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Two-sided eigenvalue estimates for subordinate processes in domains

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Abstract

Let $X=\{X_t, t\geqslant 0\}$ be a symmetric Markov process in a state space E and D an open set of E. Denote by X^D the subprocess of X killed upon leaving D. Let $S=\{S_t, t\geqslant 0\}$ be a subordinator with Laplace exponent ϕ that is independent of X. The processes $X^\phi := \{X_{S_t}, t\geqslant 0\}$ and $(X^D)^\phi := \{X^D_{S_t}, t\geqslant 0\}$ are called the subordinate processes of X and X^D, respectively. Under some mild conditions, we show that, if $\{-\mu_n, n\geqslant 1\}$ and $\{-\lambda_n, n\geqslant 1\}$ denote the eigenvalues of the generators of the subprocess of X^ϕ killed upon leaving D and of the process X^D respectively, then

$$\mu_n \leqslant \phi(\lambda_n) \quad \text{for every } n\geqslant 1.$$

We further show that, when X is a spherically symmetric α-stable process in $\mathbf{R}^d$ with $\alpha\in(0,2]$ and $D\subset\mathbf{R}^d$ is a bounded domain satisfying the exterior cone condition, there is a constant $c=c(D)>0$ such that

$$c\,\phi(\lambda_n) \leqslant \mu_n \leqslant \phi(\lambda_n) \quad \text{for every } n\geqslant 1.$$

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The above constant c can be taken as $1/2$ if D is a bounded convex domain in $\mathbf{R}^d$. In particular, when X is Brownian motion in $\mathbf{R}^d$, S is an $\alpha/2$-subordinator (i.e., $\phi(\lambda)=\lambda^{\alpha/2}$) with $\alpha \in (0,2)$, and D is a bounded domain in $\mathbf{R}^d$ satisfying the exterior cone condition, $\{-\lambda_n,\, n\geqslant 1\}$ and $\{-\mu_n,\, n\geqslant 1\}$ are the eigenvalues for the Dirichlet Laplacian in D and for the generator of the spherically symmetric α-stable process killed upon exiting the domain D, respectively. In this case, we have

$$c\,\lambda_n^{\alpha/2} \leqslant \mu_n \leqslant \lambda_n^{\alpha/2} \quad \text{for every } n\geqslant 1.$$

When D is a bounded convex domain in $\mathbf{R}^d$, we further show that

$$c_1^{\alpha}\,\mathrm{Inr}(D)^{-\alpha} \leqslant \mu_1 \leqslant c_2^{\alpha}\,\mathrm{Inr}(D)^{-\alpha},$$

where $\mathrm{Inr}(D)$ is the inner radius of D and $c_2>c_1>0$ are two constants depending only on the dimension d.

Keywords: Eigenvalues; Subordination; Subordinator; Bernstein function; Complete Bernstein function; Borel right process; Lévy process; Brownian motion; Spherically symmetric stable process; Dirichlet form; Semigroup; Resolvent

1. Introduction

For $\alpha \in (0,\, 2]$, a spherically symmetric α-stable process X in $\mathbf{R}^d$ is a Lévy process such that

$$\mathbf{E}\left[e^{i\,\xi\cdot(X_t-X_0)}\right] = e^{-t|\xi|^{\alpha}} \quad \text{for every } \xi\in\mathbf{R}^d.$$

When $\alpha=2$, X is a Brownian motion in $\mathbf{R}^d$ (which is sometimes denoted by W in the sequel) running twice fast as the standard Brownian motion in $\mathbf{R}^d$. However, when $\alpha\in(0,2)$, the sample paths of X are discontinuous. The L^2-infinitesimal generator of X is $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ so it is non-local when $0<\alpha<2$. When $\alpha=1$, X is called a Cauchy process in $\mathbf{R}^d$. Recently there have been lots of interests in studying discontinuous spherically symmetric stable processes due to its importance in theory as well as in applications. In particular, many of the “fine” and now well-known results for Brownian motion ($\alpha=2$) have been extended to these processes. These results include, among other things, sharp estimates on the Green functions and Poisson kernels [12,23], intrinsic ultracontractivity [11,24], conditional gauge theorems [8,11,14,33], the boundary Harnack principle [6,33], the identification of Martin boundary for various domains [7,13,33], and isoperimetric-type inequalities for heat kernels, Green functions, the lowest eigenvalue, and electrostatic capacities [2,25,4]. We refer the reader to [10] for a survey of some of these results. Despite the results mentioned above, until recently, many of the more detailed and refined spectral theoretic properties for which there is also an extensive literature in the case of Brownian motion were completely open for discontinuous spherically symmetric stable processes. For instance, in contrast to the Brownian motion case, the exact value of the first eigenvalue $-\mu_1$ of the one-dimensional Cauchy process killed upon leaving the interval $(-1, 1)$ is still unknown.

In their recent paper [1], Bañuelos and Kulczycki conducted a detailed study of the spectral properties of the Cauchy process in $\mathbf{R}^d$, especially the case when $d = 1$. For instance, they established the following estimate for the first eigenvalue $-\mu_1$ of the one-dimensional Cauchy process killed upon leaving the interval $(-1, 1)$: $1 \leqslant \mu_1 \leqslant 3\pi/8 \approx 1.178$. In the multidimensional case, they obtained an upper bound for the Dirichlets eigenvalues of the Cauchy process in a bounded Lipschitz domain D in terms of the Dirichlet eigenvalues of Brownian motion in D. Let D be a bounded domain in $\mathbf{R}^d$ and X^D the subprocess of X killed upon leaving D. The L^2-generator of X is $\Delta^{\alpha/2}$ in D with zero exterior condition. It is known that the generator of X^D has discrete spectrum with eigenvalues $\{-\mu_n^{(\alpha)}, n \geqslant 1\}$, where

$$0 < \mu_1^{(\alpha)} < \mu_2^{(\alpha)} \leqslant \mu_3^{(\alpha)} \leqslant \cdots \quad \text{and} \quad \lim_{n\to\infty} \mu_n^{(\alpha)} = \infty.$$

When $\alpha = 2$, $\{-\mu_n^{(2)}, n \geqslant 1\}$ are the eigenvalues for the Dirichlet Laplacian in D and will be denoted as $\{-\lambda_n, n \geqslant 1\}$. It is shown in [1, Theorem 3.14] that if D is a bounded Lipschitz domain in $\mathbf{R}^d$, then

$$\mu_n^{(1)} \leqslant \sqrt{\lambda_n} \quad \text{for every } n \geqslant 1, \tag{1.1}$$

by relating them to a mixed Steklov problem for the Laplacian in a domain in $\mathbf{R}^{d+1}$. By considering a higher order mixed Steklov problem, DeBlassie [18, Theorem 1.3] and [19] showed that

$$\mu_n^{(\alpha)} \leqslant \lambda_n^{\alpha/2} \quad \text{for every } n \geqslant 1, \tag{1.2}$$

holds for rational $\alpha \in (0, 2)$ and for bounded domain D satisfying certain regularity conditions.

The results above immediately lead to the following natural question: Does (1.2) hold for all $\alpha \in (0, 2)$?

Besides the spherically symmetric stable processes, there are various other classes of Lévy processes, like the relativistic stable processes [15,29] and geometric stable processes [31], which also have widespread applications. The study of the spectral theoretic properties of these processes are also very important in both theory and applications. So one can ask the following more general question: Can (1.2) be generalized to other nice Lévy processes?

Note that for $\alpha \in (0, 2)$, the spherically symmetric α-stable process X can be obtained from the Brownian motion W by a subordination. More precisely, let $S = \{S_t : t \geqslant 0\}$

be an $\alpha/2$-subordinator, that is, an increasing Lévy process taking values in $[0, \infty)$ with $S_0 = 0$ such that

$$\mathbf{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda)) \quad \text{for } \lambda > 0,$$

with $\phi(\lambda) = \lambda^{\alpha/2}$. The function ϕ is called the Laplace exponent of the subordinator S. Let S be independent of W. Then $\{W_{S_t}, t \geqslant 0\}$ is a spherically symmetric α-stable process in $\mathbf{R}^d$. So inequality (1.2) can be rewritten as

$$\mu_n^{(\alpha)} \leqslant \phi(\lambda_n) \quad \text{for every } n \geqslant 1. \tag{1.3}$$

The purpose of this paper is to show that inequality (1.3) holds with ϕ being the Laplace exponent of a general subordinator S and with the process obtained by subordinating X with S in place of the spherically symmetric α-stable process. See Theorem 3.4 for the precise statement. With this theorem, known upper bounds on the eigenvalues of the Laplacian immediately give rise to upper bounds on the eigenvalues of the generator of subordinate Brownian motions. As a consequence of the above general result, we get that inequality (1.2) holds for every $\alpha \in (0, 2)$ and for every bounded domain $D \subset \mathbf{R}^d$. Our proof uses a quadratic form approach and is more direct than those in [1,18,19]. The proof actually works when Brownian motion is replaced by a general symmetric Markov process X, and our Theorem 3.4 is stated and proved in this generality. Whenever we have upper bounds on the eigenvalues of the generator of the general symmetric Markov process X, Theorem 3.4 automatically gives us some upper bounds on the eigenvalues of the generator of the subordinate process. For instance, in the case when X is a Brownian motion on a sphere or a diffusion in $\mathbf{R}^d$ in divergence form, there are various estimates on the eigenvalues of the generator of X, our Theorem 3.4 immediately translates them into estimates on the eigenvalues of subordinate processes.

We further show that, when X is a spherically symmetric α-stable process in $\mathbf{R}^d$ with $\alpha \in (0, 2]$ and $D \subset \mathbf{R}^d$ is a bounded domain satisfying the exterior cone condition, there is a constant $c = c(D) > 0$ such that

$$c\,\phi(\lambda_n) \leqslant \mu_n \leqslant \phi(\lambda_n) \quad \text{for every } n \geqslant 1. \tag{1.4}$$

The above constant c can be taken as $1/2$ if D is a bounded convex domain in $\mathbf{R}^d$. In particular, when X is Brownian motion in $\mathbf{R}^d$, S is an $\alpha/2$-subordinator (i.e., $\phi(\lambda) = \lambda^{\alpha/2}$) with $\alpha \in (0, 2)$, and D is a bounded domain in $\mathbf{R}^d$ satisfying the exterior cone condition, $\{-\lambda_n, n \geqslant 1\}$ and $\{-\mu_n, n \geqslant 1\}$ are the eigenvalues for the Dirichlet Laplacian in D and for the generator of the spherically symmetric α-stable process killed upon exiting the domain D, respectively. In this case, we have

$$c\,\lambda_n^{\alpha/2} \leqslant \mu_n \leqslant \lambda_n^{\alpha/2} \quad \text{for every } n \geqslant 1.$$

The above constant c can be taken as $1/2$ if D is a bounded convex domain in $\mathbf{R}^d$.

It is well known (see Theorems 1.5.4 and 1.5.8 in [17]) that when X is a Brownian motion in $\mathbf{R}^d$ and $D \subset \mathbf{R}^d$ is a bounded convex domain, there are positive constants c_1 and c_2 depending only on dimension d such that the first Dirichlet eigenvalue for X^D has the following geometric estimate:

$$c_1 \,\mathrm{Inr}(D)^{-2} \leqslant \lambda_1 \leqslant c_2 \,\mathrm{Inr}(D)^{-2}, \tag{1.5}$$

where $\mathrm{Inr}(D) := \sup\{\mathrm{dist}(x, \partial D) : x \in D\}$ is the inner radius of D. Consequently, since the Laplace exponent ϕ is an increasing function on $\mathbf{R}_+$, we have from (1.4) that for any bounded convex domain D,

$$\frac{1}{2}\,\phi\left(c_1 \,\mathrm{Inr}(D)^{-2}\right) \leqslant \mu_1 \leqslant \phi\left(c_2 \,\mathrm{Inr}(D)^{-2}\right). \tag{1.6}$$

See Theorem 4.7 below for details. In particular, applying the above to $\phi(r) = r^{\alpha/2}$ with $\alpha \in (0, 2)$, we have the following estimate for the first eigenvalue $\mu_1^{(\alpha)}$ of the symmetric α-stable process in a bounded convex domain D in terms of its inner radius $\mathrm{Inr}(D)$:

$$\frac{1}{2}\, c_1^{\alpha/2} \,\mathrm{Inr}(D)^{-\alpha} \leqslant \mu_1^{(\alpha)} \leqslant c_2^{\alpha/2} \,\mathrm{Inr}(D)^{-\alpha}. \tag{1.7}$$

The remainder of this paper is organized as follows. In Section 2, we recall the definitions of subordinators and subordination of general symmetric Markov processes, and review some basic facts about them that will be used in this paper. In Section 3, we give the proof of (1.3) for general symmetric Markov process X and for subordinators whose Laplace exponents are complete Bernstein functions. In Section 4, we establish the lower bound estimate (1.4) for a class of symmetric Markov processes X on a general state space E and for $D \subset E$, which includes the case of X being a spherically symmetric stable processes in $\mathbf{R}^d$ and D a bounded domains in $\mathbf{R}^d$ satisfying the exterior cone condition. Some examples are given in Section 5 to illustrate the main results of this paper.

In this paper, we use ":=" as a way of definition, which is read as "is defined to be".

2. Preliminaries

Let $S = \{S_t : t \geqslant 0\}$ be a subordinator, that is, an increasing Lévy process taking values in $[0, \infty)$ with $S_0 = 0$. The law of S is characterized by

$$\mathbf{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda)) \quad \text{for } \lambda > 0. \tag{2.1}$$

The function $\phi : (0, \infty) \to \mathbf{R}_+$ is called the Laplace exponent of S, and has the representation

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \nu(dt)\,. \tag{2.2}$$

Here $b \geqslant 0$, and ν is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (t \wedge 1)\, \nu(dt) < \infty .$$

The constant b is called the drift and ν the Lévy measure of the subordinator S.

Let E be a Lusin space (i.e., a space that is homeomorphic to a Borel subset of a compact metric space) and $\mathcal{B}(E)$ be the Borel σ-algebra on E, and let m be a σ-finite measure on $\mathcal{B}(E)$ with $\mathrm{supp}[m] = E$. Let $X = (\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \mathbf{P}_x, x \in E)$ be an m-symmetric Borel right process on E. Here, a Borel right process on Lusin space E is a right continuous, strong Markov process with no branching points and with a Borel measurable resolvent. The shift operators θ_t, $t \geqslant 0$, satisfy $X_s \circ \theta_t = X_{s+t}$ identically for $s, t \geqslant 0$. Adjoined to the state space E is an isolated point $\partial \notin E$; the process X retires to ∂ at its "lifetime" $\zeta := \inf\{t \geqslant 0 : X_t = \partial\}$. Denote $E \cup \{\partial\}$ by E_∂. Denote by $p(t, x, dy)$ the transition probability of X. The transition semigroup $\{P_t,\ t \geqslant 0\}$ and the resolvent $\{G_\beta,\ \beta \geqslant 0\}$ of X are defined by

$$P_t f(x) := \mathbf{E}_x[f(X_t)] = \mathbf{E}_x[f(X_t);\ t < \zeta],$$

$$G_\beta f(x) := \int_0^\infty e^{-\beta t} P_t f(x)\, dt = \mathbf{E}_x\left[\int_0^\infty e^{-\beta t} f(X_t)\right] = \mathbf{E}_x\left[\int_0^\zeta e^{-\beta t} f(X_t)\right].$$

(Here and in the sequel, unless mentioned otherwise, we use the convention that a function defined on E takes the value 0 at the cemetery point ∂.) The L^2-generator of $\{P_t,\ t \geqslant 0\}$ will be denoted as $(\mathcal{A}, \mathrm{Dom}(\mathcal{A}))$. Define $\mathcal{F} := \mathrm{Dom}(\sqrt{-\mathcal{A}})$ and

$$\mathcal{E}(u, v) := (\sqrt{-\mathcal{A}}u,\ \sqrt{-\mathcal{A}}v)_{L^2(E,m)} \quad \text{for } u, v \in \mathrm{Dom}(\sqrt{-\mathcal{A}}).$$

The bilinear form $(\mathcal{E}, \mathcal{F})$ is called the Dirichlet form of X. Note that in general $\mathcal{E}$ is only a semi-inner product but $(\mathcal{F}, \mathcal{E}_1)$ is a Hilbert space, where

$$\mathcal{E}_1(u, v) := \mathcal{E}(u, v) + (u,\ v)_{L^2(E,m)} \quad \text{for } u, v \in \mathcal{F}.$$

We refer our readers to [20] for basic notions in the theory of Dirichlet forms, including the notion of quasi-everywhere.

If X and S are independent, the process

$$X_t^\phi := X_{S_t}, \quad t \geqslant 0$$

is called the subordinate process of X via S, which is also an m-symmetric Markov process on E. The transition probability $p^\phi(t, x, dy)$ of X^ϕ is given by

$$p^\phi(t, x, B) = \int_0^\infty p(s, x, B)\, \eta_t(ds) \quad \text{for } t > 0,\ x \in E \text{ and } B \in \mathcal{B}(E), \tag{2.3}$$

where η_t is the distribution of S_t. Let $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ be the Dirichlet form corresponding to X^ϕ, then $\mathcal{F} \subset \mathcal{F}^\phi$. When the drift b of S is strictly positive,

$$\mathcal{F}^\phi = \mathcal{F} \tag{2.4}$$

and for $u \in \mathcal{F}$,

$$\begin{aligned}\mathcal{E}^\phi(u,u) &= b\mathcal{E}(u,u) + \int_0^\infty (u - P_s u, u)_{L^2(E,m)} v(ds) \\ &= b\mathcal{E}(u,u) + \int_{E\times E} (u(x) - u(y))^2 J^\phi(dx,dy) \\ &\quad + \int_E u^2(x)\kappa^\phi(dx),\end{aligned} \tag{2.5}$$

where

$$J^\phi(dx,dy) = \frac{1}{2} m(dx) \int_0^\infty p(s,x,dy) v(ds),$$

and

$$\kappa^\phi(dx) = m(dx) \int_0^\infty (1 - p(s,x,E)) v(ds).$$

When S has no drift,

$$\mathcal{F}^\phi = \left\{ u \in L^2(E,m) : \int_0^\infty (u - P_s u, u)_{L^2(E,m)} v(ds) < \infty \right\} \tag{2.6}$$

and for any $u \in \mathcal{F}^\phi$,

$$\begin{aligned}\mathcal{E}^\phi(u,u) &= \int_0^\infty (u - P_s u,\ u)_{L^2(E,m)}\, v(ds) \\ &= \int_{E\times E} (u(x) - u(y))^2 J^\phi(dx,dy) + \int_E u^2(x)\kappa^\phi(dx),\end{aligned} \tag{2.7}$$

with J^ϕ and κ^ϕ given above. For the above facts regarding $(\mathcal{E}^\phi, \mathcal{F}^\phi)$, see [27]. Let $\mathcal{A}$ and $\mathcal{A}^\phi$ be the L^2-infinitesimal generator of X and X^ϕ, respectively. The following relation is due to R.S. Phillips (see [30, Theorem 2.3]):

$$\begin{aligned}&\mathrm{Dom}(\mathcal{A}) \subset \mathrm{Dom}(\mathcal{A}^\phi) \quad \text{and} \\ &\mathcal{A}^\phi u = b\mathcal{A}u + \int_0^\infty (P_s u - u)\, v(ds) \quad \text{for } u \in \mathrm{Dom}(\mathcal{A}).\end{aligned} \tag{2.8}$$

It is known (see Lemma 2.4 of Schilling [30]) that $\mathrm{Dom}(\mathcal{A})$ is a core for the generator $\mathcal{A}^{\phi}$. So in particular, $\mathrm{Dom}(\mathcal{A})$ is dense in $(\mathcal{F}^{\phi}, \mathcal{E}_1^{\phi})$.

Recall that a C^{∞} function $\phi : (0, \infty) \to \mathbf{R}_+$ is called a Bernstein function if $(-1)^n D^n \phi \leqslant 0$ for every $n \geqslant 1$. It is well known that a function $\phi : (0, \infty) \to \mathbf{R}$ satisfying $\lim_{\lambda \to 0} \phi(\lambda) = 0$ is a Bernstein function if and only if it has the representation given by (2.2). A function $\phi : (0, \infty) \to \mathbf{R}_+$ is called a complete Bernstein function if there exists a Bernstein function ψ such that

$$\phi(\lambda) = \lambda^2 \mathcal{L}\psi(\lambda), \quad \lambda > 0,$$

where $\mathcal{L}$ stands for the Laplace transform. It is well known that every complete Bernstein function is a Bernstein function. For this and other basic properties of complete Bernstein functions, we refer the reader to [21].

Most of the familiar Bernstein functions are complete Bernstein functions. The following are some examples of complete Bernstein functions: (i) $\lambda^{\alpha}, \alpha \in (0, 1]$; (ii) $(\lambda+a)^{\alpha} - a^{\alpha}, \alpha \in (0, 1), a > 0$; (iii) $\log(1+\lambda^{\alpha}), \alpha \in (0, 1]$; (iv) $1-(1+\lambda)^{\alpha-1}, \alpha \in (0, 1)$. The first family of complete Bernstein functions corresponds to α-stable subordinators $(0 < \alpha < 1)$ and the pure drift $(\alpha = 1)$, the second family to relativistic α-stable subordinators, and the third family to geometric stable subordinators $(0 < \alpha < 1)$ and the gamma subordinator $(\alpha = 1)$. The Lévy measure for the fourth one is $\frac{1}{\Gamma(1-\alpha)} e^{-x} x^{-\alpha} \, dx$, which has finite total mass on $[0, \infty)$, thus the corresponding subordinator is a compounded Poisson process. An example of a Bernstein function which is not complete Bernstein is the Bernstein function $1 - e^{-\lambda}$ of the Poisson process.

The family of complete Bernstein functions satisfies the following properties: (i) if ϕ_1 and ϕ_2 are complete Bernstein functions and a_1 and a_2 are nonnegative constants, then so are $a_1\phi_1 + a_2\phi_2$ and $\phi_1 \circ \phi_2$; (ii) if ϕ is a non-zero complete Bernstein function, then so are $\lambda/\phi(\lambda)$ and $1/\phi(\lambda^{-1})$; (iii) if ϕ_1 and ϕ_2 are non-zero complete Bernstein functions and $\beta \in (0, 1)$, then $\phi_1^{\beta}(\lambda)\phi_2^{1-\beta}(\lambda)$ is also a complete Bernstein function; (iv) if ϕ_1 and ϕ_2 are non-zero complete Bernstein functions and $\beta \in (-1, 0) \cup (0, 1)$, then $(\phi_1^{\beta}(\lambda) + \phi_2^{\beta}(\lambda))^{1/\beta}$ is also a complete Bernstein function. For these properties of complete Bernstein functions, one can see [26].

When the Laplace exponent ϕ of the subordinator S is a complete Bernstein function, it has the representation

$$\phi(\lambda) = b\lambda + \int_0^{\infty} (1 - e^{-\lambda t}) m(t) \, dt,$$

where $b \geqslant 0$, $m(t) = \int_{0+}^{\infty} e^{-ts} \rho(ds)$ for some positive measure ρ with $\int_{0+}^{\infty} \frac{1}{s(1+s)} \rho(ds) < \infty$. There is a nice representation of $\mathcal{A}^{\phi}$ in terms of the L^2-generator $\mathcal{A}$ and the resolvent $\{G_{\beta}, \beta > 0\}$ of X:

$$\mathcal{A}^{\phi} f = b\mathcal{A} f + \int_{0+}^{\infty} \frac{1}{\beta} (\beta G_{\beta} f - f) \, \rho(d\beta) \quad \text{for } f \in \mathrm{Dom}(\mathcal{A}), \tag{2.9}$$

(see [30, Theorem 4.1]).

3. A general upper bound eigenvalue estimate

For an open subset D of E, let X^D denote the process obtained by killing the process X upon exiting from D. The process X^D is an m-symmetric Markov process on D. The Dirichlet form of X^D is $(\mathcal{E}, \mathcal{F}^D)$ with

$$\mathcal{F}^D = \{u \in \mathcal{F} : u = 0 \ \mathcal{E}\text{-q.e. on } E \setminus D\}. \tag{3.1}$$

Here $\mathcal{E}$-q.e. stands for quasi everywhere with the respect to the Dirichlet form $(\mathcal{E}, \mathcal{F})$. We are going to use $\{P_t^D,\ t \geqslant 0\}$ and $\{G_\beta^D,\ \beta \geqslant 0\}$ to denote the semigroup and resolvent of X^D, respectively.

Lemma 3.1. *For any $\beta > 0$ and every $f \in L^2(E, m)$ with $f = 0$ m-a.e. on $E \setminus D$, we have*

$$(G_\beta f,\ f)_{L^2(E,m)} \geqslant (G_\beta^D f, f)_{L^2(E,m)}. \tag{3.2}$$

Proof. For every $\beta > 0$ and every $f \in L^2(E, m)$ with $f = 0$ m-a.e. on $E \setminus D$, by the strong Markov property of X,

$$G_\beta f(x) = G_\beta^D f(x) + \mathbf{H}_\beta^D G_\beta f(x) \quad \text{for } x \in D,$$

which is the $\mathcal{E}_\beta$-orthogonal decomposition of $G_\beta f$ into $G_\beta^D f \in \mathcal{F}^D$ and its complement (cf. [20]). Here

$$\mathbf{H}_\beta^D \phi(x) := \mathbf{E}_x \left[e^{-\beta \tau_D} \phi(X_{\tau_D}) \right],$$

where $\tau_D := \inf\{t \geqslant 0 : X_t \notin D\}$. Hence

$$\begin{aligned}
(G_\beta f - G_\beta^D f,\ f)_{L^2(E,m)} &= (\mathbf{H}_\beta^D G_\beta f,\ f)_{L^2(E,m)} \\
&= \frac{1}{\beta} \mathcal{E}_\beta(\mathbf{H}_\beta^D G_\beta f, G_\beta f) \\
&= \frac{1}{\beta} \mathcal{E}_\beta(\mathbf{H}_\beta^D G_\beta f, \mathbf{H}_\beta^D G_\alpha f) \geqslant 0.
\end{aligned}$$

This establishes (3.2). □

Remark 3.2. Note that the semigroup version of inequality (3.2) is not true. It is shown in DeBlassie [19] that for one-dimensional Brownian motion X, $E = \mathbf{R}$

and $D=(0,\infty)$, the following is false:

$$(P_1 f,\ f)_{L^2(\mathbf{R},dx)} \geqslant (P_1^D f,\ f)_{L^2(\mathbf{R},dx)} \quad \text{for} f\in L^2(\mathbf{R},dx) \text{ with } f=0 \text{ on } \mathbf{R}\setminus D.$$

Suppose that $S=\{S_t,\ t\geqslant 0\}$ is a subordinator independent of X with Laplace exponent ϕ. We consider the subordinate process $X^\phi := \{X_{S_t},\ t\geqslant 0\}$ on E and its subprocess $(X^\phi)^D$ killed upon leaving D. The Dirichlet form of X^ϕ is $(\mathcal{E}^\phi,\mathcal{F}^\phi)$, where $\mathcal{E}^\phi$ and $\mathcal{F}^\phi$ are given in the previous section. The Dirichlet form of $(X^D)^\phi$ is $(\mathcal{E}^\phi,(\mathcal{F}^D)^\phi)$, where $(\mathcal{F}^D)^\phi$ is defined the same way through X^D as $\mathcal{F}^\phi$ was defined through X.

Theorem 3.3. *Suppose that the Laplace exponent ϕ of S is a complete Bernstein function. Then for any open subset D of E, we have $(\mathcal{F}^D)^\phi\subset(\mathcal{F}^\phi)^D$ and*

$$\mathcal{E}^\phi(u,u)\leqslant(\mathcal{E}^D)^\phi(u,u) \quad \textit{for every } u\in(\mathcal{F}^D)^\phi.$$

Proof. Let $\mathcal{A}^D$ denote the L^2-generator of X^D. For $f\in\mathrm{Dom}(\mathcal{A})$, it follows from (2.9) that

$$\begin{aligned}\mathcal{E}^\phi(f,f) &= (-A^\phi f,\ f)_{L^2(E,m)}\\ &= b\mathcal{E}(f,f)+\int_{0+}^\infty (f-\beta G_\beta f,\ f)_{L^2(E,m)}\,\rho(d\beta).\end{aligned}$$

Since $\mathrm{Dom}(\mathcal{A})$ is dense in $(\mathcal{F},\mathcal{E}_1)$,

$$\mathcal{E}^\phi(f,f)=b\mathcal{E}(f,f)+\int_{0+}^\infty (f-\beta G_\beta f,\ f)_{L^2(E,m)}\,\rho(d\beta) \quad \text{for every } f\in\mathcal{F}.$$

Similarly, we have

$$(\mathcal{E}^D)^\phi(f,f)=b\mathcal{E}^D(f,f)+\int_{0+}^\infty (f-\beta G_\beta^D f,\ f)_{L^2(D,m)}\,\rho(d\beta) \quad \text{for every } f\in\mathcal{F}^D.$$

As $\mathcal{F}^D\subset\mathcal{F}$, we deduce from above and (3.2) that for $f\in\mathcal{F}^D$,

$$(\mathcal{E}^D)^\phi(f,f)-\mathcal{E}^\phi(f,f)=\beta\int_{0+}^\infty (G_\beta f-G_\beta^D f,\ f)_{L^2(E,m)}\,\rho(d\beta)\geqslant 0.$$

Since $\mathrm{Dom}(\mathcal{A}^D)\subset\mathcal{F}^D$ is dense in $\left((\mathcal{F}^D)^\phi,\mathcal{E}_1^\phi\right)$ (see the two lines following (2.8)), we conclude from above that $(\mathcal{F}^D)^\phi\subset\mathcal{F}^\phi$ and hence $(\mathcal{F}^D)^\phi\subset(\mathcal{F}^\phi)^D$. Moreover $\mathcal{E}^\phi(f,f)\leqslant(\mathcal{E}^D)^\phi(f,f)$ holds for every $f\in(\mathcal{F}^D)^\phi$. $\square$

For any open subset D of E, we use $(X^{\phi})^D$ to denote the process obtained by killing the process X^{ϕ} upon exiting D. The transition semigroup of $(X^{\phi})^D$ will be denoted by $\{P_t^{\phi,D} : t \geqslant 0\}$.

The basic assumptions of this paper are as follows:

(I). For some (and hence for every) $t > 0$ both P_t^D and $\mathcal{A}^D$, the generator for the killed process X^D, have discrete spectra with eigenvalues $\{e^{-\lambda_n t};\ n \geqslant 1\}$ and $\{-\lambda_n;\ n \geqslant 1\}$, respectively, arranged in decreasing order and repeated according to multiplicity. When there are only finitely many, say N, eigenvalues, we put $\lambda_{N+1} = \lambda_{N+2} = \cdots = \infty$.

(II). For some (and hence for every) $t > 0$, both $P_t^{\phi,D}$ and $\mathcal{A}^{\phi,D}$, the generator for the killed process $(X^{\phi})^D$, have discrete spectra with eigenvalues $\{e^{-\mu_n t};\ n \geqslant 1\}$ and $\{-\mu_n;\ n \geqslant 1\}$, respectively, arranged in decreasing order and repeated according to multiplicity. When there are only finitely many, say N, eigenvalues, we put $\mu_{N+1} = \mu_{N+2} = \cdots = \infty$.

Theorem 3.4. *Suppose that the Laplace exponent ϕ of the subordinator S is a complete Bernstein function and D is an open subset of E. If assumptions* **(I)** *and* **(II)** *are satisfied, then*

$$\mu_n \leqslant \phi(\lambda_n) \quad \text{for every } n \geqslant 1.$$

Proof. Since under assumption (I) the generator of X^D has discrete spectrum $\{\lambda_n, n \geqslant 1\}$, the generator of its subordinate process $(X^D)^{\phi}$ has discrete spectrum $\{\phi(\lambda_n), n \geqslant 1\}$ with the same corresponding eigenfunctions as that for X^D. By the mini–max principle for eigenvalues (see [9, p. 17], or [16, Section 1.1.10], or the paragraph before Example 3 in Section XIII.2 of [28]), for every $n \geqslant 1$,

$$\begin{aligned}
\mu_n &= \inf_{\substack{L:\ \text{subspace of } (\mathcal{F}^{\phi})^D \\ \text{with } \dim L = n}} \sup \left\{ \mathcal{E}^{\phi}(u,u) : u \in L \text{ and } (u,u)_{L^2(D,m)} = 1 \right\} \\
&\leqslant \inf_{\substack{L:\ \text{subspace of } (\mathcal{F}^{D})^{\phi} \\ \text{with } \dim L = n}} \sup \left\{ (\mathcal{E}^{D})^{\phi}(u,u) : u \in L \text{ and } (u,u)_{L^2(D,m)} = 1 \right\} \\
&= \phi(\lambda_n).
\end{aligned}$$

This proves the theorem. $\square$

One case in which the assumptions **(I)** and **(II)** are automatically satisfied is when X is a symmetric continuous time Markov chain on a countable state space E and when D is a finite subset of E. In this paper we are mostly interested in the case when the state space of X is uncountable.

In the remainder of this section, we assume the Borel right process X has a transition density $p(t,x,y)$ with respect to the measure m for every $x \in E$. We are going to give some sufficient conditions for assumptions **(I)** and **(II)** to hold. It is well-known in this case that X^D has a transition density $p^D(t,x,y)$ with respect to m and that

$$p^D(t,x,y)\leqslant p(t,x,y), \quad t>0, \ x,y\in D.$$

If

$$\int_D p^D(t,x,x)m(dx)<\infty \quad \text{for some} \quad t>0. \tag{3.3}$$

Then for the $t>0$ in the display above, the operator P_t^D is Hilbert–Schmidt and hence is compact. So the operator P_t^D has discrete spectrum. It follows from the spectral theory for self-adjoint operators that, whenever condition (3.3) is satisfied, then for *any* $t>0$, both P_t^D and $\mathcal{A}^D$, the generator for the killed process X^D, have discrete spectra. So condition (3.3) is a sufficient condition for assumption **(I)**.

Suppose now that the Laplace exponent ϕ for subordinator $S=\{S_t,\ t\geqslant 0\}$ is unbounded, which is equivalent to assume that S is not a compound Poisson process (see [3, Proposition I.2 and Corollary I.3]). Then for every $t>0$, the distribution η_t of S_t does not charge $\{0\}$ and so the subordinate process X^ϕ has a transition density function given by

$$p^\phi(t,x,y):=\int_{0+}^\infty p(s,x,y)\eta_t(ds).$$

For any open subset D of E, the process $(X^\phi)^D$ obtained by killing the process upon exiting D has a density $p^{\phi,D}$ such that

$$\mathbf{E}_x\left[f((X^\phi)_t^D)\right]=\int_D p^{\phi,D}(t,x,y)f(y)\,dy \quad \text{for any Borel function } f\geqslant 0 \text{ on } D.$$

Clearly,

$$p^{\phi,D}(t,x,y)\leqslant p^\phi(t,x,y), \quad t>0, \ x,y\in D.$$

If

$$\int_D p^{\phi,D}(t,x,x)m(dx)<\infty \quad \text{for some} \quad t>0, \tag{3.4}$$

then the operator $P_t^{\phi,D}$ defined by

$$P_t^{\phi,D}f(x):=\int_D p^{\phi,D}(t,x,y)f(y)m(dy)$$

is a Hilbert–Schmidt operator in $L^2(D,m)$ and hence is compact. It follows again from the spectral theory for self-adjoint operators that, whenever condition (3.4) is satisfied, then for *any* $t > 0$, both $P_t^{\phi,D}$ and $\mathcal{A}^{\phi,D}$, the generator for the killed process $(X^\phi)^D$, have discrete spectra. So conditions (3.4) is a sufficient condition for assumption **(II)**.

Since $p^{\phi,D}(t,x,x) \leqslant p^\phi(t,x,x)$ for every $t > 0$ and $x \in \mathbf{R}^d$, a sufficient condition for (3.4) to hold is

$$\int_D p^\phi(t,x,x)m(dx) = \int_0^\infty \left(\int_D p(s,x,x)m(dx) \right) \mu_t(ds) < \infty \quad \text{for some } t > 0. \tag{3.5}$$

Remark 3.5. (i) When X is a Brownian motion in $\mathbf{R}^d$ running twice as fast as the standard Brownian motion and S an independent subordinator with Laplace exponent $\phi(\lambda) = \lambda^{\alpha/2}$ for $\alpha \in (0, 2)$, X^ϕ is just the spherically symmetric α-stable process in $\mathbf{R}^d$. It is known that there is a constant $c > 0$ such that

$$p(t,x,y) \leqslant c\, t^{-d/2} \quad \text{and} \quad p^\phi(t,x,y) \leqslant c\, t^{-d/\alpha} \quad \text{for every } t > 0 \text{ and } x, y \in \mathbf{R}^d.$$

Hence for any bounded domain D of $\mathbf{R}^n$, both conditions (3.3) and (3.5) are satisfied. Thus assumptions **(I)** and **(II)** hold and so Theorem 3.4 applies. In this case, $\{-\lambda_n, n \geqslant 1\}$ are the eigenvalues for Δ in D with Dirichlet boundary condition (or, equivalently, for the generator of the Brownian motion X killed upon leaving D) and $\{-\mu_n, n \geqslant 1\}$ are the eigenvalues for $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ in D with Dirichlet exterior condition (or, equivalently, for the generator of the spherically symmetric α-stable process X^ϕ killed upon leaving D). By Theorem 3.4, we have

$$\mu_n \leqslant \lambda_n^{\alpha/2} \quad \text{for every } n \geqslant 1. \tag{3.6}$$

This not only recovers but also extends a result of Bañuelos and Kulczycki [1] and the main result of DeBlassie [18,19]. When D is a bounded Lipschitz domain and $\alpha = 1$, the above inequality was proved by Bañuelos and Kulczycki [1, Theorem 3.14] by relating them to a mixed Steklov problem for the Laplacian in a domain in $\mathbf{R}^{d+1}$. By considering a higher-order mixed Steklov problem, DeBlassie [18, Theorem 1.3] and [19] showed inequality (3.6) holds for *rational* $\alpha \in (0, 2)$ and for bounded domain D satisfying certain regularity conditions.

(ii) Our Theorem 3.4 generalizes the main result, Theorem 1.3, of DeBlassie [18,19] in two directions. First our base process is a general symmetric Markov process, while the base process in [18] is a Brownian motion. Secondly, our subordinator is very general, while [18] only deals with stable subordinators with rational exponent. □

4. Lower bound eigenvalue estimates

In this section, X is an m-symmetric right process on a Lusin space E and D is an open subset of E. We use X^D to denote the process obtained by killing X upon exiting

D. We use $\{P_t,\ t\geqslant 0\}$ and $\{P_t^D,\ t\geqslant 0\}$ to denote the transition semigroups of X and X^D, respectively.

Let S be a subordinator with Laplace exponent ϕ with a drift b which can be strictly positive or zero. We assume that S is independent of X. The Dirichlet form $(\mathcal{E}^\phi, \mathcal{F}^\phi)$ of the subordinate process X^ϕ is given by (2.4)–(2.6). Note that

$$\mathcal{E}^\phi(u,u) = b\mathcal{E}(u,u) + \int_{E\times E} (u(x)-u(y))^2 J^\phi(dx,dy) + \int_E u(x)^2 \kappa^\phi(x) m(dx), \tag{4.1}$$

where

$$J^\phi(dx,dy) := \frac{1}{2}\, m(dx) \int_0^\infty p(s,x,dy)\nu(ds) \quad \text{and}$$
$$\kappa^\phi(x) = \int_0^\infty (1-p(s,x,E))\nu(ds). \tag{4.2}$$

Here ν the Lévy measure for the subordinator S given by (2.2). The Dirichlet form for the subprocess $(X^\phi)^D$ of X^ϕ killed upon leaving D is $\left(\mathcal{E}^\phi,\ (\mathcal{F}^\phi)^D\right)$ with

$$(\mathcal{F}^\phi)^D = \left\{ u\in\mathcal{F}^\phi : u=0 \quad \mathcal{E}^\phi\text{-q.e. on } E\setminus D \right\}.$$

Note that for $u\in(\mathcal{F}^\phi)^D$, $\mathcal{E}^\phi(u,u)$ can be rewritten as

$$\mathcal{E}^\phi(u,u) = b\mathcal{E}(u,u) + \int_{D\times D} (u(x)-u(y))^2 J^\phi(dx,dy) + \int_D u(x)^2 (\kappa^\phi)_D(x) m(dx), \tag{4.3}$$

where

$$(\kappa^\phi)_D(x) = \int_0^\infty \int_{E\setminus D} p(s,x,dy)\nu(ds) + \kappa^\phi(x) \quad \text{for } x\in D.$$

Let $((\mathcal{E}^D)^\phi, (\mathcal{F}^D)^\phi)$ be the Dirichlet form for subordinate process $(X^D)^\phi$ of X^D. Then for any $u\in(\mathcal{F}^D)^\phi$,

$$(\mathcal{E}^D)^\phi(u,u) = b\mathcal{E}(u,u) + \int_{D\times D} (u(x)-u(y))^2 (J_D)^\phi(dx,dy) + \int_D u^2(x)(\kappa_D)^\phi(x) m(dx) \tag{4.4}$$

where

$$(J_D)^{\phi}(dx, dy) = \frac{1}{2} m(dx) \int_0^{\infty} p^D(s, x, dy)\nu(ds) \quad \text{for } x, y \in D,$$

and

$$(\kappa_D)^{\phi}(x) = \int_0^{\infty} (1 - P_s^D 1(x))\nu(ds) \quad \text{for } x \in D.$$

From the expressions for J^{ϕ} and $(J_D)^{\phi}$ above one easily concludes that

$$(J_D)^{\phi}(\cdot, \cdot) \leqslant J^{\phi}(\cdot, \cdot). \tag{4.5}$$

Let $\{P_t^{\phi, D}, t \geqslant 0\}$ and $\{P_t^{D, \phi}, t \geqslant 0\}$ denote the transition semigroups of $(X^{\phi})^D$ and $(X^D)^{\phi}$ in $L^2(D)$, respectively.

Proposition 4.1. *Suppose that S is a subordinator with Laplace exponent ϕ. If there exists $C \in (0, 1)$ such that*

$$\mathbf{P}_x(X_t \in D) \leqslant C \quad \text{for every } t > 0 \text{ and every } x \in E \setminus D, \tag{4.6}$$

then

$$(1 - C)(\kappa_D)^{\phi}(x) \leqslant (\kappa^{\phi})_D(x) \leqslant (\kappa_D)^{\phi}(x) \quad \text{for } x \in D. \tag{4.7}$$

Proof. Recall that ν stands for the Lévy measure of S. Since for any $x \in D$, we have

$$\begin{aligned}(\kappa^{\phi})_D(x) &= \int_0^{\infty} \int_{E \setminus D} p(s, x, dy)\nu(ds) + \int_0^{\infty} (1 - p(s, x, E))\nu(dx) \\ &= \int_0^{\infty} (1 - P_s 1_D(x))\nu(ds)\end{aligned}$$

and

$$(\kappa_D)^{\phi}(x) = \int_0^{\infty} (1 - P_s^D 1(x))\nu(ds),$$

the inequality on the right of (4.7) follows immediately from the fact that $p^D(s, x, \cdot) \leqslant p(s, x, \cdot)$ for all $(s, x) \in (0, \infty) \times D$.

Now we prove the inequality on the left. Let τ_D be the first exit time of D for the process X and let $u(x,s) = \mathbf{P}_x(X_s \in D)$. According to assumption (4.6), $u(x,t) \leqslant C$ for every $x \in E \setminus D$ and $t > 0$. Hence by the strong Markov property of X, we have for any $x \in D$ and any $t > 0$ we have

$$\mathbf{P}_x(\tau_D \leqslant t, X_t \in D) = \mathbf{E}_x\left[1_{\{\tau_D \leqslant t\}} u(X_{\tau_D}, \tau_D)\right] \leqslant C\mathbf{P}_x(\tau_D \leqslant t).$$

Therefore for any $s > 0$ and any $x \in D$,

$$\begin{aligned} 1 - P_s 1_D(x) &= 1 - P_s^D 1(x) - \mathbf{P}_x(\tau_D \leqslant t, X_t \in D) \\ &\geqslant 1 - P_s^D 1(x) - C\mathbf{P}_x(\tau_D \leqslant t) \\ &= (1 - C)(1 - P_s^D 1(x)). \end{aligned}$$

Now the inequality on the left of (4.7) follows immediately. □

Recall that a domain $D \subset \mathbf{R}^d$ is said to satisfy the exterior cone condition if there exist a cone K in $\mathbf{R}^d$ with vertex at the origin and positive number $R > 0$ such that for each point $x \in \partial D$ there exist a translation and a rotation taking the cone K into a cone K_x with vertex at x such that

$$K_x \cap B(x, R) \subset \mathbf{R}^d \setminus D.$$

As consequence of the result above, we get the following

Proposition 4.2. *Suppose that X is a spherically symmetric α-stable in $\mathbf{R}^d$ with $\alpha \in (0, 2]$ killed upon leaving an open set $E \subset \mathbf{R}^d$, and S is a subordinator with Laplace exponent ϕ.*

(i) *If D is the intersection with E of a bounded domain in $\mathbf{R}^d$ satisfying the exterior cone condition, then there exist a constant $C \in (0, 1)$ such that*

$$(1 - C)(\kappa_D)^{\phi}(x) \leqslant (\kappa^{\phi})_D(x) \leqslant (\kappa_D)^{\phi}(x).$$

(ii) *If D is the intersection with E of a bounded convex domain in $\mathbf{R}^d$, then*

$$\frac{1}{2}(\kappa_D)^{\phi}(x) \leqslant (\kappa^{\phi})_D(x) \leqslant (\kappa_D)^{\phi}(x).$$

Proof. (i) When D is the intersection with E of a bounded domain in $\mathbf{R}^d$ satisfying the exterior cone condition, it is easy to check (see the paragraph after the proof of Proposition 2.1 in [32]) that there exists $C \in (0, 1)$ such that condition (4.6) is satisfied. Assertion (i) now follows from Proposition 4.1.

(ii) It follows from the spherical symmetry of the symmetric α-stable process in $\mathbf{R}^d$ that when D is the intersection with E of a bounded convex domain in $\mathbf{R}^d$,

$$\mathbf{P}_x(X_t \in D) < \frac{1}{2}, \qquad \text{for every } t > 0 \text{ and every } x \in E \setminus D.$$

Assertion (ii) now also follows from Proposition 4.1. □

Theorem 4.3. *Suppose that the Laplace exponent ϕ of S is a complete Bernstein function. If condition* (4.6) *is satisfied for some $C \in (0, 1)$, then $(\mathcal{F}^D)^\phi = (\mathcal{F}^\phi)^D$ and*

$$(1 - C)(\mathcal{E}^D)^\phi(u, u) < \mathcal{E}^\phi(u, u) \leqslant (\mathcal{E}^D)^\phi(u, u), \quad u \in (\mathcal{F}^\phi)^D. \tag{4.8}$$

Proof. We know from Theorem 3.3 that $(\mathcal{F}^D)^\phi \subset (\mathcal{F}^\phi)^D$. On the other hand, by (2.4)–(2.7), (4.3)–(4.4) and Proposition 4.1, $(\mathcal{F}^\phi)^D \subset (\mathcal{F}^D)^\phi$. Thus $(\mathcal{F}^D)^\phi = (\mathcal{F}^\phi)^D$. Inequality (4.8) follows immediately by combining Proposition 4.1 with (4.5), and Theorem 3.3. □

Let D be an open subset of E. Assume that S is a subordinator and that assumption **(II)** holds. Then there exists an orthonormal basis of eigenfunctions $\{\psi_n : n \in \mathbb{N}\}$ for the operator $(P^\phi)_t^D$ in $L^2(D, m)$ with corresponding eigenvalues $\{e^{-\mu_n t} : n \in \mathbb{N}\}$ satisfying

$$0 < \mu_1 < \mu_2 \leqslant \mu_3 \leqslant \cdots$$

and $\mu_n \to \infty$ as $n \to \infty$.

By using Theorem 4.3 and the mini–max principle for eigenvalues (see the proof of Theorem 3.4 above), we immediately get the following result.

Theorem 4.4. *Suppose that the Laplace exponent ϕ of S is a complete Bernstein function and that assumption* **(II)** *holds. If* (4.6) *is satisfied for some $C \in (0, 1)$, then*

$$(1 - C)\phi(\lambda_n) \leqslant \mu_n \leqslant \phi(\lambda_n) \quad \textit{for every } n \geqslant 1.$$

By combining Proposition 4.2 with the theorem above we immediately get the following.

Theorem 4.5. *Suppose that X is a spherically symmetric α-stable in $\mathbf{R}^d$ with $\alpha \in (0, 2]$, that the Laplace exponent ϕ of S is a complete Bernstein function and that assumption* **(II)** *holds.*

(i) *If D is a bounded domain in $\mathbf{R}^d$ satisfying the exterior cone condition, then there exist a constant $C \in (0, 1)$ such that*

$$(1 - C)\phi(\lambda_n) \leqslant \mu_n \leqslant \phi(\lambda_n) \quad \textit{for every } n \geqslant 1.$$

(ii) *If D is a bounded convex domain in* $\mathbf{R}^d$, *then*

$$\frac{1}{2}\,\phi(\lambda_n)\leqslant\mu_n\leqslant\phi(\lambda_n)\quad \textit{for every } n\geqslant 1.$$

Remark 4.6. Clearly by Proposition 4.2, the above theorem remains true if we replace X by a spherically symmetric α-stable in $\mathbf{R}^d$ with $\alpha\in(0,2]$ killed upon leaving an open set $E\subset\mathbf{R}^d$ and replace D by the intersection $E\cap D$.

Theorem 4.7. *Suppose that X is a Brownian motion in $\mathbf{R}^d$ with infinitesimal generator Δ, that the Laplace exponent ϕ of S is a complete Bernstein function and that assumption* (II) *holds. Let D be a bounded convex domain D in $\mathbf{R}^d$. Then there are positive constants $c_2>c_1>0$ depending only on the dimension d such that*

$$\frac{1}{2}\,\phi\left(c_1\,\mathrm{Inr}(D)^{-2}\right)\leqslant\mu_1\leqslant\phi\left(c_2\,\mathrm{Inr}(D)^{-2}\right), \tag{4.9}$$

where $\mathrm{Inr}(D):=\sup\{\mathrm{dist}(x,\partial D)\,:\,x\in D\}$ *is the inner radius of D. In particular, we have the following estimate for the first eigenvalue $\mu_1^{(\alpha)}$ of the symmetric α-stable process in a bounded convex domain D in terms of its inner radius* $\mathrm{Inr}(D)$:

$$\frac{1}{2}\,c_1^{\alpha/2}\,\mathrm{Inr}(D)^{-\alpha}\leqslant\mu_1^{(\alpha)}\leqslant c_2^{\alpha/2}\,\mathrm{Inr}(D)^{-\alpha}. \tag{4.10}$$

Proof. Let λ_1 be the first eigenvalue for the killed Brownian motion X^D in D. It follows from Theorems 1.5.4 and 1.5.8 in [17] that there exist positive constants c_1 and c_2 depending only on d such that

$$c_1\,\mathrm{Inr}(D)^{-2}\leqslant\lambda_1\leqslant c_2\,\mathrm{Inr}(D)^{-2},$$

for every bounded convex domain D. Since the Laplace exponent ϕ is an increasing function on $\mathbf{R}_+$, inequality (4.9) follows from Theorem 4.5(ii). Applying (4.9) to $\phi(r)=r^{\alpha/2}$ yields (4.10). □

5. Examples

In this section, we present some examples to illustrate the main results of this paper. In the first 3 examples we assume that X is a Brownian motion in $\mathbf{R}^d$ running twice as fast as the standard Brownian motion. Let D be a bounded domain in $\mathbf{R}^d$. Since the transition density function $p(t,x,y)$ of X satisfies $p(t,x,y)\leqslant c\,t^{-d/2}$ for every $t>0$ and $x,y\in\mathbf{R}^d$, condition (3.3) is satisfied and so assumption **(I)** holds. Let $\{-\lambda_n,n\geqslant 1\}$ be the eigenvalues for Δ in D with Dirichlet boundary condition (or, equivalently, for the generator of the Brownian motion X killed upon leaving D). Let S be a subordinator with Laplace exponent ϕ. We assume that X and S are independent.

Example 5.1. Let $\phi(\lambda) = \lambda^{\alpha/2}$ for $\alpha \in (0, 2)$, which is a complete Bernstein function. Then X^ϕ is just the spherically symmetric α-stable process in $\mathbf{R}^d$. Let $\{-\mu_n, n \geqslant 1\}$ be the eigenvalues for $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ in D with Dirichlet exterior condition (or, equivalently, for the generator of the spherically symmetric α-stable process X^ϕ killed upon leaving D). We see from (3.6) that

$$\mu_n \leqslant \lambda_n^{\alpha/2} \quad \text{for every } n \geqslant 1.$$

Moreover, by Theorem 4.5, we have that

$$c\, \lambda_n^{\alpha/2} \leqslant \mu_n \leqslant \lambda_n^{\alpha/2} \quad \text{for every } n \geqslant 1. \tag{5.1}$$

When D is a bounded convex domain in $\mathbf{R}^d$, the above constant c can be taken as 1/2.

When $d = 1$ and $D = (0, L) \subset \mathbf{R}$, a direct computation shows that X^D has eigenvalues $\{(n\pi)^2/L^2;\ n = 1, 2, \ldots\}$ and the corresponding eigenfunctions $\{\sin(\frac{n\pi}{L}x);\ n = 1, 2, \ldots\}$. So (5.1) says that

$$\tfrac{1}{2}(n\pi)^\alpha/L^\alpha \leqslant \mu_n \leqslant (n\pi)^\alpha/L^\alpha \quad \text{for every } n \geqslant 1.$$

In particular for $\alpha = 1$ and $L = 2$,

$$n\pi/4 \leqslant \mu_n \leqslant n\pi/2 \quad \text{for every } n \geqslant 1.$$

But this is not sharp when $n = 1, 2, 3$. By a detailed analysis of the corresponding mixed Steklov problem it is shown in [1, Corollary 2.2, Theorems 5.3 and 5.4] that for $L = 2$, $\alpha = 1$,

$$1 \leqslant \mu_1 \leqslant \frac{3\pi}{8}, \quad 2 \leqslant \mu_2 \leqslant \pi, \quad 3.4 \leqslant \mu_3 \leqslant \frac{3\pi}{2}.$$

Nevertheless our estimate (5.1) is valid for every $n \geqslant 1$, for every $\alpha \in (0, 2)$, and for any bounded domain $D \subset \mathbf{R}^d$ with any dimension $d \geqslant 1$.

It is interesting to compare the estimate (5.1) with the Weyl's type asymptotics for eigenvalues $\{\lambda_n, n \geqslant 1\}$ and $\{\mu_n, n \geqslant 1\}$. It follows from Theorem 2.3 of [5] that if D is a bounded domain with ∂D having zero Lebesgue measure, then

$$\lim_{n\to\infty} \frac{\lambda_n}{n^{2/d}} = c(d, 2)\, |D|^{2/d} \in (0, \infty) \tag{5.2}$$

and

$$\lim_{n\to\infty} \frac{\mu_n}{n^{\alpha/d}} = c(d, \alpha)\, |D|^{\alpha/d} \in (0, \infty), \tag{5.3}$$

where $|D|$ denotes the Lebesgue measure of the domain D. □

Example 5.2. Let $\phi(\lambda) = (\lambda + a)^{\alpha/2} - a^{\alpha/2}$, where $a > 0$ and $\alpha \in (0, 2)$, which is an unbounded complete Bernstein function. Then X^ϕ is a relativistic α-stable process in $\mathbf{R}^d$. According to (3.7) of [15], the Dirichlet form for X^ϕ is $(\mathcal{E}^\phi, W^{\alpha/2,2}(\mathbf{R}^d))$ and there is a constant $c_5 > 1$ such that

$$c_5^{-1}\mathcal{E}_1(u, u) \leqslant \mathcal{E}_1^\phi(u, u) \leqslant c_5\, \mathcal{E}_1(u, u) \quad \text{for } u \in W^{\alpha/2,2}(\mathbf{R}^d),$$

where $(\mathcal{E}, W^{\alpha/2,2}(\mathbf{R}^d))$ is the Dirichlet form for the spherical symmetric α-stable process in $\mathbf{R}^d$. Since for every $u \in W^{\alpha/2,2}(\mathbf{R}^d)$,

$$\|u\|_{L^2(\mathbf{R}^d)}^{2+\frac{\alpha}{d}} \leqslant c_6\, \mathcal{E}(u, u)\|u\|_{L^1(\mathbf{R}^d)}^{\frac{\alpha}{d}} \leqslant c_7\, \mathcal{E}_1^\phi(u, u)\|u\|_{L^1(\mathbf{R}^d)}^{\frac{\alpha}{d}}.$$

So by Theorem 2.4.6 of [17],

$$e^{-t}\, p^\phi(t, x, y) \leqslant c_8\, t^{-d/\alpha} \quad \text{for every } t > 0 \text{ and } x, y \in \mathbf{R}^d.$$

Thus if D is a bounded domain in $\mathbf{R}^d$, then both conditions (3.3) and (3.5) are satisfied and therefore both assumptions **(I)** and **(II)** hold. In this case, $\{-\lambda_n, n \geqslant 1\}$ are the eigenvalues for Δ in D with Dirichlet boundary condition (or, equivalently, for the generator of the Brownian motion X killed upon leaving D) and $\{-\mu_n, n \geqslant 1\}$ are the eigenvalues for $a^{\alpha/2} - (a - \Delta)^{\alpha/2}$ in D with Dirichlet exterior condition (or, equivalently, for the generator of the relativistic α-stable process X^ϕ killed upon leaving D). By Theorem 3.4, we have

$$\mu_n \leqslant (\lambda_n + a)^{\alpha/2} - a^{\alpha/2} \quad \text{for every } n \geqslant 1. \tag{5.4}$$

Again by Theorem 4.5 if D is a bounded domain satisfying the exterior cone condition, then for any $\alpha \in (0, 2)$, there is a constant $c = c(D) > 0$ such that

$$c\left((\lambda_n + a)^{\alpha/2} - a^{\alpha/2}\right) \leqslant \mu_n \leqslant (\lambda_n + a)^{\alpha/2} - a^{\alpha/2} \quad \text{for every } n \geqslant 1. \tag{5.5}$$

When D is a bounded convex domain in $\mathbf{R}^d$, the above constant c can be taken as 1/2. Hence it follows from (5.2) and (5.5) that there are positive constants $c_1 = c_1(D)$ and $c_2 = c_2(D)$ such that

$$c_1\, n^{\alpha/d} \leqslant \mu_n \leqslant c_2\, n^{\alpha/d} \quad \text{for every } n \geqslant 1. \qquad \square \tag{5.6}$$

Example 5.3. Let $\phi(\lambda) = \log(1 + \lambda^{\alpha/2}), \alpha \in (0, 2]$. $\phi(\lambda)$ is an unbounded complete Bernstein function, the process X^ϕ is a geometric α-stable process for $\alpha \in (0, 2)$ and a variance gamma process for $\alpha = 2$. Geometric stable distributions were first introduced in [22] and they have played an important role in heavy-tail modeling of economic data. For recent results on the potential theory of geometric stable processes, please see [31]. Since $\phi(\lambda) = \Psi_2 \circ \psi_1(\lambda)$, where $\psi_1(\lambda) = \lambda^{\alpha/2}$ and $\psi_2(\lambda) = \log(1+\lambda)$, it is easy to see that the process X^ϕ can be obtained by subordinating a spherically symmetric α-stable process Y with an independent gamma subordinator. Let $P^Y(t,x,y)$ denote the transition density function of Y and note that the gamma subordinator has transition density function $\frac{1}{\Gamma(t)} u^{t-1} e^{-u}$. Suppose that D is a bounded domain in $\mathbf{R}^d$. For $0 < \alpha \leqslant 2$, we have

$$\int_D p^\phi(t,x,x) m(dx) = \int_0^\infty \left(\int_D p^Y(s,x,x) m(dx) \right) \frac{1}{\Gamma(t)} s^{t-1} e^{-s} ds$$
$$\leqslant \frac{c_1}{\Gamma(t)} \int_0^\infty s^{t-\frac{d}{\alpha}-1} e^{-s} ds < \infty$$

whenever $t > \frac{d}{\alpha}$. So condition (3.5) is satisfied for any $0 < \alpha \leqslant 2$ and for any bounded domain D in $\mathbf{R}^d$. Condition (3.3) is obviously satisfied for any bounded domain D in $\mathbf{R}^d$. Let $\{-\mu_n, n \geqslant 1\}$ be the eigenvalues for the generator of X^ϕ killed upon leaving D. By Theorem 3.4, we have

$$\mu_n \leqslant \log(1+\lambda_n^{\alpha/2}) \quad \text{for every } n \geqslant 1.$$

Moreover, if D is a bounded domain satisfying the exterior cone condition, there is a constant $c = c(D) > 0$ depending on D only such that

$$c \log(1+\lambda_n^{\alpha/2}) \leqslant \mu_n \leqslant \log(1+\lambda_n^{\alpha/2}) \quad \text{for every } n \geqslant 1.$$

When D is a bounded convex domain in $\mathbf{R}^d$, the above constant c can be taken as 1/2. □

Obviously more examples can be given in similar lines of the examples above by replacing X by other kind of symmetric processes, such as spherically symmetric α-stable process, symmetric Lévy process, symmetric diffusions with infinitesimal generators of divergence form, etc., or their subprocesses killed upon leaving a domain. We will only present one example of this nature.

Example 5.4. Suppose that $0 < \alpha < \beta \leqslant 2$, X is a spherically symmetric β-stable process and that Y is a spherically symmetric α-stable process. Let D be a bounded open subset of $\mathbf{R}^d$. Since the transition density function $p(t,x,y)$ of X satisfies $p(t,x,y) \leqslant c\, t^{-d/\beta}$ for every $t > 0$ and $x, y \in \mathbf{R}^d$, condition (3.3) is satisfied and thus the spectrum of the generator of X^D is discrete. Let $\{-\lambda_n, n \geqslant 1\}$ be the eigenvalues for $\Delta^{\beta/2}$ in D with zero exterior conditions (or, equivalently, for the generator of the X killed upon leaving D). Similarly we know that the spectrum of the generator of Y^D is also discrete. Let $\{-\mu_n, n \geqslant 1\}$ be the eigenvalues for $\Delta^{\alpha/2}$ in D with zero exterior conditions. Note that the process Y can be realized as a subordinate process of

X via an independent subordinator with Laplace exponent $\phi(\lambda) = \lambda^{\alpha/\beta}$. So by Theorem 3.4, we have

$$\mu_n \leqslant \lambda_n^{\alpha/\beta} \quad \text{for every } n \geqslant 1.$$

Moreover, if D is a bounded domain satisfying the exterior cone condition, there is a constant $c = c(D) > 0$ depending on D only such that

$$c\,\lambda_n^{\alpha/\beta} \leqslant \mu_n \leqslant \lambda_n^{\alpha/\beta} \quad \text{for every } n \geqslant 1.$$

When D is a bounded convex domain in $\mathbf{R}^d$, the above constant c can be taken as 1/2. □

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Note added in proof

In connection with Theorem 3.3, when S is a general subordinator with Laplace exponent ϕ (which is not necessarily complete Bernstein), we always have $(\mathcal{F}^D)^\phi \subset (\mathcal{F}^\phi)^D$ but with

$$\mathcal{E}^\phi(u,u) \leqslant 4(\mathcal{E}^D)^\phi(u,u) \qquad \text{for every } u \in (\mathcal{F}^D)^\phi. \tag{A.1}$$

This is because when $u \in (\mathcal{F}^D)^\phi$ is a non-negative function, since $P_s^D u \leqslant P_s u$ m-a.e. on D, we conclude from (2.4)-(2.7) that $u \in \mathcal{F}^\phi$ with

$$\mathcal{E}^\phi(u,u) \leqslant (\mathcal{E}^D)^\phi(u,u). \tag{A.2}$$

For general $u \in (\mathcal{F}^D)^\phi$, u^+ and u^- are in $(\mathcal{F}^D)^\phi$ with

$$\mathcal{E}^\phi(u^+,u^+) \leqslant (\mathcal{E}^D)^\phi(u^+,u^+) \quad \text{and} \quad \mathcal{E}^\phi(u^-,u^-) \leqslant (\mathcal{E}^D)^\phi(u^-,u^-).$$

It follows that $u = u^+ - u^- \in \mathcal{F}^\phi$ and, noting the contraction property

$$(\mathcal{E}^D)^\phi(u^+,u^+) \leq (\mathcal{E}^D)^\phi(u,u) \quad \text{and} \quad (\mathcal{E}^D)^\phi(u^-,u^-) \leq (\mathcal{E}^D)^\phi(u,u),$$

we get that (A.1) holds. Consequently, $(\mathcal{F}^D)^\phi \subset (\mathcal{F}^\phi)^D$. In their recent preprint titled "Killing transform and subordinate process", P. He and J. Ying showed that (A.1) holds with multiplicative constant 3 instead of 4 using a different method. We remark that using the above result (A.1), the conclusions of Theorems 4.3-4.5 hold for general subordinator S (that is, its Laplace exponent ϕ is a general Bernstein function only) except that the upper bound estimates for $\mathcal{E}^{\phi}(u,u)$ in (4.8) and for eigenvalue μ_n need to have a multiplicative factor of 4 respectively. Using the following formula for the first eigenvalue

$$\mu_1 = \inf\{\mathcal{E}^{\phi}(|u|,|u|) : u \in (\mathcal{E}^{\phi})^D \text{ with } (u,u)_{L^2(D,m)} = 1\}$$

and the analogous formula for the first eigenvalue $\phi(\lambda_1)$ for $((\mathcal{E}^D)^{\phi}, (\mathcal{F}^D)^{\phi})$, we see by (A.2) that

$$\mu_1 \leqslant \phi(\lambda_1)$$

for general subordinator S with Laplace exponent ϕ. Thus the first eigenvalue estimates in Theorem 4.4–4.5 (i.e., with $n = 1$) and, consequently, in Theorem 4.7 hold in fact for general subordinator S.

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