a sequence in $\mathcal{C}_D$ converging strongly to u in $L^2(D; m)$ such that

$$\lim_{n\to\infty} \mathcal{E}^{\phi_n}(u_n, u_n) = \mathcal{E}^\phi(u, u).$$

For $u \in L^2(D; m) \setminus \mathcal{F}^{\phi,D}$, since $\mathcal{E}^{\phi}(u, u) = \infty$, it trivially holds that

$$\limsup_{n\to\infty} \mathcal{E}^{\phi_n}(u, u) \leq \mathcal{E}(u, u).$$

This shows that the condition (ii) in Definition 2.1 is satisfied. The theorem is now proved. □

Proof of Theorem 1.2. Combining Theorem 4.3 with Propositions 2.1-2.2 and Theorem 2.3 establishes Theorem 1.2. □

5. Examples

In this section, we present some examples to illustrate the main results of this paper. For simplicity, we assume throughout this section that, unless otherwise specified, X is a Brownian motion in $\mathbf{R}^d$ running twice as fast as the standard Brownian motion. More examples can be given in similar lines of this section by replacing X by other kind of symmetric processes, such as spherically symmetric α-stable process, symmetric Lévy process, symmetric diffusions with infinitesimal generators of divergence form.

Example 5.1. Let $\phi_\alpha(\lambda) = \lambda^{\alpha/2}$ for $\alpha \in (0, 2]$. Then X^{ϕ_α} is just a spherically symmetric α-stable process in $\mathbf{R}^d$, whose transition density function has an upper bound estimate $c\, t^{-d/\alpha}$. Let D be an open subset of $\mathbf{R}^d$ of finite Lebesgue measure. Let $X^{\phi_\alpha,D}$ denote the subprocess of X^{ϕ_α} killed upon leaving D. For any $\alpha \in (0, 2]$, the semigroup $P_t^{\phi_\alpha,D}$ of $X^{\phi_\alpha,D}$ is a Hilbert-Schmidt operator in $L^2(D)$ and hence is compact. This implies that, for $\alpha \in (0, 2]$, the generator of $X^{\phi_\alpha,D}$ has discrete spectrum $\{-\lambda_k^{(\alpha)};\ k \geq 1\}$. It follows from Theorem 2.3 and Theorem 4.3 that $\lambda_k^{(\alpha)}$ is continuous in $\alpha \in (0, 2]$ for every $k \geq 1$.

The Dirichlet form of X^{ϕ_α} is $\big(\mathcal{E}^{(\alpha)}, W^{\alpha/2,2}(\mathbf{R}^d)\big)$, where

$$W^{\alpha/2,2}(\mathbf{R}^d) = \left\{u \in L^2(\mathbf{R}^d) : \int_{\mathbf{R}^d} |\widehat{u}(\xi)|^2 |\xi|^\alpha \, d\xi < \infty\right\} \tag{5.1}$$

$$\mathcal{E}^{(\alpha)}(u, v) = \int_{\mathbf{R}^d} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}|\xi|^\alpha \, d\xi, \quad u, v \in W^{\alpha/2,2}(\mathbf{R}^d). \tag{5.2}$$

The Dirichlet form for the part process $X^{\phi_\alpha,D}$ is $\left(\mathcal{E}^{(\alpha)}, W_0^{\alpha/2,2}(D)\right)$, where

$$W_0^{\alpha/2,2}(D) = \left\{u \in W^{\alpha/2,2}(\mathbf{R}^d) : u = 0 \text{ a.e. on } D^c\right\}.$$

Note that for $\alpha \in (0, 2]$ and $u \in W^{\alpha/2,2}(\mathbf{R}^d)$,

$$2^{-\alpha} \int_{\mathbf{R}^d} |\widehat{u}(\xi)|^2 (1 + |\xi|)^\alpha \, d\xi \leq \mathcal{E}_1^{(\alpha)}(u, u) \leq 2 \int_{\mathbf{R}^d} |\widehat{u}(\xi)|^2 (1 + |\xi|)^\alpha \, d\xi. \tag{5.3}$$

Here $\mathcal{E}_1^{(\alpha)}(u, u) := \mathcal{E}^{(\alpha)}(u, u) + \int_{\mathbf{R}^d} u(x)^2 \, dx$. Then for any $\alpha > \beta$ in $(0, 2]$, there is a constant $c > 0$, independent of $\alpha > \beta$, such that

$$\mathcal{E}_1^{(\beta)}(u, u) \leq c\, \mathcal{E}_1^{(\alpha)}(u, u) \quad \text{for every } u \in W^{\alpha/2,2}(D), \tag{5.4}$$

and consequently we have $W_0^{\alpha/2,2}(D) \subset W_0^{\beta/2,2}(D)$. Now assume that $\{\alpha_n;\ n \geq 1\}$ is a sequence in $(0, 2]$ converging to $\alpha \in (0, 2]$. For every $k \geq 1$, let $\psi_k^{(\alpha_n)}$ be a unit eigenfunction of the generator of $X^{\phi_{\alpha_n}, D}$ with eigenvalue $-\lambda_k^{(\alpha_n)}$. Let $\alpha_0 := \inf\{\alpha_n; n \geq 1\}$, which is in $(0, 2]$. Then by (5.4),

$$\begin{aligned}\mathcal{E}_1^{(\alpha_0)}(\psi_k^{(\alpha_n)}, \psi_k^{(\alpha_n)}) &\leq c\mathcal{E}_1^{(\alpha_n)}(\psi_k^{(\alpha_n)}, \psi_k^{(\alpha_n)}) \\ &= c\,(\lambda_k^{(\alpha_n)} + 1)\|\psi_k^{(\alpha_n)}\|^2_{L^2(D)} = c\,(\lambda_k^{(\alpha_n)} + 1).\end{aligned}$$

Hence

$$\sup_{n\geq 1} \mathcal{E}_1^{(\alpha_0)}\left(\psi_k^{(\alpha_n)}, \psi_k^{(\alpha_n)}\right) < \infty.$$

It is well-known (see [1, Theorem 7.3.2]) that the embedding $W_0^{\alpha_0/2,2}(D) \to L^2(D)$ is compact. This implies that $\{\psi_k^{(\alpha_n)};\ n \geq 1\}$ is relatively compact in $L^2(D)$ and, by Theorem 4.3(ii), any of its limit points is a unit eigenfunction of the generator of $X^{\phi_\alpha, D}$ with eigenvalue $-\lambda_k^{(\alpha)}$. □

Remark 5.2. Note that in the example above, D is an open set of $\mathbf{R}^d$ which may not be connected; that is, D may not be a domain of $\mathbf{R}^d$. DeBlassie and Méndez-Hernández has proved in [6] that $\lambda_k^{(\alpha)}$ is right continuous in $\alpha \in (0, 2)$ when D is a domain of $\mathbf{R}^d$ having finite Lebesgue measure and is continuous in $\alpha \in (0, 2)$ when D is a bounded Lipschitz domain. When D is a bounded Lipschitz domain, they also proved that $\{\psi_k^{(\alpha_n)},\ n \geq 1\}$ is relatively compact in $C(\overline{D})$, the space of continuous functions on $\overline{D}$ equipped with the supremum norm. □

Example 5.1 is in fact a special case of next example.

Example 5.3. Let $\phi(\lambda) = (\lambda + \gamma)^{\alpha/2} - \gamma^{\alpha/2}$, where $\alpha \in (0, 2]$ and $\gamma \geq 0$. When $\gamma > 0$, X^ϕ is a relativistic α-stable process in $\mathbf{R}^d$, and when $\gamma = 0$, X^ϕ is a spherical symmetric α-stable process in $\mathbf{R}^d$. Note that

$$\min\left\{\frac{\alpha}{4}, \frac{1}{\gamma^{\alpha/2} + 1}\right\}(|\xi|^\alpha + 1) \leq \phi(|\xi|^2) + 1 \leq |\xi|^\alpha + 1 \quad \text{for every } \xi \in \mathbf{R}^d. \tag{5.5}$$

We deduce that the Dirichlet form for X^ϕ is $(\mathcal{E}^\phi, W^{\alpha/2,2}(\mathbf{R}^d))$, where

$$\mathcal{E}^\phi(u, v) = \int_{\mathbf{R}^d} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\phi(|\xi|^2)\, d\xi \quad \text{for } u, v \in W^{\alpha/2,2}(\mathbf{R}^d). \tag{5.6}$$

It follows from (5.5) that for $u \in W^{\alpha/2,2}(\mathbf{R}^d)$,

$$\min\left\{\frac{\alpha}{4}, \frac{1}{\gamma^{\alpha/2} + 1}\right\}\mathcal{E}_1^{(\alpha)}(u, u) \leq \mathcal{E}_1^\phi(u, u) \leq \mathcal{E}_1^{(\alpha)}(u, u), \tag{5.7}$$

where $\mathcal{E}^{(\alpha)}$, defined by (5.2), is the Dirichlet form of a spherically symmetric α-stable process in $\mathbf{R}^d$. It is well known that the transition density function $p^{(\alpha)}(t, x, y)$ of a spherically symmetric α-stable process in $\mathbf{R}^d$ has an upper bound

$ct^{-d/\alpha}$ and so by Theorem 2.4.6 of [5], there is a constant $c_1 = c_1(d, \alpha) > 0$ such that

$$\|u\|_{L^2(\mathbf{R}^d)}^{2+\frac{\alpha}{d}} \le c_1\, \mathcal{E}^{(\alpha)}(u, u)\, \|u\|_{L^1(\mathbf{R}^d)}^{\frac{\alpha}{d}} \quad \text{for every } u \in W^{\alpha/2,2}(\mathbf{R}^d) \cap L^1(\mathbf{R}^d).$$

Consequently by (5.7), there is a constant $c_2 = c_2(d, \alpha, \gamma) > 0$ such that

$$\|u\|_{L^2(\mathbf{R}^d)}^{2+\frac{\alpha}{d}} \le c_2\, \mathcal{E}_1^{\phi}(u, u)\, \|u\|_{L^1(\mathbf{R}^d)}^{\frac{\alpha}{d}} \quad \text{for every } u \in W^{\alpha/2,2}(\mathbf{R}^d) \cap L^1(\mathbf{R}^d).$$

Now let $p^{\phi}(t, x, y)$ is the transition density of the Lévy process X^{ϕ}. Then $e^{-t} p^{\phi}(t, x, y)$ is the transition density of the process associated with the Dirichlet form $(\mathcal{E}_1^{\phi}, W^{\alpha/2,2}(\mathbf{R}^d))$. Thus applying Theorem 2.4.6 of [5] to the Dirichlet form $(\mathcal{E}_1^{\phi}, W^{\alpha/2,2}(\mathbf{R}^d))$, we get

$$e^{-t} p^{\phi}(t, x, y) \le c_3(d, \alpha, \gamma)\, t^{-d/\alpha} \quad \text{for every } t > 0 \text{ and } x, y \in \mathbf{R}^d,$$

Thus if D is an open set in $\mathbf{R}^d$ having finite Lebesgue measure and $\alpha \in (0, 2]$, the semigroup $P_t^{\phi,D}$ of $X^{\phi,D}$ is a Hilbert-Schmidt operator in $L^2(D)$ and hence is compact. This implies that the generator of $X^{\phi,D}$ has discrete spectrum $\{-\lambda_k^{\alpha,\gamma};\ k \ge 1\}$. It follows from Theorem 2.3 and Theorem 4.3 that $\lambda_k^{\alpha,\gamma}$ is jointly continuous in $(\alpha, \gamma) \in (0, 2] \times [0, \infty)$.

The Dirichlet form for the subprocess $X^{\phi,D}$ of X^{ϕ} killed upon leaving D is $\left(\mathcal{E}^{\phi}, W_0^{\alpha/2,2}(D)\right)$. Now let (α_n, γ_n) be a sequence in $(0, 2] \times [0, \infty)$ converging to some (α, γ) in $(0, 2] \times [0, \infty)$. For $k \ge 1$, let $\psi_k^{\alpha_n,\gamma_n}$ be a unit eigenfunction of the generator of $X^{\phi_{\alpha_n,\gamma_n},D}$ with eigenvalue $-\lambda_k^{\alpha_n,\gamma_n}$, where

$$\phi_{\alpha_n,\gamma_n}(\lambda) := (\lambda + \gamma_n)^{\alpha_n/2} - \gamma_n^{\alpha_n/2}.$$

Then by (5.7) and by the same argument as that in the second part of Example 5.1, we can conclude that $\{\psi_k^{\alpha_n,\gamma_n};\ n \ge 1\}$ is relatively compact in $L^2(D)$. By Theorem 1.2(ii), any of its limit point is a unit eigenfunction of the generator of $X^{\phi_{\alpha,\gamma},D}$ with eigenvalue $-\lambda_k^{\alpha,\gamma}$. □

Example 5.4. Let $\phi(\lambda) = \log(1 + \lambda^{\alpha/2})$, $\alpha \in (0, 2]$, which is a Bernstein function. The process X^{ϕ} is a geometric α-stable process for $\alpha \in (0, 2)$ and a variance gamma process for $\alpha = 2$. Geometric stable distributions were first introduced in [8] and they have played an important role in heavy-tail modeling of economic data. Since $\phi(\lambda) = \phi_2 \circ \phi_1(\lambda)$, where $\phi_1(\lambda) = \lambda^{\alpha/2}$ and $\phi_2(\lambda) = \log(1 + \lambda)$, it is easy to see that the process X^{ϕ} can be obtained by subordinating a spherically symmetric α-stable process Y with an independent gamma subordinator. Let $P^Y(t, x, y)$ denote the transition density function of Y and note that the gamma subordinator has transition density function $\frac{1}{\Gamma(t)} u^{t-1} e^{-u}$. Suppose that D is an open set in $\mathbf{R}^d$ having finite Lebesgue measure. For $0 < \alpha \le 2$, we have

$$\begin{aligned}\int_D p^{\phi}(t, x, x) m(dx) &= \int_0^{\infty} \left(\int_D p^Y(s, x, x)\, dx \right) \frac{1}{\Gamma(t)} s^{t-1} e^{-s} ds \\ &\le \frac{c_1}{\Gamma(t)} \int_0^{\infty} s^{t-\frac{d}{\alpha}-1} e^{-s} ds < \infty\end{aligned}$$

whenever $t > \frac{d}{\alpha}$. So for $t > \frac{d}{\alpha}$, the semigroup $P_t^{\phi,D}$ of $X^{\phi,D}$ is a Hilbert-Schmidt operator in $L^2(D)$ and hence is compact. This implies that the generator of $X^{\phi,D}$ has discrete spectrum $\{-\lambda_k^{(\alpha)};\ k \geq 1\}$. It follows from Theorem 2.3 and Theorem 4.3 that $\lambda_k^{(\alpha)}$ is continuous in $\alpha \in (0,\ 2]$. □

The next example makes a connection between Example 5.3 and Example 5.4.

Example 5.5. Let $\phi_\alpha(\lambda) = \frac{(\lambda+1)^\alpha - 1}{\alpha}$ with $\alpha \in (0, 1]$. Clearly

$$\lim_{\alpha \downarrow 0} \phi_\alpha(\lambda) = \ln(1+\lambda) := \phi_0(\lambda) \quad \text{for every } \lambda \geq 0. \tag{5.8}$$

As mentioned previously, X^{ϕ_α} is a relativistic (2α)-stable process in $\mathbf{R}^d$ when $0 < \alpha \leq 1$ and X^{ϕ_0} is a variance gamma process in $\mathbf{R}^d$. Let D be an open set of $\mathbf{R}^d$ having finite Lebesgue measure. Then by Examples 5.3 and 5.4, the generator of $X^{\phi_\alpha, D}$ has discrete spectrum $\{-\lambda_k^{(\alpha)};\ k \geq 1\}$ for every $\alpha \in [0, 1]$. It follows from Theorem 2.3, Theorem 4.3 and (5.8) that for each fixed $k \geq 1$, $\lambda_k^{(\alpha)}$ is continuous in $\alpha \in [0,\ 1]$. □

Note that when X is a symmetric diffusions in $\mathbf{R}^d$ with uniformly elliptic divergence form generator, it is well-known that its transition density function has Aronson estimate (see, e.g., [5, (3.1.3)] and this in particular implies that for $\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0,\ 2]$,

$$p^\phi(t,x,y) \leq c\, t^{-d/\alpha} \quad \text{for } (t,x,y) \in (0,\ \infty) \times \mathbf{R}^d \times \mathbf{R}^d.$$

When X is a spherically symmetric β-stable process in $\mathbf{R}^d$ for some $\beta \in (0, 2]$ and $\phi(\lambda) = \lambda^{\alpha/2}$ with $\alpha \in (0,\ 2]$. The subordinate process X^ϕ is a spherically symmetric $\frac{\alpha\beta}{2}$-stable process in $\mathbf{R}^d$ and so its transition density function p^ϕ has upper bound estimate

$$p^\phi(t,x,y) \leq c\, t^{-2d/(\alpha\beta)} \quad \text{for } (t,x,y) \in (0,\ \infty) \times \mathbf{R}^d \times \mathbf{R}^d.$$

So the conclusion of all the examples in this section hold with trivial modification in justification when X is a symmetric diffusions in $\mathbf{R}^d$ with uniformly elliptic divergence form generator or a spherically symmetric β-stable process in $\mathbf{R}^d$.

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