Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 332 (2007) 57–80
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Estimates on Green functions and Schrödinger-type equations for non-symmetric diffusions with measure-valued drifts

Panki Kim [a,*,1], Renming Song [b,2,3]

[a] *Department of Mathematics, Seoul National University, Seoul 151-742, Republic of Korea*
[b] *Department of Mathematics, University of Illinois, Urbana, IL 61801, USA*

Received 5 May 2006

Available online 13 November 2006

Submitted by L. Chen

Abstract

In this paper, we establish sharp two-sided estimates for the Green functions of non-symmetric diffusions with measure-valued drifts in bounded Lipschitz domains. As consequences of these estimates, we get a 3G type theorem and a conditional gauge theorem for these diffusions in bounded Lipschitz domains.

Informally the Schrödinger-type operators we consider are of the form $L + \mu \cdot \nabla + \nu$ where L is a uniformly elliptic second order differential operator, μ is a vector-valued signed measure belonging to $\mathbf{K}_{d,1}$ and ν is a signed measure belonging to $\mathbf{K}_{d,2}$. In this paper, we establish two-sided estimates for the heat kernels of Schrödinger-type operators in bounded $C^{1,1}$-domains and a scale invariant boundary Harnack principle for the positive harmonic functions with respect to Schrödinger-type operators in bounded Lipschitz domains.

Keywords: Brownian motion; Diffusion; Diffusion process; Non-symmetric diffusion; Kato class; Measure-valued drift; Transition density; Green function; Lipschitz domain; 3G theorem; Schrödinger operator; Heat kernel; Boundary Harnack principle; Harmonic function

* Corresponding author. Fax: 82 2 887 4694.
E-mail addresses: pkim@snu.ac.kr (P. Kim), rsong@math.uiuc.edu (R. Song).
URLs: http://www.math.snu.ac.kr/~pkim (P. Kim), http://www.math.uiuc.edu/~rsong (R. Song).
[1] The research of this author is supported by Research Settlement Fund for the new faculty of Seoul National University.
[2] The research of this author is supported in part by a joint US–Croatia grant INT 0302167.
[3] Fax: 1 217 333 9576.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.10.014

1. Introduction

This paper is a natural continuation of [11–13], where diffusion (Brownian motion) with measure-valued drift was discussed. For a vector-valued signed measure μ belonging to $\mathbf{K}_{d,1}$, a diffusion with measure-valued drift μ is a diffusion process whose generator can be informally written as $L + \mu \cdot \nabla$, where L is a uniformly elliptic second order differential operator. In this paper we consider Schrödinger-type operators $L + \mu \cdot \nabla + \nu$ (see below for the definition) and discuss their properties.

In this paper we always assume that $d \geqslant 3$. First we recall the definition of the Kato class $\mathbf{K}_{d,\alpha}$ for $\alpha \in (0, 2]$. For any function f on $\mathbf{R}^d$ and $r > 0$, we define

$$M_f^\alpha(r) = \sup_{x\in\mathbf{R}^d} \int\limits_{|x-y|\leqslant r} \frac{|f|(y)\,dy}{|x-y|^{d-\alpha}}, \quad 0 < \alpha \leqslant 2.$$

In this paper, we mean, by a signed measure, the difference of two nonnegative measures at most one of which can have infinite total mass. For any signed measure ν on $\mathbf{R}^d$, we use ν^+ and ν^- to denote its positive and negative parts, and $|\nu| = \nu^+ + \nu^-$ its total variation. For any signed measure ν on $\mathbf{R}^d$ and any $r > 0$, we define

$$M_\nu^\alpha(r) = \sup_{x\in\mathbf{R}^d} \int\limits_{|x-y|\leqslant r} \frac{|\nu|(dy)}{|x-y|^{d-\alpha}}, \quad 0 < \alpha \leqslant 2.$$

Definition 1.1. Let $0 < \alpha \leqslant 2$. We say that a function f on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if $\lim_{r\downarrow 0} M_f^\alpha(r) = 0$. We say that a signed Radon measure ν on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if $\lim_{r\downarrow 0} M_\nu^\alpha(r) = 0$. We say that a d-dimensional vector valued function $V = (V^1, \ldots, V^d)$ on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if each V^i belongs to the Kato class $\mathbf{K}_{d,\alpha}$. We say that a d-dimensional vector valued signed Radon measure $\mu = (\mu^1, \ldots, \mu^d)$ on $\mathbf{R}^d$ belongs to the Kato class $\mathbf{K}_{d,\alpha}$ if each μ^i belongs to the Kato class $\mathbf{K}_{d,\alpha}$.

Rigorously speaking a function f in $\mathbf{K}_{d,\alpha}$ may not give rise to a signed measure ν in $\mathbf{K}_{d,\alpha}$ since it may not give rise to a signed measure at all. However, for the sake of simplicity we use the convention that whenever we write that a signed measure ν belongs to $\mathbf{K}_{d,\alpha}$ we are implicitly assuming that we are covering the case of all the functions in $\mathbf{K}_{d,\alpha}$ as well.

Throughout this paper we assume that $\mu = (\mu^1, \ldots, \mu^d)$ is fixed with each μ^i being a signed measure on $\mathbf{R}^d$ belonging to $\mathbf{K}_{d,1}$. We also assume that the operator L is either L_1 or L_2 where

$$L_1 := \frac{1}{2}\sum_{i,j=1}^d \partial_i(a_{ij}\partial_j) \quad \text{and} \quad L_2 := \frac{1}{2}\sum_{i,j=1}^d a_{ij}\partial_i\partial_j$$

with $\mathbf{A} := (a_{ij})$ being C^1 and uniformly elliptic. We do not assume that $\mathbf{A}$ is symmetric.

Informally, when $\mathbf{A}$ is symmetric, a diffusion process X in $\mathbf{R}^d$ with drift μ is a diffusion process in $\mathbf{R}^d$ with generator $L+\mu\cdot\nabla$. When each μ^i is given by $U^i(x)\,dx$ for some function U^i, X is a diffusion in $\mathbf{R}^d$ with generator $L + U \cdot \nabla$ and it is a solution to the stochastic differential equation $dX_t = dY_t + U(X_t)\cdot dt$ where Y is a diffusion in $\mathbf{R}^d$ with generator L. For a precise definition of a (non-symmetric) diffusion X with drift μ in $\mathbf{K}_{d,1}$, we refer to Section 6 in [12] and Section 1 in [13]. The existence and uniqueness of X were established in [1] (see Remark 6.1 in [1]). In this paper, we will always use X to denote a diffusion process with drift μ.

In [11–13], we have already studied some potential theoretical properties of the process X. More precisely, we have established two-sided estimates for the heat kernel of the killed diffusion process X^D and sharp two-sided estimates on the Green function of X^D when D is a bounded $C^{1,1}$ domain; proved a scale invariant boundary Harnack principle for the positive harmonic functions of X in bounded Lipschitz domains; and identified the Martin boundary X^D in bounded Lipschitz domains.

In this paper, we will first establish sharp two-sided estimates for the Green function of X^D when D is a bounded Lipschitz domain. As consequences of these estimates, we get a 3G type theorem and a conditional gauge theorem for X in bounded Lipschitz domains. We also establish two-sided estimates for the heat kernels of Schrödinger-type operators in bounded $C^{1,1}$-domains and a scale invariant boundary Harnack principle for the positive harmonic functions with respect to Schrödinger-type operators in bounded Lipschitz domains. The results of this paper will be used in proving the intrinsic ultracontractivity of the Schrödinger semigroup of X^D in [14].

Throughout this paper, for two real numbers a and b, we denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The distance between x and ∂D is denote by $\rho_D(x)$. In this paper we will use the following convention: the values of the constants r_i, $i = 1, \ldots, 6$, C_0, C_1, M, M_i, $i = 1, \ldots, 5$, and ε_1 will remain the same throughout this paper, while the values of the constants $c, c_1, c_2, \ldots$ may change from one appearance to another. In this paper, we use ":=" to denote a definition, which is read as "is defined to be."

2. Green function estimates and 3G theorem

In this section we will establish sharp two-sided estimates for the Green function and a 3G theorem for X in bounded Lipschitz domains. We will first establish some preliminary results for the Green function $G_D(x, y)$ of X^D. Once we have these results, the proof of the Green function estimates is similar to the ones in [3,5,10]. The main difference is that the Green function $G_D(x, y)$ is not (quasi-)symmetric.

For any bounded domain D, we use τ_D to denote the first exit time of D, i.e., $\tau_D = \inf\{t > 0\colon X_t \notin D\}$. Given a bounded domain $D \subset \mathbf{R}^d$, we define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \geqslant \tau_D(\omega)$, where ∂ is a cemetery state. The process X^D is called a killed diffusion with drift μ in D. Throughout this paper, we use the convention $f(\partial) = 0$.

It is shown in [12] that, for any bounded domain D, X^D has a jointly continuous and strictly positive transition density function $q^D(t, x, y)$ (see Theorem 2.4 in [12]). In [12], we also showed that there exist positive constants c_1 and c_2 depending on D via its diameter such that for any $(t, x, y) \in (0, \infty) \times D \times D$,

$$q^D(t, x, y) \leqslant c_1 t^{-\frac{d}{2}} e^{-\frac{c_2 |x-y|^2}{2t}} \tag{2.1}$$

(see Lemma 2.5 in [12]). Let $G_D(x, y)$ be the Green function of X^D, i.e.,

$$G_D(x, y) := \int_0^\infty q^D(t, x, y)\, dt.$$

By (2.1), $G_D(x, y)$ is finite for $x \neq y$ and

$$G_D(x, y) \leqslant \frac{c}{|x - y|^{d-2}} \tag{2.2}$$

for some $c = c(\mathrm{diam}(D)) > 0$.

By Theorem 3.7 in [12], we see that there exist constants $r_1 = r_1(d,\mu) > 0$ and $c = c(d,\mu) > 1$ depending on μ only via the rate at which $\max_{1\leqslant i\leqslant d} M_{\mu^i}(r)$ goes to zero such that for $r \leqslant r_1$, $z \in \mathbf{R}^d$,

$$c^{-1}|x-y|^{-d+2} \leqslant G_{B(z,r)}(x,y) \leqslant c|x-y|^{-d+2}, \quad x,y \in \overline{B(z,2r/3)}. \tag{2.3}$$

Definition 2.1. Suppose U is an open subset of $\mathbf{R}^d$.

(1) A Borel function u defined on U is said to be harmonic with respect to X in U if

$$u(x) = \mathbf{E}_x\big[u(X_{\tau_B})\big], \quad x \in B, \tag{2.4}$$

for every bounded open set B with $\overline{B} \subset U$.

(2) A Borel function u defined on $\overline{U}$ is said to be regular harmonic with respect to X in U if u is harmonic with respect to X in U and (2.4) is true for $B = U$.

Every positive harmonic function in a bounded domain D is continuous in D (see Proposition 2.10 in [12]). Moreover, for every open subset U of D, we have

$$\mathbf{E}_x\big[G_D(X_{T_U},y)\big] = G_D(x,y), \quad (x,y) \in D \times U, \tag{2.5}$$

where $T_U := \inf\{t > 0\colon X_t \in U\}$. In particular, for every $y \in D$ and $\varepsilon > 0$, $G_D(\cdot,y)$ is regular harmonic in $D \setminus B(y,\varepsilon)$ with respect to X (see Theorem 2.9(1) in [12]).

We recall here the scale invariant Harnack inequality from [11].

Theorem 2.2. [11, Corollary 5.8] *There exist $r_2 = r_2(d,\mu) > 0$ and $c = c(d,\mu) > 0$ depending on μ only via the rate at which $\max_{1\leqslant i\leqslant d} M^1_{\mu^i}(r)$ goes to zero such that for every positive harmonic function f for X in $B(x_0,r)$ with $r \in (0,r_2)$, we have*

$$\sup_{y\in B(x_0,r/2)} f(y) \leqslant c \inf_{y\in B(x_0,r/2)} f(y).$$

Recall that $r_1 > 0$ is the constant from (2.3).

Lemma 2.3. *For any bounded domain D, there exists $c = c(D,\mu) > 0$ such that for every $r \in (0, r_1 \wedge r_2]$ and $B(z,r) \subset D$, we have for every $x \in D \setminus \overline{B(z,r)}$,*

$$\sup_{y\in B(z,r/2)} G_D(y,x) \leqslant c \inf_{y\in B(z,r/2)} G_D(y,x) \tag{2.6}$$

and

$$\sup_{y\in B(z,r/2)} G_D(x,y) \leqslant c \inf_{y\in B(z,r/2)} G_D(x,y). \tag{2.7}$$

Proof. Fix $x \in D \setminus \overline{B(z,r)}$. Since $G_D(\cdot,x)$ is harmonic for X in $B(z,r)$, (2.6) follows from Theorem 2.2. So we only need to show (2.7).

Since $r < r_1$, by (2.2) and (2.3), there exist $c_1 = c_1(D) > 1$ and $c_2 = c_2(d) > 1$ such that for every $y, w \in \overline{B(z,3r/4)}$,

$$c_2^{-1}\frac{1}{|w-y|^{d-2}} \leqslant G_{B(z,r)}(w,y) \leqslant G_D(w,y) \leqslant c_1\frac{1}{|w-y|^{d-2}}.$$

Thus for $w \in \partial B(z, \frac{3r}{4})$ and $y_1, y_2 \in B(z, \frac{r}{2})$, we have

$$G_D(w, y_1) \leqslant c_1 \left(\frac{|w-y_2|}{|w-y_1|} \right)^{d-2} \frac{1}{|w-y_2|^{d-2}} \leqslant 4^{d-2} c_2 c_1 G_D(w, y_2). \tag{2.8}$$

On the other hand, by (2.5), we have

$$G_D(x, y) = \mathbf{E}_x \big[G_D(X_{T_{B(z, \frac{3r}{4})}}, y) \big], \quad y \in B\left(z, \frac{r}{2}\right). \tag{2.9}$$

Since $X_{T_{B(z, \frac{3r}{4})}} \in \partial B(z, \frac{3r}{4})$, combining (2.8)–(2.9), we get

$$G_D(x, y_1) \leqslant 4^{d-2} c_2 c_1 \mathbf{E}_x \big[G_D(X_{T_{B(z, \frac{3r}{4})}}, y_2) \big] = 4^{d-2} c_2 c_1 G_D(x, y_2), \quad y_1, y_2 \in B\left(z, \frac{r}{2}\right).$$

In fact, (2.7) is true for every $x \in D$. $\square$

Recall that a bounded domain D is said to be Lipschitz if there are a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $Q \in \partial D$, there are a Lipschitz function $\phi_Q : \mathbf{R}^{d-1} \to \mathbf{R}$ satisfying $|\phi_Q(x) - \phi_Q(z)| \leqslant \Lambda_0 |x - z|$, and an orthonormal coordinate system CS_Q with origin at Q such that

$$B(Q, R_0) \cap D = B(Q, R_0) \cap \big\{ y = (y_1, \dots, y_{d-1}, y_d) =: (\tilde{y}, y_d) \text{ in } CS_Q\colon\ y_d > \phi_Q(\tilde{y}) \big\}.$$

The pair (R_0, Λ_0) is called the characteristic of the Lipschitz domain D.

Any bounded Lipschitz domain satisfies the κ-fat property: there exists $\kappa_0 \in (0, 1/2]$ depending on Λ_0 such that for each $Q \in \partial D$ and $r \in (0, R_0)$ (by choosing R_0 smaller if necessary), $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa_0 r)$.

In this section, we fix a bounded Lipschitz domain D with its characteristic (R_0, Λ_0) and κ_0. Without loss of generality, we may assume that the diameter of D is less than 1.

We recall here the scale invariant boundary Harnack principle for X^D in bounded Lipschitz domains from [12].

Theorem 2.4. [12, Theorem 4.6] *Suppose D is a bounded Lipschitz domain. Then there exist constants M_1, $c > 1$ and $r_3 > 0$, depending on μ only via the rate at which* $\max_{1 \leqslant i \leqslant d} M^1_{\mu^i}(r)$ *goes to zero such that for every $Q \in \partial D$, $r < r_3$ and any nonnegative functions u and v which are harmonic with respect to X^D in $D \cap B(Q, M_1 r)$ and vanish continuously on $\partial D \cap B(Q, M_1 r)$, we have*

$$\frac{u(x)}{v(x)} \leqslant c \frac{u(y)}{v(y)} \quad \textit{for any } x, y \in D \cap B(Q, r). \tag{2.10}$$

For any $Q \in \partial D$, we define

$$\Delta_Q(r) := \big\{ y \text{ in } CS_Q\colon\ \phi_Q(\tilde{y}) + 2r > y_d > \phi_Q(\tilde{y}),\ |\tilde{y}| < 2(M_1 + 1) r \big\},$$
$$\partial_1 \Delta_Q(r) := \big\{ y \text{ in } CS_Q\colon\ \phi_Q(\tilde{y}) + 2r \geqslant y_d > \phi_Q(\tilde{y}),\ |\tilde{y}| = 2(M_1 + 1) r \big\},$$
$$\partial_2 \Delta_Q(r) := \big\{ y \text{ in } CS_Q\colon\ \phi_Q(\tilde{y}) + 2r = y_d,\ |\tilde{y}| \leqslant 2(M_1 + 1) r \big\},$$

where CS_Q is the coordinate system with origin at Q in the definition of Lipschitz domains and ϕ_Q is the Lipschitz function there. Let $M_2 := 2(1+M_1)\sqrt{1+\Lambda_0^2}+2$ and $r_4 := M_2^{-1}(R_0 \wedge r_1 \wedge r_2 \wedge r_3)$. If $z \in \overline{\Delta_Q(r)}$ with $r \leqslant r_4$, then

$$|Q-z| \leqslant \left|\left(\tilde{z}, \phi_Q(\tilde{z})\right) - (\tilde{z}, 0)\right| + 2r \leqslant 2r(1+M_1)\sqrt{1+\Lambda_0^2} + 2r = M_2 r \leqslant M_2 r_4 \leqslant R_0.$$

So $\overline{\Delta_Q(r)} \subset B(Q, M_2 r) \cap D \subset B(Q, R_0) \cap D$.

Lemma 2.5. *There exists a constant $c > 1$ such that for every $Q \in \partial D$, $r < r_4$, and any non-negative functions u and v which are harmonic in $D \setminus B(Q,r)$ and vanish continuously on $\partial D \setminus B(Q,r)$, we have*

$$\frac{u(x)}{u(y)} \leqslant c\frac{v(x)}{v(y)} \quad \textit{for any } x, y \in D \setminus B(Q, M_2 r). \tag{2.11}$$

Proof. Throughout this proof, we fix a point Q on ∂D, $r < r_4$, $\Delta_Q(r)$, $\partial_1\Delta_Q(r)$ and $\partial_2\Delta_Q(r)$. Fix a $\tilde{y}_0 \in \mathbf{R}^{d-1}$ with $|\tilde{y}_0| = 2(M_1+1)r$. Since $|(\tilde{y}_0, \phi_Q(\tilde{y}_0))| > r$, u and v are harmonic with respect to X in $D \cap B((\tilde{y}_0, \phi_Q(\tilde{y}_0)), 2M_1 r)$ and vanish continuously on $\partial D \cap B((\tilde{y}_0, \phi_Q(\tilde{y}_0)), 2M_1 r)$. Therefore by Theorem 2.4,

$$\frac{u(x)}{u(y)} \leqslant c_1\frac{v(x)}{v(y)} \quad \text{for any } x, y \in \partial_1\Delta_Q(r) \text{ with } \tilde{x} = \tilde{y} = \tilde{y}_0, \tag{2.12}$$

for some constant $c_1 > 0$. Since $\operatorname{dist}(D \cap B(Q,r), \partial_2\Delta_Q(r)) > cr$ for some $c := c(\Lambda_0)$, the Harnack inequality (Theorem 2.2) and a Harnack chain argument imply that there exists a constant $c_2 > 1$ such that

$$c_2^{-1} < \frac{u(x)}{u(y)}, \frac{v(x)}{v(y)} < c_2, \quad \text{for any } x, y \in \partial_2\Delta_Q(r). \tag{2.13}$$

In particular, (2.13) is true with $y := (\tilde{y}_0, \phi_Q(\tilde{y}_0) + 2r)$, which is also in $\partial_1\Delta_Q(r)$. Thus (2.12) and (2.13) imply that

$$c_3^{-1}\frac{u(x)}{u(y)} \leqslant \frac{v(x)}{v(y)} \leqslant c_3\frac{u(x)}{u(y)}, \quad x, y \in \partial_1\Delta_Q(r) \cup \partial_2\Delta_Q(r), \tag{2.14}$$

for some constant $c_3 > 0$. Now, by applying the maximum principle (Lemma 7.2 in [11]) twice, we get that (2.14) is true for every $x \in D \setminus \Delta_Q(r) \supset D \setminus B(Q, M_2 r)$. □

Combining Theorem 2.4 and Lemma 2.5, we get a uniform boundary Harnack principle for $G_D(x,y)$ in both variables. Recall κ_0 is the κ-fat constant of D.

Lemma 2.6. *There exist constants $c > 1$, $M > 1/\kappa_0$ and $r_0 \leqslant r_4$ such that for every $Q \in \partial D$, $r < r_0$, we have for $x, y \in D \setminus B(Q,r)$ and $z_1, z_2 \in D \cap B(Q, r/M)$,*

$$\frac{G_D(x,z_1)}{G_D(y,z_1)} \leqslant c\frac{G_D(x,z_2)}{G_D(y,z_2)} \quad \textit{and} \quad \frac{G_D(z_1,x)}{G_D(z_1,y)} \leqslant c\frac{G_D(z_2,x)}{G_D(z_2,y)}. \tag{2.15}$$

Fix $z_0 \in D$ with $r_0/M < \rho_D(z_0) < r_0$ and let $\varepsilon_1 := r_0/(12M)$. For $x, y \in D$, we let $r(x,y) := \rho_D(x) \vee \rho_D(y) \vee |x-y|$ and

$$\mathcal{B}(x,y) := \left\{A \in D\colon\ \rho_D(A) > \frac{1}{M}r(x,y),\ |x-A| \vee |y-A| < 5r(x,y)\right\}$$

if $r(x,y) < \varepsilon_1$, and $\mathcal{B}(x,y) := \{z_0\}$ otherwise.

By a Harnack chain argument we get the following from (2.2) and (2.3).

Lemma 2.7. *There exists a positive constant C_0 such that $G_D(x,y) \leqslant C_0|x-y|^{-d+2}$, for all $x,y \in D$, and $G_D(x,y) \geqslant C_0^{-1}|x-y|^{-d+2}$ if $2|x-y| \leqslant \rho_D(x) \vee \rho_D(y)$.*

Let $C_1 := C_0 2^{d-2}\rho_D(z_0)^{2-d}$. The above lemma implies that $G_D(\cdot, z_0)$ and $G_D(z_0,\cdot)$ are bounded above by C_1 on $D \setminus B(z_0, \rho_D(z_0)/2)$. Now we define

$$g_1(x) := G_D(x,z_0) \wedge C_1 \quad \text{and} \quad g_2(y) := G_D(z_0,y) \wedge C_1.$$

Using Lemma 2.3 and a Harnack chain argument, we get the following.

Lemma 2.8. *For every $y \in D$ and $x_1, x_2 \in D \setminus B(y, \rho_D(y)/2)$ with $|x_1 - x_2| \leqslant k(\rho_D(x_1) \wedge \rho_D(x_2))$, there exists $c := c(D,k) > 0$ independent of y and x_1, x_2 such that*

$$G_D(x_1,y) \leqslant c G_D(x_2,y) \quad \textit{and} \quad G_D(y,x_1) \leqslant c G_D(y,x_2). \tag{2.16}$$

The next two lemmas follow easily from the result above.

Lemma 2.9. *There exists $c = c(D) > 0$ such that for every $x, y \in D$,*

$$c^{-1} g_1(A_1) \leqslant g_1(A_2) \leqslant c g_1(A_1) \quad \textit{and} \quad c^{-1} g_2(A_1) \leqslant g_2(A_2) \leqslant c g_2(A_1),$$
$$A_1, A_2 \in \mathcal{B}(x,y).$$

Lemma 2.10. *There exists $c = c(D) > 0$ such that for every $x \in \{y \in D;\ \rho_D(y) \geqslant \varepsilon_1/(8M^3)\}$, $c^{-1} \leqslant g_i(x) \leqslant c$, $i = 1,2$.*

Using Lemma 2.3, the proof of the next lemma is routine (for example, see Lemma 6.7 in [8]). So we omit the proof.

Lemma 2.11. *For any given $c_1 > 0$, there exists $c_2 = c_2(D, c_1, \mu) > 0$ such that for every $|x-y| \leqslant c_1(\rho_D(x) \wedge \rho_D(y))$,*

$$G_D(x,y) \geqslant c_2 |x-y|^{-d+2}.$$

In particular, there exists $c = c(D,\mu) > 0$ such that for every $|x-y| \leqslant (8M^3/\varepsilon_1)(\rho_D(x) \wedge \rho_D(y))$,

$$c^{-1}|x-y|^{-d+2} \leqslant G_D(x,y) \leqslant c|x-y|^{-d+2}.$$

With the preparations above, the following two-sided estimates for G_D is a direct generalization of the estimates of the Green function for symmetric processes (see [5] for a symmetric jump process case).

Theorem 2.12. *There exists $c := c(D) > 0$ such that for every $x, y \in D$,*

$$c^{-1}\frac{g_1(x)g_2(y)}{g_1(A)g_2(A)}|x-y|^{-d+2} \leqslant G_D(x,y) \leqslant c\frac{g_1(x)g_2(y)}{g_1(A)g_2(A)}|x-y|^{-d+2} \tag{2.17}$$

for every $A \in \mathcal{B}(x,y)$.

Proof. Since the proof is an adaptation of the proofs of Proposition 6 in [3] and Theorem 2.4 in [10], we only give a sketch of the proof for the case $\rho_D(x) \leqslant \rho_D(y) \leqslant \frac{1}{2M}|x-y|$.

In this case, we have $r(x,y) = |x-y|$. Let $r := \frac{1}{2}(|x-y| \wedge \varepsilon_1)$. Choose $Q_x, Q_y \in \partial D$ with $|Q_x - x| = \rho_D(x)$ and $|Q_y - y| = \rho_D(y)$. Pick points $x_1 = A_{r/M}(Q_x)$ and $y_1 = A_{r/M}(Q_y)$ so that $x, x_1 \in B(Q_x, r/M)$ and $y, y_1 \in B(Q_y, r/M)$. Then one can easily check that $|z_0 - Q_x| \geqslant r$ and $|y - Q_x| \geqslant r$. So by the first inequality in (2.15), we have

$$c_1^{-1}\frac{G_D(x_1,y)}{g_1(x_1)} \leqslant \frac{G_D(x,y)}{g_1(x)} \leqslant c_1\frac{G_D(x_1,y)}{g_1(x_1)}$$

for some $c_1 > 1$. On the other hand, since $|z_0 - Q_y| \geqslant r$ and $|x_1 - Q_y| \geqslant r$, applying the second inequality in (2.15),

$$c_1^{-1}\frac{G_D(x_1,y_1)}{g_2(y_1)} \leqslant \frac{G_D(x_1,y)}{g_2(y)} \leqslant c_1\frac{G_D(x_1,y_1)}{g_2(y_1)}.$$

Putting the four inequalities above together we get

$$c_1^{-2}\frac{G_D(x_1,y_1)}{g_1(x_1)g_2(y_1)} \leqslant \frac{G_D(x,y)}{g_1(x)g_2(y)} \leqslant c_1^2\frac{G_D(x_1,y_1)}{g_1(x_1)g_2(y_1)}.$$

Moreover, $\frac{1}{3}|x-y| < |x_1-y_1| < 2|x-y|$ and $|x_1 - y_1| \leqslant (8M^3/\varepsilon_1)(\rho_D(x_1) \wedge \rho_D(y_1))$. Thus by Lemma 2.11, we have

$$\frac{1}{2^{d-2}c_2c_1^2}\frac{|x-y|^{-d+2}}{g_1(x_1)g_2(y_1)} \leqslant \frac{G_D(x,y)}{g_1(x)g_2(y)} \leqslant 3^{d-2}c_2c_1^2\frac{|x-y|^{-d+2}}{g_1(x_1)g_2(y_1)}$$

for some $c_2 > 1$.

If $r = \varepsilon_1/2$, then $r(x,y) = |x-y| \geqslant \varepsilon_1$. Thus $g_1(A) = g_2(A) = g_1(z_0) = g_2(z_0) = C_1$ and $\rho_D(x_1), \rho_D(y_1) \geqslant r/M = \varepsilon_1/(2M)$. So by Lemma 2.10,

$$C_1^{-2}c_3^{-2} \leqslant \frac{g_1(A)g_2(A)}{g_1(x_1)g_2(y_1)} \leqslant C_1^2c_3^2$$

for some $c_3 > 1$.

If $r < \varepsilon_1/2$, then $r(x,y) = |x-y| < \varepsilon_1$ and $r = \frac{1}{2}r(x,y)$. Hence $\rho_D(x_1), \rho_D(y_1) \geqslant r/M = r(x,y)/(2M)$. Moreover, $|x_1 - A|, |y_1 - A| \geqslant 6r(x,y)$. So by applying the first inequality in (2.16) to g_1, and the second inequality in (2.16) to g_2 (with $k = 12M$),

$$c_4^{-1} \leqslant \frac{g_1(A)}{g_1(x_1)} \leqslant c_4 \quad \text{and} \quad c_4^{-1} \leqslant \frac{g_2(A)}{g_2(y_1)} \leqslant c_4$$

for some constant $c_4 = c_4(D) > 0$. □

Lemma 2.13 *(Carleson's estimate). For any given* $0 < N < 1$, *there exists constant* $c > 1$ *such that for every* $Q \in \partial D$, $r < r_0$, $x \in D \setminus B(Q,r)$ *and* $z_1, z_2 \in D \cap B(Q, r/M)$ *with* $B(z_2, Nr) \subset D \cap B(Q, r/M)$,

$$G_D(x,z_1) \leqslant cG_D(x,z_2) \quad \text{and} \quad G_D(z_1,x) \leqslant cG_D(z_2,x). \tag{2.18}$$

Proof. Recall that CS_Q is the coordinate system with origin at Q in the definition of Lipschitz domains. Let $\bar{y} := (\tilde{0}, r)$. Since $z_1, z_2 \in D \cap B(Q, r/M)$, by (2.2),

$$G_D(\bar{y}, z_1) \leqslant c_1 r^{-d+2} \quad \text{and} \quad G_D(z_1, \bar{y}) \leqslant c_1 r^{-d+2}$$

for some constant $c_1 > 0$. On the other hand, since $\rho_D(\bar{y}) \geqslant c_2 r$ for some constant $c_2 > 0$ and $\rho_D(z_2) \geqslant Nr$, by Lemma 2.11,

$$G_D(\bar{y}, z_2) \geqslant c_3|\bar{y} - z_2|^{-d+2} \geqslant c_4 r^{-d+2} \quad \text{and} \quad G_D(z_2, \bar{y}) \geqslant c_3|\bar{y} - z_2|^{-d+2} \geqslant c_4 r^{-d+2}$$

for some constants $c_3, c_4 > 0$. Thus from (2.15) with $y = \bar{y}$, we get

$$G_D(x, z_1) \leqslant c_5\left(\frac{c_1}{c_4}\right)G_D(x, z_2) \quad \text{and} \quad G_D(z_1, x) \leqslant c_5\left(\frac{c_1}{c_4}\right)G_D(z_2, x)$$

for some constant $c_5 > 0$. □

Recall that, for $r \in (0, R_0)$, $A_r(Q)$ is a point in $D \cap B(Q, r)$ such that $B(A_r(Q), \kappa_0 r) \subset D \cap B(Q, r)$. For every $x, y \in D$, we denote by Q_x, Q_y points on ∂D such that $\rho_D(x) = |x - Q_x|$ and $\rho_D(y) = |y - Q_y|$, respectively. It is easy to check that if $r(x, y) < \varepsilon_1$,

$$A_{r(x,y)}(Q_x), A_{r(x,y)}(Q_y) \in \mathcal{B}(x, y). \tag{2.19}$$

In fact, by the definition of $A_{r(x,y)}(Q_x)$, $\rho_D(A_{r(x,y)}(Q_x)) \geqslant \kappa_0 r(x, y) > r(x, y)/M$. Moreover,

$$\left|x - A_{r(x,y)}(Q_x)\right| \leqslant |x - Q_x| + \left|Q_x - A_{r(x,y)}(Q_x)\right| \leqslant \rho_D(x) + r(x, y) \leqslant 2r(x, y)$$

and $|y - A_{r(x,y)}(Q_x)| \leqslant |x - y| + |x - A_{r(x,y)}(Q_x)| \leqslant 3r(x, y)$.

Lemma 2.14. *There exists $c > 0$ such that the following hold:*

(1) *If $Q \in \partial D$, $0 < s \leqslant r < \varepsilon_1$ and $A = A_r(Q)$, then*

$$g_i(x) \leqslant c g_i(A) \quad \text{for every } x \in D \cap B(Q, Ms) \cap \left\{y \in D\colon \rho_D(y) > \frac{s}{M}\right\}, \ i = 1, 2.$$

(2) *If $x, y, z \in D$ satisfy $|x - z| \leqslant |y - z|$, then*

$$g_i(A) \leqslant c g_i(B) \quad \text{for every } (A, B) \in \mathcal{B}(x, y) \times \mathcal{B}(y, z), \ i = 1, 2.$$

Proof. This is an easy consequence of the Carleson's estimates (Lemma 2.13), (2.19) and Lemmas 2.9–2.11 (see p. 467 in [10]). Since the proof is similar to the proof on p. 467 in [10], we omit the details. □

The next result is called a generalized triangle property.

Theorem 2.15. *There exists a constant $c > 0$ such that for every $x, y, z \in D$,*

$$\frac{G_D(x, y)G_D(y, z)}{G_D(x, z)} \leqslant c\left(\frac{g_1(y)}{g_1(x)}G_D(x, y) \vee \frac{g_2(y)}{g_2(z)}G_D(y, z)\right). \tag{2.20}$$

Proof. Let $A_{x,y} \in \mathcal{B}(x, y)$, $A_{y,z} \in \mathcal{B}(y, z)$ and $A_{z,x} \in \mathcal{B}(z, x)$. If $|x - y| \leqslant |y - z|$ then $|x - z| \leqslant |x - y| + |y - z| \leqslant 2|y - z|$. So by (2.17) and Lemma 2.14(2), we have

$$\frac{G_D(y, z)}{G_D(x, z)} \leqslant c_1^2 \frac{g_1(A_{x,z})g_2(A_{x,z})}{g_1(A_{y,z})g_2(A_{y,z})} \frac{|x - z|^{d-2}}{|y - z|^{d-2}} \frac{g_1(y)}{g_1(x)} \leqslant c_1^2 c_2 2^{d-2} \frac{g_1(y)}{g_1(x)}$$

for some $c_1, c_2 > 0$. Similarly if $|x - y| \geqslant |y - z|$, then

$$\frac{G_D(x, y)}{G_D(x, z)} \leqslant c_1^2 \frac{g_1(A_{x,z})g_2(A_{x,z})}{g_1(A_{x,y})g_2(A_{x,y})} \frac{|x - z|^{d-2}}{|x - y|^{d-2}} \frac{g_1(y)}{g_1(x)} \leqslant c_1^2 c_2 2^{d-2} \frac{g_2(y)}{g_2(z)}.$$

Thus

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leqslant c_1^2 c_2 2^{d-2}\left(\frac{g_1(y)}{g_1(x)}G_D(x,y) \vee \frac{g_2(y)}{g_2(z)}G_D(y,z)\right). \qquad \square$$

Lemma 2.16. *There exists $c>0$ such that for every $x,y \in D$ and $A \in \mathcal{B}(x,y)$,*

$$g_i(x) \vee g_i(y) \leqslant c g_i(A), \quad i=1,2.$$

Proof. If $r(x,y) \geqslant \varepsilon_1$, the lemma is clear. If $r(x,y) < \varepsilon_1$, from Lemma 2.14(1), it is easy to see that

$$g_i(x) \leqslant c g_i\big(A_{r(x,y)}(Q_x)\big)$$

for some $c>0$, where Q_x is a point on ∂D such that $\rho_D(x) = |x - Q_x|$. Thus the lemma follows from Lemmas 2.9 and (2.19). $\square$

Now we are ready to prove the 3G theorem.

Theorem 2.17. *There exists a constant $c>0$ such that for every $x,y,z \in D$,*

$$\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} \leqslant c\frac{|x-z|^{d-2}}{|x-y|^{d-2}|y-z|^{d-2}}. \tag{2.21}$$

Proof. Let $A_{x,y} \in \mathcal{B}(x,y)$, $A_{y,z} \in \mathcal{B}(y,z)$ and $A_{z,x} \in \mathcal{B}(z,x)$. By (2.17), the left-hand side of (2.21) is less than or equal to

$$\left(\frac{g_1(y)g_1(A_{x,z})}{g_1(A_{x,y})g_1(A_{y,z})}\right)\left(\frac{g_2(y)g_2(A_{x,z})}{g_2(A_{x,y})g_2(A_{y,z})}\right)\frac{|x-z|^{d-2}}{|x-y|^{d-2}|y-z|^{d-2}}.$$

If $|x-y| \leqslant |y-z|$, by Lemmas 2.14 and 2.16, we have

$$\frac{g_1(y)}{g_1(A_{x,y})} \leqslant c_1, \quad \frac{g_2(y)}{g_2(A_{x,y})} \leqslant c_1, \quad \frac{g_1(A_{x,z})}{g_1(A_{y,z})} \leqslant c_2 \quad \text{and} \quad \frac{g_2(A_{x,z})}{g_2(A_{y,z})} \leqslant c_2$$

for some constants $c_1, c_2 > 0$. Similarly, if $|x-y| \geqslant |y-z|$, then

$$\frac{g_1(y)}{g_1(A_{y,z})} \leqslant c_1, \quad \frac{g_2(y)}{g_2(A_{y,z})} \leqslant c_1, \quad \frac{g_1(A_{x,z})}{g_1(A_{x,y})} \leqslant c_2 \quad \text{and} \quad \frac{g_2(A_{x,z})}{g_2(A_{x,y})} \leqslant c_2. \qquad \square$$

Combining the main results of this section, we get the following inequality.

Theorem 2.18. *There exist constants $c_1, c_2 > 0$ such that for every $x,y,z \in D$,*

$$\begin{aligned}\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} &\leqslant c_1\left(\frac{g_1(y)}{g_1(x)}G_D(x,y) \vee \frac{g_2(y)}{g_2(z)}G_D(y,z)\right)\\ &\leqslant c_2\big(|x-y|^{-d+2} \vee |y-z|^{-d+2}\big).\end{aligned} \tag{2.22}$$

Proof. We only need to prove the second inequality. Applying Theorem 2.12, we get that there exists $c_1>0$ such that

$$\frac{g_1(y)}{g_1(x)}G_D(x,y) \leqslant c_1\frac{g_1(y)g_2(y)}{g_1(A)g_2(A)}|x-y|^{-d+2}$$

and

$$\frac{g_2(y)}{g_2(z)} G_D(y,z) \leqslant c_1 \frac{g_1(y)g_2(y)}{g_1(B)g_2(B)} |x-y|^{-d+2}$$

for every $(A,B) \in \mathcal{B}(x,y) \times \mathcal{B}(y,z)$. Applying Lemma 2.16, we arrive at the desired assertion. □

3. Schrödinger semigroups for X^D

In this section, we will assume that D is a bounded Lipschitz domain. We first recall some notions from [13]. A measure ν on D is said to be a smooth measure of X^D if there is a positive continuous additive functional (PCAF in abbreviation) A of X^D such that for any $x \in D$, $t > 0$ and bounded nonnegative function f on D,

$$\mathbf{E}_x \int_0^t f\big(X_s^D\big)\,dA_s = \int_0^t \int_D q^D(s,x,y) f(y) \nu(dy)\,ds. \tag{3.1}$$

The additive functional A is called the PCAF of X^D with Revuz measure ν.

For a signed measure ν, we use ν^+ and ν^- to denote its positive and negative parts of ν, respectively. A signed measure ν is called smooth if both ν^+ and ν^- are smooth. For a signed smooth measure ν, if A^+ and A^- are the PCAFs of X^D with Revuz measures ν^+ and ν^-, respectively, the additive functional $A := A^+ - A^-$ of is called the continuous additive functional (CAF in abbreviation) of X^D with (signed) Revuz measure ν. When $\nu(dx) = c(x)\,dx$, A_t is given by $A_t = \int_0^t c(X_s^D)\,ds$.

We recall now the definition of the Kato class.

Definition 3.1. A signed smooth measure ν is said to be in the class $\mathbf{S}_\infty(X^D)$ if for any $\varepsilon > 0$ there is a Borel subset $K = K(\varepsilon)$ of finite $|\nu|$-measure and a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{(x,z)\in(D\times D)\setminus d} \int_{D\setminus K} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |\nu|(dy) \leqslant \varepsilon \tag{3.2}$$

and for all measurable set $B \subset K$ with $|\nu|(B) < \delta$,

$$\sup_{(x,z)\in(D\times D)\setminus d} \int_{B} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |\nu|(dy) \leqslant \varepsilon, \tag{3.3}$$

where d is the diagonal of the set $D \times D$. A function q is said to be in the class $\mathbf{S}_\infty(X^D)$ if $q(x)\,dx$ is in $\mathbf{S}_\infty(X^D)$.

It follows from Proposition 7.1 of [13] and Theorem 2.17 above that $\mathbf{K}_{d,2}$ is contained in $\mathbf{S}_\infty(X^D)$. In fact, by Theorem 2.18 we have the following result. Recall that $g_1(x) = G_D(x,z_0) \wedge C_1$ and $g_2(y) = G_D(z_0,y) \wedge C_1$.

Proposition 3.2. *If a signed smooth measure ν satisfies*

$$\sup_{x\in D} \lim_{r\downarrow 0} \int_{D\cap\{|x-y|\leqslant r\}} \frac{g_1(y)}{g_1(x)} G_D(x,y) |\nu|(dy) = 0$$

and

$$\sup_{x\in D}\lim_{r\downarrow 0}\int_{D\cap\{|x-y|\leqslant r\}}\frac{g_2(y)}{g_2(x)}G_D(y,x)|\nu|(dy)=0,$$

then $\nu\in\mathbf{S}_\infty(X^D)$.

Proof. This is a direct consequence of Theorem 2.18. □

In the remainder of this section, we will fix a signed measure $\nu\in S_\infty(X^D)$ and we will use A to denote the CAF of X^D with Revuz measure ν. For simplicity, we will use $e_A(t)$ to denote $\exp(A_t)$. The CAF A gives rise to a Schrödinger semigroup:

$$Q_t^D f(x):=\mathbf{E}_x\big[e_A(t)f\big(X_t^D\big)\big].$$

The function $x\mapsto\mathbf{E}_x[e_A(\tau_D)]$ is called the gauge function of ν. We say ν is *gaugeable* if $\mathbf{E}_x[e_A(\tau_D)]$ is finite for some $x\in D$. In the remainder of this section we will assume that ν is gaugeable. It is shown in [13], by using the duality and the gauge theorems in [4,7], that the gauge function $x\mapsto\mathbf{E}_x[e_A(\tau_D)]$ is bounded on D (see Section 7 in [13]).

For $y\in D$, let $X^{D,y}$ denote the h-conditioned process obtained from X^D with $h(\cdot)=G_D(\cdot,y)$ and let $\mathbf{E}_x^y$ denote the expectation for $X^{D,y}$ starting from $x\in D$. We will use τ_D^y to denote the lifetime of $X^{D,y}$. We know from [13] that $\mathbf{E}_x^y[e_A(\tau_D^y)]$ is continuous in $D\times D$ (also see Theorem 3.4 in [6]) and

$$\sup_{(x,y)\in(D\times D)\setminus d}\mathbf{E}_x^y\big[|A|_{\tau_D^y}\big]<\infty \tag{3.4}$$

(also see [4] and [7]) and therefore by Jensen's inequality

$$\inf_{(x,y)\in(D\times D)\setminus d}\mathbf{E}_x^y\big[e_A\big(\tau_D^y\big)\big]>0. \tag{3.5}$$

We also know from Section 7 in [13] that

$$V_D(x,y):=\mathbf{E}_x^y\big[e_A\big(\tau_D^y\big)\big]G_D(x,y) \tag{3.6}$$

is the Green function of $\{Q_t^D\}$, that is, for any nonnegative function f on D,

$$\int_D V_D(x,y)f(y)\,dy=\int_0^\infty Q_t^D f(x)\,dt$$

(see also Lemma 3.5 of [4]). (3.4)–(3.6) and the continuity of $\mathbf{E}_x^y[e_A(\tau_D^y)]$ imply that $V_D(x,y)$ is comparable to $G_D(x,y)$ and $V_D(x,y)$ is continuous on $(D\times D)\setminus d$. Thus there exists a constant $c>0$ such that for every $x,y,z\in D$,

$$\frac{V_D(x,y)V_D(y,z)}{V_D(x,z)}\leqslant c\frac{|x-z|^{d-2}}{|x-y|^{d-2}|y-z|^{d-2}}. \tag{3.7}$$

4. Two-sided heat kernel estimates for $\{Q_t^D\}$

In this section, we will establish two-sided estimates for the heat kernel of Q_t^D in bounded $C^{1,1}$ domains.

Recall that a bounded domain D in $\mathbf{R}^d$ is said to be a $C^{1,1}$ domain if there are a localization radius $r_0 > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there are a $C^{1,1}$-function $\phi = \phi_Q : \mathbf{R}^{d-1} \to \mathbf{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \leqslant \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \leqslant \Lambda|x - z|$, and an orthonormal coordinate system $y = (y_1, \ldots, y_{d-1}, y_d) := (\tilde{y}, y_d)$ such that $B(Q, r_0) \cap D = B(Q, r_0) \cap \{y\colon\ y_d > \phi(\tilde{y})\}$.

We will always assume in this section that D is a bounded $C^{1,1}$ domain. Since we will follow the method in [11] (see also [17]), the proof of this section will be a little sketchy.

First, we recall some results from [11]. For every bounded $C^{1,1}$ domain D and any $T > 0$, there exist positive constants c_i, $i = 1, \ldots, 4$, such that

$$c_1 \psi_D(t,x,y) t^{-\frac{d}{2}} e^{-\frac{c_2|x-y|^2}{t}} \leqslant q^D(t,x,y) \leqslant c_3 \psi_D(t,x,y) t^{-\frac{d}{2}} e^{-\frac{c_4|x-y|^2}{t}} \tag{4.1}$$

for all $(t,x,y) \in (0,T] \times D \times D$, where

$$\psi_D(t,x,y) := \left(1 \wedge \frac{\rho_D(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho_D(y)}{\sqrt{t}}\right)$$

(see (4.27) in [11]).

For any $z \in \mathbf{R}^d$ and $0 < r \leqslant 1$, let

$$D_r^z := z + rD, \quad \psi_{D_r^z}(t,x,y) := \left(1 \wedge \frac{\rho_{D_r^z}(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho_{D_r^z}(y)}{\sqrt{t}}\right),$$
$$(t,x,y) \in (0,\infty) \times D_r^z \times D_r^z,$$

where $\rho_{D_r^z}(x)$ is the distance between x and ∂D_r^z. Then, for any $T > 0$, there exist positive constants t_0 and c_j, $5 \leqslant j \leqslant 8$, independent of z and r such that

$$c_5 t^{-\frac{d}{2}} \psi_{D_r^z}(t,x,y) e^{-\frac{c_6|x-y|^2}{2t}} \leqslant q^{D_r^z}(t,x,y) \leqslant c_7 t^{-\frac{d}{2}} \psi_{D_r^z}(t,x,y) e^{-\frac{c_8|x-y|^2}{2t}} \tag{4.2}$$

for all $(t,x,y) \in (0, t_0 \wedge (r^2 T)] \times D_r^z \times D_r^z$ (see (5.1) in [11]). We will sometimes suppress the indices from D_r^z when there is no possibility of confusion.

For the remainder of this paper, we will assume that ν is in the Kato class $\mathbf{K}_{d,2}$. Using the estimates above and the joint continuity of the density $q^D(t,x,y)$ (Theorem 2.4 in [12]), it is routine (see, for example, [8, Theorem 3.17], [2, Theorem 3.1] and [4, p. 4669]) to show that Q_t^D has a jointly continuous density $r^D(t,\cdot,\cdot)$ (also see Theorem 2.4 in [12]). So we have

$$\mathbf{E}_x\left[e_A(t) f\left(X_t^D\right)\right] = \int_D f(y) r^D(t,x,y)\,dy, \tag{4.3}$$

where A is the CAF of X^D with Revuz measure ν in D.

Theorem 4.1. *The density $r^D(t,x,y)$ satisfies the following equation*

$$r^D(t,x,y) = q^D(t,x,y) + \int_0^t \int_D r^D(s,x,z) q^D(t-s,z,y) \nu(dz)\,ds \tag{4.4}$$

for all $(t,x,y) \in (0,\infty) \times D \times D$.

Proof. Recall that A is the CAF of X^D with Revuz measure ν in D. Let θ be the usual shift operator for Markov processes.

Since for any $t>0$,

$$e_A(t)=e^{A_t}=1+\int_0^t e^{A_t-A_s}\,dA_s=1+\int_0^t e^{A_{t-s}\circ\theta_s}\,dA_s,$$

we have

$$\mathbf{E}_x\big[e_A(t)f\big(X_t^D\big)\big]=\mathbf{E}_x\big[f\big(X_t^D\big)\big]+\mathbf{E}_x\left[f\big(X_t^D\big)\int_0^t e^{A_{t-s}\circ\theta_s}\,dA_s\right] \tag{4.5}$$

for all $(t,x)\in(0,\infty)\times D$ and all bounded Borel-measurable functions f in D.

By the Markov property and Fubini's theorem, we have

$$\begin{aligned}\mathbf{E}_x\left[f\big(X_t^D\big)\int_0^t e^{A_{t-s}\circ\theta_s}\,dA_s\right]&=\int_0^t \mathbf{E}_x\big[f\big(X_t^D\big)e^{A_{t-s}\circ\theta_s}\,dA_s\big]\\&=\int_0^t \mathbf{E}_x\big[\mathbf{E}_{X_s^D}\big[f\big(X_{t-s}^D\big)e_A(t-s)\big]\,dA_s\big].\end{aligned}$$

Thus by (3.1) and (4.3),

$$\mathbf{E}_x\left[f\big(X_t^D\big)\int_0^t e^{A_{t-s}\circ\theta_s}\,dA_s\right]=\int_D f(y)\int_0^t\int_D r^D(s,x,z)q^D(t-s,z,y)\nu(dz)\,ds\,dy. \tag{4.6}$$

Since $r^D(s,\cdot,\cdot)$ and $q^D(t-s,\cdot,\cdot)$ are jointly continuous, combining (4.5)–(4.6), we have proved the theorem. □

The proof of the next lemma is almost identical to that of Lemma 3.1 in [18]. We omit the proof.

Lemma 4.2. *For any $a>0$, there exists a positive constant c depending only on a and d such that for any $(t,x,y)\in(0,\infty)\times\mathbf{R}^d\times\mathbf{R}^d$,*

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^d} s^{-\frac{d}{2}}e^{-\frac{a|x-z|^2}{2s}}(t-s)^{-\frac{d}{2}}e^{-\frac{a|z-y|^2}{t-s}}|\nu|(dz)\,ds\\&\quad\leqslant ct^{-\frac{d}{2}}e^{-\frac{a|x-y|^2}{2t}}\sup_{u\in\mathbf{R}^d}\int_0^t\int_{\mathbf{R}^d} s^{-\frac{d}{2}}e^{-\frac{a|u-z|^2}{4s}}|\nu|(dz)\,ds\end{aligned}$$

and

$$\begin{aligned}&\int_0^t\int_{\mathbf{R}^d} s^{-\frac{d+1}{2}}e^{-\frac{a|x-z|^2}{2s}}(t-s)^{-\frac{d}{2}}e^{-\frac{a|z-y|^2}{t-s}}|\nu|(dz)\,ds\\&\quad\leqslant ct^{-\frac{d+1}{2}}e^{-\frac{a|x-y|^2}{2t}}\sup_{u\in\mathbf{R}^d}\int_0^t\int_{\mathbf{R}^d} s^{-\frac{d}{2}}e^{-\frac{a|u-z|^2}{4s}}|\nu|(dz)\,ds.\end{aligned}$$

Lemma 4.3. *For any $a > 0$, there exists a positive constant c depending only on a and d such that for any $(t, x, y) \in (0, \infty) \times D \times D$,*

$$\int_0^t \int_D \left(1 \wedge \frac{\rho(x)}{\sqrt{s}}\right)\left(1 \wedge \frac{\rho(z)}{\sqrt{s}}\right) s^{-\frac{d}{2}} e^{-\frac{a|x-z|^2}{2s}} \left(1 \wedge \frac{\rho(y)}{\sqrt{t-s}}\right)(t-s)^{-\frac{d}{2}} e^{-\frac{a|z-y|^2}{t-s}} |\nu|(dz)\, ds$$

$$\leqslant c\left(1 \wedge \frac{\rho(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\rho(y)}{\sqrt{t}}\right) t^{-\frac{d}{2}} e^{-\frac{a|x-y|^2}{2t}} \sup_{u \in \mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d}{2}} e^{-\frac{a|u-z|^2}{4s}} |\nu|(dz)\, ds. \tag{4.7}$$

Proof. With Lemma 4.2 in hand, we can follow the proof of Theorem 2.1 in [15, pp. 389–391] to get the present lemma. So we skip the details. □

Recall that

$$M^1_{\mu^i}(r) = \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leqslant r} \frac{|\mu^i|(dy)}{|x-y|^{d-1}} \quad \text{and}$$
$$M^2_\nu(r) = \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leqslant r} \frac{|\nu|(dy)}{|x-y|^{d-2}}, \quad r > 0,\ i = 1, \ldots, d.$$

Theorem 4.4.

(1) *For each $T > 0$, there exist positive constants c_j, $1 \leqslant j \leqslant 4$, depending on μ and ν only via the rate at which $\max_{1 \leqslant i \leqslant d} M^1_{\mu^i}(r)$ and $M^2_\nu(r)$ go to zero such that*

$$c_1 t^{-\frac{d}{2}} \psi_D(t, x, y) e^{-\frac{c_2|x-y|^2}{2t}} \leqslant r^D(t, x, y) \leqslant c_3 t^{-\frac{d}{2}} \psi_D(t, x, y) e^{-\frac{c_4|x-y|^2}{2t}}. \tag{4.8}$$

(2) *There exist $T_1 = T_1(D) > 0$ such that for any $T > 0$, there exist positive constants t_1 and c_j, $5 \leqslant j \leqslant 8$, independent of z and r such that*

$$c_5 t^{-\frac{d}{2}} \psi_{D^z_r}(t, x, y) e^{-\frac{c_6|x-y|^2}{2t}} \leqslant r^{D^z_r}(t, x, y) \leqslant c_7 t^{-\frac{d}{2}} \psi_{D^z_r}(t, x, y) e^{-\frac{c_8|x-y|^2}{2t}} \tag{4.9}$$

for all $r \in (0, 1]$ and $(t, x, y) \in (0, t_1 \wedge (r^2(T \wedge T_1))] \times D^z_r \times D^z_r$.

Proof. We only give the proof of (4.9). The proof of (4.8) is similar. Fix $T > 0$ and $z \in \mathbf{R}^d$. Let $D_r := D^z_r$, $\rho_r(x) := \rho_{D^z_r}(x)$ and $\psi_r(t, x, y) := \psi_{D^z_r}(t, x, y)$. We define $\tilde{I}_k(t, x, y)$ recursively for $k \geqslant 0$ and $(t, x, y) \in (0, \infty) \times D \times D$:

$$I^r_0(t, x, y) := q^{D_r}(t, x, y),$$
$$I^r_{k+1}(t, x, y) := \int_0^t \int_{D_r} I^r_k(s, x, z) q(z) q^{D_r}(t-s, z, y)\, dz\, ds.$$

Then iterating the above gives

$$r^{D_r}(t, x, y) = \sum_{k=0}^{\infty} I^r_k(t, x, y), \quad (t, x, y) \in (0, \infty) \times D_r \times D_r. \tag{4.10}$$

Let

$$N_\nu^2(t) := \sup_{u\in\mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d}{2}} e^{-\frac{|u-z|^2}{2s}} |\nu|(dz)\,ds, \quad t > 0.$$

It is well known (see, for example, Proposition 2.1 in [11]) that for any $r > 0$, there exist $c_1 = c_1(d,r)$ and $c_2 = c_2(d)$ such that

$$N_\nu^2(t) \leqslant (c_1 t + c_2) M_\nu^2(r), \quad \text{for every } t \in (0,1). \tag{4.11}$$

We claim that there exist positive constants c_3, c_4 and A depending only on the constants in (4.2) and (4.7) such that for $k = 0, 1, \ldots$ and $(t,x,y) \in (0, t_0 \wedge (r^2T)] \times D_r \times D_r$,

$$\left|I_k^r(t,x,y)\right| \leqslant c_3 \psi_r(t,x,y) t^{-\frac{d}{2}} e^{-\frac{A|x-y|^2}{2t}} \left(c_4 N_\nu^2\left(\frac{2t}{A}\right)\right)^k, \quad 0 < r \leqslant 1. \tag{4.12}$$

We will prove the above claim by induction. By (4.2), there exist constants t_0, c_3 and A such that

$$\left|I_0^r(t,x,y)\right| = \left|q^{D_r}(t,x,y)\right| \leqslant c_3 \psi_r(t,x,y) t^{-\frac{d}{2}} e^{-\frac{A|x-y|^2}{2t}} \tag{4.13}$$

for $(t,x,y) \in (0, t_0 \wedge (r^2T)] \times D_r \times D_r$. On the other hand, by Lemma 4.3, there exists a positive constant c_5 depending only on A and d such that

$$\begin{aligned}
&\int_0^t \int_{D_r} \psi_r(s,x,z) s^{-\frac{d}{2}} e^{-\frac{A|x-z|^2}{2s}} \left(1 \wedge \frac{\rho_r(y)}{\sqrt{t-s}}\right) (t-s)^{-\frac{d}{2}} e^{-\frac{A|z-y|^2}{t-s}} |\nu|(dz)\,ds \\
&\quad \leqslant c_5 \psi_r(t,x,y) t^{-\frac{d}{2}} e^{-\frac{A|x-y|^2}{2t}} \sup_{u\in\mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d}{2}} e^{-\frac{A|u-z|^2}{4s}} |\nu|(dz)\,ds.
\end{aligned} \tag{4.14}$$

So there exists $c_6 = c_6(d) > 0$ such that

$$\begin{aligned}
\left|I_1^r(t,x,y)\right| &\leqslant c_3^2 c_5 \psi_r(t,x,y) t^{-\frac{d}{2}} e^{-\frac{A|x-y|^2}{2t}} \sup_{u\in\mathbf{R}^d} \int_0^t \int_{\mathbf{R}^d} s^{-\frac{d}{2}} e^{-\frac{A|u-z|^2}{4s}} |\nu|(dz)\,ds \\
&\leqslant c_3^2 c_5 c_6 A^{\frac{d}{2}} \psi_r(t,x,y) t^{-\frac{d}{2}} e^{-\frac{A|x-y|^2}{2t}} N_\nu^2\left(\frac{2t}{A}\right)
\end{aligned}$$

for $(t,x,y) \in (0, t_0 \wedge (r^2T)] \times D_r \times D_r$. Therefore (4.12) is true for $k = 0, 1$ with $c_4 := c_3^2 c_5 c_6 A^{\frac{d}{2}}$. Now we assume (4.12) is true up to k. Then by (4.13)–(4.14), we have

$$\begin{aligned}
\left|I_{k+1}^r(t,x,y)\right| &\leqslant \int_0^t \int_{D_r} \left|I_k^r(s,x,z)\right| q^{D_r}(t-s,z,y) |\nu|(dz)\,ds \\
&\leqslant \int_0^t \int_{D_r} c_3 \psi_r(s,x,z) s^{-\frac{d}{2}} e^{-\frac{A|x-z|^2}{2s}} \left(c_4 N_\nu^2\left(\frac{2s}{A}\right)\right)^k \\
&\qquad \times c_3 \left(1 \wedge \frac{\rho_r(y)}{\sqrt{t-s}}\right) (t-s)^{-\frac{d}{2}} e^{-\frac{A|z-y|^2}{t-s}} |\nu|(dz)\,ds
\end{aligned}$$

$$\leqslant c_3^2\left(c_4N_\nu^2\left(\frac{2t}{A}\right)\right)^k\int_0^t\int_{D_r}\psi_r(s,x,z)s^{-\frac{d}{2}}e^{-\frac{A|x-z|^2}{2s}}\left(1\wedge\frac{\rho_r(y)}{\sqrt{t-s}}\right)$$
$$\times(t-s)^{-\frac{d}{2}}e^{-\frac{M|z-y|^2}{t-s}}|\nu|(dz)\,ds$$
$$\leqslant c_3^2\left(c_4N_\nu^2\left(\frac{2t}{A}\right)\right)^k c_5c_6A^{\frac{d}{2}}\psi_r(t,x,y)t^{-\frac{d}{2}}e^{-\frac{A|x-y|^2}{2t}}N_\nu^2\left(\frac{2t}{A}\right)$$
$$\leqslant c_3\psi_r(t,x,y)t^{-\frac{d}{2}}e^{-\frac{A|x-y|^2}{2t}}\left(c_4N_\nu^2\left(\frac{2t}{A}\right)\right)^{k+1}.$$

So the claim is proved.

Choose $t_1<(1\wedge t_0)$ small so that

$$c_4N_\nu^2\left(\frac{2t_1}{A}\right)<\frac{1}{2}. \tag{4.15}$$

By (4.11), t_1 depends on ν only via the rate at which $M_\nu^2(r)$ goes to zero. (4.10) and (4.12) imply that for $(t,x,y)\in(0,t_1\wedge(r^2T)]\times D_r\times D_r$,

$$r^{D_r}(t,x,y)\leqslant\sum_{k=0}^\infty\left|I_k^r(t,x,y)\right|\leqslant 2c_3\psi_r(t,x,y)t^{-\frac{d}{2}}e^{-\frac{A|x-y|^2}{2t}}. \tag{4.16}$$

Now we are going to prove the lower estimate of $r^{D_r}(t,x,y)$. Combining (4.10), (4.12) and (4.15) we have for every $(t,x,y)\in(0,t_1\wedge(r^2T)]\times D_r\times D_r$,

$$\left|r^{D_r}(t,x,y)-q^{D_r}(t,x,y)\right|\leqslant\sum_{k=1}^\infty\left|I_k^r(t,x,y)\right|\leqslant c_3c_4N_\nu^2\left(\frac{2t_1}{A}\right)\psi_r(t,x,y)t^{-\frac{d}{2}}e^{-\frac{A|x-y|^2}{2t}}.$$

Since there exist c_7 and $c_8\leqslant 1$ depending on T such that

$$q^{D_r}(t,x,y)\geqslant 2c_8\psi_r(t,x,y)t^{-\frac{d}{2}}e^{-\frac{c_7|x-y|^2}{2t}},$$

we have for $|x-y|\leqslant\sqrt{t}$ and $(t,x,y)\in(0,t_1\wedge(r^2T)]\times D\times D$,

$$r^{D_r}(t,x,y)\geqslant\left(2c_8e^{-2c_7}-c_3c_4N_\nu^2\left(\frac{2t_1}{A}\right)\right)\psi(t,x,y)t^{-\frac{d}{2}}. \tag{4.17}$$

Now we choose $t_2\leqslant t_1$ small so that

$$c_3c_4N_\nu^2\left(\frac{2t_2}{A}\right)<c_8e^{-2c_7}. \tag{4.18}$$

Note that t_2 depends on ν only via the rate at which $M_\nu^2(r)$ goes to zero. So for $(t,x,y)\in(0,t_2\wedge(r^2T)]\times D\times D$ and $|x-y|\leqslant\sqrt{t}$, we have

$$r^{D_r}(t,x,y)\geqslant c_8e^{-2c_7}\psi_r(t,x,y)t^{-\frac{d}{2}}. \tag{4.19}$$

It is easy to check (see pp. 420–421 of [19]) that there exists a positive constant T_0 depending only on the characteristic of the bounded $C^{1,1}$ domain D such that for any $\hat{t}\leqslant T_0$ and $x,y\in D$ with $\rho_D(x)\geqslant\sqrt{\hat{t}}$, $\rho_D(y)\geqslant\sqrt{\hat{t}}$, one can find an arclength-parameterized curve $l\subset D$ connecting x and y such that the length $|l|$ of l is equal to $\lambda_1|x-y|$ with $\lambda_1\leqslant\lambda_0$, a constant depending only on the characteristic of the bounded $C^{1,1}$ domain D. Moreover, l can be chosen so that

$$\rho_D\big(l(s)\big)\geqslant\lambda_2\sqrt{\hat{t}},\quad s\in\big[0,|l|\big],$$

for some positive constant λ_2 depending only on the characteristic of the bounded $C^{1,1}$ domain D. Thus for any $t = r^2\hat{t} \leqslant r^2 T_0$ and $x, y \in D_r$ with $\rho_r(x) \geqslant \sqrt{t}$, $\rho_r(y) \geqslant \sqrt{t}$, one can find an arclength-parameterized curve $l \subset D_r$ connecting x and y such that the length $|l|$ of l is equal to $\lambda_1|x-y|$ and

$$\rho_r\big(l(s)\big) \geqslant \lambda_2\sqrt{t}, \quad s \in \big[0, |l|\big].$$

Using this fact and (4.19), and following the proof of Theorem 2.7 in [9], we can show that there exists a positive constant c_9 depending only on d and the characteristic of the bounded $C^{1,1}$ domain D such that

$$r^{D_r}(t,x,y) \geqslant \frac{1}{2} c_8 e^{-2c_7} \psi_r(t,x,y) t^{-\frac{d}{2}} e^{-\frac{c_9|x-y|^2}{t}} \tag{4.20}$$

for all $t \in (0, t_2 \wedge r^2(T \wedge T_0)]$ and $x, y \in D_r$ with $\rho_r(x) \geqslant \sqrt{t}$, $\rho_r(y) \geqslant \sqrt{t}$.

It is easy to check that there exists a positive constant $T_1 \leqslant T_0$ depending only on the characteristic of the bounded $C^{1,1}$ domain D such that for $\hat{t} \leqslant T_1$ and arbitrary $x, y \in D$, one can find $x_1, y_1 \in D$ be such that $\rho_D(x_1) \geqslant \sqrt{\hat{t}}$, $\rho_D(y_1) \geqslant \sqrt{\hat{t}}$ and $|x - x_0| \leqslant \sqrt{\hat{t}}$, $|y - y_0| \leqslant \sqrt{\hat{t}}$. Thus for any $t = r^2\hat{t} \leqslant r^2 T_1$ and arbitrary $x, y \in D_r$, one can find $x_1, y_1 \in D_r$ be such that $\rho_r(x_1) \geqslant \sqrt{t}$, $\rho_r(y_1) \geqslant \sqrt{t}$ and $|x - x_0| \leqslant \sqrt{t}$, $|y - y_0| \leqslant \sqrt{t}$. Now using (4.17) and (4.20) one can repeat the last paragraph of the proof of Theorem 2.1 in [15] to show that there exists a positive constant c_{10} depending only on d and the characteristic of the bounded $C^{1,1}$ domain D such that

$$r^{D_r}(t,x,y) \geqslant c_8 c_{10} e^{-2c_7} \psi_r(t,x,y) t^{-\frac{d}{2}} e^{-\frac{2c_9|x-y|^2}{t}} \tag{4.21}$$

for all $(t,x,y) \in (0, t_2 \wedge r^2(T \wedge T_1)] \times D_r \times D_r$.

Using (4.1) instead of (4.2) the proof of (4.8) up to $t \leqslant t_3$ for some t_3 depending on T and D is similar (and simpler) to the proof of (4.9). To prove (4.8) for a general $T > 0$, we can apply the Chapman–Kolmogorov equation and use the argument in the proof of Theorem 3.9 in [16]. We omit the details. □

Remark 4.5. Theorem 4.4(2) will be used in [14] to prove a parabolic Harnack inequality, a parabolic boundary Harnack inequality and the intrinsic ultracontractivity of the semigroup Q_t^D.

5. Uniform 3G type estimates for small Lipschitz domains

Recall that $r_1 > 0$ is the constant from (2.3) and $r_3 > 0$ is the constant from Theorem 2.2. The next lemma is a scale invariant version of Lemma 2.3. The proof is similar to the proof of Lemma 2.3.

Lemma 5.1. *There exists $c = c(d, \mu) > 0$ such that for every $r \in (0, r_1 \wedge r_3]$, $Q \in \mathbf{R}^d$ and open subset U with $B(z,l) \subset U \subset B(Q,r)$, we have for every $x \in U \setminus \overline{B(z,l)}$,*

$$\sup_{y \in B(z,l/2)} G_U(y,x) \leqslant c \inf_{y \in B(z,l/2)} G_U(y,x) \tag{5.1}$$

and

$$\sup_{y \in B(z,l/2)} G_U(x,y) \leqslant c \inf_{y \in B(z,l/2)} G_U(x,y). \tag{5.2}$$

Proof. (5.1) follows from Theorem 2.2. So we only need to show (5.2). Since $r < r_1$, by (2.3), there exists $c = c(d) > 1$ such that for every $x, w \in \overline{B(z, 3l/4)}$,

$$c^{-1}\frac{1}{|w-x|^{d-2}} \leqslant G_{B(z,l)}(w,x) \leqslant G_U(w,x) \leqslant G_{B(Q,r)}(w,x) \leqslant c\frac{1}{|w-x|^{d-2}}.$$

Thus for $w \in \partial B(z, \frac{3l}{4})$ and $y_1, y_2 \in B(z, \frac{l}{2})$, we have

$$G_U(w,y_1) \leqslant c\left(\frac{|w-y_2|}{|w-y_1|}\right)^{d-2}\frac{1}{|w-y_2|^{d-2}} \leqslant 4^{d-2}c^2 G_U(w,y_2). \tag{5.3}$$

On the other hand, from (2.5), we have

$$G_U(x,y) = \mathbf{E}_x\big[G_U(X_{T_{B(z,\frac{l}{2})}}, y)\big], \quad y \in B\left(z, \frac{l}{2}\right). \tag{5.4}$$

Since $X_{T_{B(z,\frac{3l}{4})}} \in \partial B(z, \frac{3l}{4})$, combining (5.3)–(5.4), we get

$$\begin{aligned} G_U(x,y_1) &\leqslant 4^{d-2}c^2\mathbf{E}_x\big[G_U(X_{T_{B(z,\frac{3l}{4})}}, y_2)\big] \\ &= 4^{d-2}c^2 G_U(x,y_2), \quad y_1, y_2 \in B\left(z, \frac{l}{2}\right). \qquad \square \end{aligned}$$

In the remainder of this section, we fix a bounded Lipschitz domain D with characteristic (R_0, Λ_0). For every $Q \in \partial D$ we put

$$\Delta_Q(r) := \{y \text{ in } CS_Q\colon \phi_Q(\tilde{y}) + r > y_d > \phi_Q(\tilde{y}),\ |\tilde{y}| < r\},$$

where CS_Q is the coordinate system with origin at Q in the definition of Lipschitz domains and ϕ_Q is the Lipschitz function there. Define

$$r_5 := \frac{R_0}{\sqrt{1+\Lambda_0^2}+1} \wedge r_1 \wedge r_3. \tag{5.5}$$

If $z \in \overline{\Delta_Q(r)}$ with $r \leqslant r_5$, we have

$$|Q-z| \leqslant \big|\big(\tilde{z}, \phi_Q(\tilde{z})\big) - (\tilde{Q}, 0)\big| + r \leqslant \big(\sqrt{1+\Lambda_0^2}+1\big)r \leqslant R_0.$$

So $\overline{\Delta_Q(r)} \subset B(Q, R_0) \cap D$.

For any Lipschitz function $\psi : \mathbf{R}^{d-1} \to \mathbf{R}$ with Lipschitz constant Λ_0, let

$$\Delta^{\psi} := \{y\colon r_5 > y_d - \psi(\tilde{y}) > 0,\ |\tilde{y}| < r_5\}$$

so that $\Delta^{\psi} \subset B(0, R_0)$. We observe that, for any Lipschitz function $\varphi : \mathbf{R}^{d-1} \to \mathbf{R}$ with the Lipschitz constant Λ, its dilation $\varphi_r(x) := r\varphi(x/r)$ is also Lipschitz with the same Lipschitz constant Λ_0. For any $r > 0$, put $\eta = \frac{r}{r_5}$ and $\psi = (\phi_Q)_\eta$. Then it is easy to see that for any $Q \in \partial D$ and $r \leqslant r_5$,

$$\Delta_Q(r) = \eta\Delta^{\psi}.$$

Thus by choosing appropriate constants $\Lambda_1 > 1$, $R_1 < 1$ and $d_1 > 0$, we can say that for every $Q \in \partial D$ and $r \leqslant r_5$, the $\Delta_Q(r)$'s are bounded Lipschitz domains with the characteristics (rR_1, Λ_1) and the diameters of $\Delta_Q(r)$'s are less than rd_1. Since $r_5 \leqslant r_1 \wedge r_3$, Lemma 5.1 works for $G_{\Delta_Q(r)}(x,y)$ with $Q \in \partial D$ and $r \leqslant r_5$. Moreover, we can restate the scale invariant boundary Harnack principle in the following way.

Theorem 5.2. *There exist constants $M_3, c > 1$ and $s_1 > 0$, depending on μ, ν and D such that for every $Q \in \partial D$, $r < r_5$, $s < rs_1$, $w \in \partial\Delta_Q(r)$ and any nonnegative functions u and v which are harmonic with respect to X^D in $\Delta_Q(r) \cap B(w, M_3 s)$ and vanish continuously on $\partial\Delta_Q(r) \cap B(w, M_3 s)$, we have*

$$\frac{u(x)}{v(x)} \leqslant c\frac{u(y)}{v(y)} \quad \text{for any } x, y \in \Delta_Q(r) \cap B(w, s). \tag{5.6}$$

In the remainder of this section we will fix the above constants r_5, M_3, s_1, Λ_1, R_1 and $d_1 > 0$, and consider the Green functions of X in $\Delta_Q(r)$ with $Q \in \partial D$ and $r > 0$. We will prove a scale invariant 3G type estimates for these Green functions for small r. The main difficulties of the scale invariant 3G type estimates for X are the facts that X does not have rescaling property and that the Green function $G_{\Delta_Q(r)}(x, \cdot)$ is not harmonic for X. To overcome these difficulties, we first establish some results for the Green functions of X in $\Delta_Q(r)$ with $Q \in \partial D$ and r small.

Let $\delta_r^Q(x) := \operatorname{dist}(x, \partial\Delta_Q(r))$. Using Lemma 5.1 and a Harnack chain argument, the proof of the next lemma is almost identical to the proof of Lemma 6.7 in [8]. So we omit the proof.

Lemma 5.3. *For any given $c_1 > 0$, there exists $c_2 = c_2(D, c_1, \mu) > 0$ such that for every $Q \in \partial D$, $r < r_5$, $|x - y| \leqslant c_1(\delta_r^Q(x) \wedge \delta_r^Q(y))$, we have*

$$G_{\Delta_Q(r)}(x, y) \geqslant c_2|x - y|^{-d+2}.$$

Recall that $M_3 > 0$ and $s_1 > 0$ are the constants from Theorem 5.2. Let $M_4 := 2(1 + M_3) \times \sqrt{1 + \Lambda_1^2} + 2$ and $R_4 := R_1/M_4$. The next lemma is a scale invariant version of Lemma 2.5. The proof is similar to the proof of Lemma 2.5. We spell out the details for the reader's convenience.

Lemma 5.4. *There exists constant $c > 1$ such that for every $Q \in \partial D$, $r < r_5$, $s < rR_4$, $w \in \partial\Delta_Q(r)$ and any nonnegative functions u and v which are harmonic in $\Delta_Q(r) \setminus B(w, s)$ and vanish continuously on $\partial\Delta_Q(r) \setminus B(w, s)$, we have*

$$\frac{u(x)}{u(y)} \leqslant c\frac{v(x)}{v(y)} \quad \text{for any } x, y \in \Delta_Q(r) \setminus B(w, M_4 s). \tag{5.7}$$

Proof. We fix a point Q on ∂D, $r < r_5$, $s < rR_4$ and $w \in \partial\Delta_Q(r)$ throughout this proof. Let

$$\Delta^s := \{y \text{ in } CS_w\colon \varphi_w(\tilde{y}) + 2s > y_d > \varphi_w(\tilde{y}),\ |\tilde{y}| < 2(M_3 + 1)s\},$$
$$\partial_1\Delta^s := \{y \text{ in } CS_w\colon \varphi_w(\tilde{y}) + 2s \geqslant y_d > \varphi_w(\tilde{y}),\ |\tilde{y}| = 2(M_3 + 1)s\},$$
$$\partial_2\Delta^s := \{y \text{ in } CS_w\colon \varphi_w(\tilde{y}) + 2s = y_d,\ |\tilde{y}| \leqslant 2(M_3 + 1)s\},$$

where CS_w is the coordinate system with origin at w in the definition of the Lipschitz domain $\Delta_Q(r)$ and φ_w is the Lipschitz function there. If $z \in \overline{\Delta^s}$,

$$|w - z| \leqslant |(\tilde{z}, \varphi_w(\tilde{z})) - (\tilde{z}, 0)| + 2s \leqslant 2s(1 + M_3)\sqrt{1 + \Lambda^2} + 2s = M_4 s \leqslant rR_1.$$

So $\overline{\Delta^s} \subset B(Q, M_4 s) \cap D \subset B(Q, rR_1) \cap D$. For $|\tilde{y}| = 2(M_3 + 1)s$, we have $|(\tilde{y}, \varphi_w(\tilde{y}))| > s$. So u and v are harmonic with respect to X in $\Delta_Q(r) \cap B((\tilde{y}, \varphi_w(\tilde{y})), 2M_3 s)$ and vanish continuously on $\partial\Delta_Q(r) \cap B((\tilde{y}, \varphi_w(\tilde{y})), 2M_3 s)$ where $|\tilde{y}| = 2(M_3 + 1)s$. Therefore by Theorem 5.2,

$$\frac{u(x)}{u(y)} \leqslant c\frac{v(x)}{v(y)} \quad \text{for any } x, y \in \partial_1\Delta^s \text{ with } \tilde{x} = \tilde{y}. \tag{5.8}$$

Since $\mathrm{dist}(\Delta_Q(r) \cap B(w,s), \partial_2\Delta^s) > c_1 s$ for some $c_1 = c_1(D)$, if $x \in \partial_2\Delta^s$, the Harnack inequality (Theorem 2.2) and a Harnack chain argument give that there exists constant $c_2 > 1$ such that

$$c_2^{-1} < \frac{u(x)}{u(y)}, \frac{v(x)}{v(y)} < c_2. \tag{5.9}$$

In particular, (5.9) is true with $x = x_s := (\tilde{x}, \varphi_w(\tilde{x}) + 2s)$, which is also in $\partial_1\Delta^s$. Thus (5.8) and (5.9) imply that

$$c_3^{-1}\frac{u(x)}{u(y)} \leqslant \frac{v(x)}{v(y)} \leqslant c_3\frac{u(x)}{u(y)}, \quad x, y \in \partial_1\Delta^s \cup \partial_2\Delta^s, \tag{5.10}$$

for some $c_3 > 1$. Now, by applying the maximum principle (Lemma 7.2 in [11]) twice (x and y), (5.10) is true for every $x \in \Delta_Q(r) \setminus \Delta^s$. □

Combining Theorem 5.2 and Lemma 5.4, we get the following as a corollary.

Corollary 5.5. *There exists constant $c > 1$ such that for every $Q \in \partial D$, $r < r_5$, $w \in \partial(\Delta_Q(r))$, and $s < rR_4$, we have for $x, y \in \Delta_Q(r) \setminus B(w, M_4 s)$ and $z_1, z_2 \in \Delta_Q(r) \cap B(w,s)$,*

$$\frac{G_{\Delta_Q(r)}(x,z_1)}{G_{\Delta_Q(r)}(y,z_1)} \leqslant c\frac{G_{\Delta_Q(r)}(x,z_2)}{G_{\Delta_Q(r)}(y,z_2)} \quad \textit{and} \quad \frac{G_{\Delta_Q(r)}(z_1,x)}{G_{\Delta_Q(r)}(z_1,y)} \leqslant c\frac{G_{\Delta_Q(r)}(z_2,x)}{G_{\Delta_Q(r)}(z_2,y)}. \tag{5.11}$$

Corollary 5.6. *For any given $N \in (0,1)$, there exists constant $c = c(N, M_4, D) > 1$ such that for every $Q \in \partial D$, $r < r_5$, $w \in \partial(\Delta_Q(r))$ and $s < rR_4$, we have*

$$G_{\Delta_Q(r)}(x,z_1) \leqslant cG_{\Delta_Q(r)}(x,z_2) \quad \textit{and} \quad G_{\Delta_Q(r)}(z_1,x) \leqslant cG_{\Delta_Q(r)}(z_2,x) \tag{5.12}$$

for $x \in \Delta_Q(r) \setminus B(w, M_4 s)$ and $z_1, z_2 \in \Delta_Q(r) \cap B(w,s)$ with $B(z_2, Ns) \subset \Delta_Q(r) \cap B(w,s)$.

Proof. Fix $Q \in \partial D$, $r < r_5$, $w \in \partial(\Delta_Q(r))$ and $s < rR_4$. Recall from the proof of Lemma 5.4 that CS_w is the coordinate system with origin at w in the definition of the Lipschitz domain $\Delta_Q(r)$. Let $\bar{y} := (\tilde{0}, M_4 s)$. By (2.2),

$$G_{\Delta_Q(r)}(\bar{y}, z_1) \leqslant c_1|y - z_2|^{-d+2} \leqslant c_2 s^{-d+2} \quad \text{and}$$
$$G_{\Delta_Q(r)}(z_1, \bar{y}) \leqslant c_1|y - z_2|^{-d+2} \leqslant c_2 s^{-d+2}$$

for some constants $c_1, c_2 > 0$.

Note that, since $\Delta_Q(r)$'s are bounded Lipschitz domains with the characteristics (rR_1, Λ_1) and $s < rR_4$, it is easy to see that there exists a positive constant c_3 such that $\rho_r^Q(\bar{y}) \geqslant c_3 M_4 s$ and $\rho_r^Q(z_2) \geqslant Ns$. Thus by Lemma 5.3,

$$G_{\Delta_Q(r)}(y, z_2) \geqslant c_4|y - z_2|^{-d+2} \geqslant c_5 s^{-d+2} \quad \text{and}$$
$$G_{\Delta_Q(r)}(z_2, y) \geqslant c_4|y - z_2|^{-d+2} \geqslant c_5 s^{-d+2}$$

for some constants $c_4, c_5 > 0$.

Now apply (5.11) with $y = \bar{y}$ and get

$$G_{\Delta_Q(r)}(x,z_1) \leqslant c_6 G_{\Delta_Q(r)}(x,z_2) \quad \text{and} \quad G_{\Delta_Q(r)}(z_1,x) \leqslant c_6 G_{\Delta_Q(r)}(z_2,x)$$

for some $c_6 > 1$. □

With Lemma 5.1, Corollaries 5.5 and 5.6 in hand, one can follow either the argument in Section 2 of this paper or the argument on pp. 170–173 of [8] to prove the next theorem. So we skip the details.

Theorem 5.7. *There exists a constant $c > 0$ such that for every $Q \in \partial D$, $r < r_5$ and $x, y, z \in \Delta_Q(r)$,*

$$\frac{G_{\Delta_Q(r)}(x, y)G_{\Delta_Q(r)}(y, z)}{G_{\Delta_Q(r)}(x, z)} \leqslant c\big(|x - y|^{-d+2} + |y - z|^{-d+2}\big). \tag{5.13}$$

6. Boundary Harnack principle for the Schrödinger operator of X^D in bounded Lipschitz domains

Recall that ν belongs to the Kato class $\mathbf{K}_{d,2}$ and A is continuous additive functional associated with $\nu|_D$. We also recall $e_A(t) = \exp(A_t)$ and the Schrödinger semigroup

$$Q_t^D f(x) = \mathbf{E}_x\big[e_A(t) f\big(X_t^D\big)\big].$$

Using the Martin representation for Schrödinger operators (Theorem 7.5 in [6]) and the uniform 3G estimates (Theorem 5.7), we will prove the boundary Harnack principle for the Schrödinger operator of diffusions with measure-valued drifts in bounded Lipschitz domains. In the remainder of this section, we fix a bounded Lipschitz domain D with its characteristic (R_0, Λ_0). Recall

$$\Delta_Q(r) = \big\{y \text{ in } CS_Q\colon \phi_Q(\tilde{y}) + r > y_d > \phi_Q(\tilde{y}),\ |\tilde{y}| < r\big\},$$

where CS_Q is the coordinate system with origin at $Q \in \partial D$ in the definition of Lipschitz domains and ϕ_Q is the Lipschitz function there. We also recall that r_5 is the constant from (5.5) and that the diameters of $\Delta_Q(r)$'s are less than rd_1.

For $Q \in \partial D$, $r < r_5$ and $y \in \Delta_Q(r)$, let $X^{Q,r,y}$ denote the h-conditioned process obtained from $X^{\Delta_Q(r)}$ with $h(\cdot) = G_{\Delta_Q(r)}(\cdot, y)$ and let $\mathbf{E}_x^{Q,r,y}$ denote the expectation for $X^{Q,r,y}$ starting from $x \in \Delta_Q(r)$. Now define the conditional gauge function

$$u_r^Q(x, y) := \mathbf{E}_x^{Q,r,y}\big[e_{A^\nu}\big(\tau_{\Delta_Q(r)}^y\big)\big].$$

By Theorem 5.7,

$$\begin{aligned}\mathbf{E}_x^{Q,r,y}\big[A\big(\tau_{\Delta_Q(r)}^y\big)\big] &\leqslant \int_{\Delta_Q(r)} \frac{G_{\Delta_Q(r)}(x, a)G_{\Delta_Q(r)}(a, y)}{G_{\Delta_Q(r)}(x, y)}\nu(da) \\ &\leqslant c\int_{\Delta_Q(r)} \big(|x - a|^{-d+2} + |a - y|^{-d+2}\big)\nu(da), \quad r < r_5.\end{aligned}$$

Since the above constant is independent of $r < r_5$, we have

$$\begin{aligned}&\sup_{x,y\in\Delta_Q(r)} \mathbf{E}_x^{Q,r,y}\big[A\big(\tau_{\Delta_Q(r)}^y\big)\big] \\ &\quad\leqslant c \sup_{x\in\mathbf{R}^d} \int_{|x-a|\leqslant rd_1} \frac{|\nu|(da)}{|x - a|^{d-2}} = cM_\nu^2(rd_1) < \infty, \quad r < r_5,\ Q \in \partial D.\end{aligned}$$

Thus $\nu \in \mathbf{S}_\infty(X^{\Delta_Q(r)})$ for every $r < r_5$ and there exists $r_6 \leqslant r_5$ such that

$$\sup_{x,y\in\Delta_Q(r)} \mathbf{E}_x^{Q,r,y}\big[A\big(\tau^y_{\Delta_Q(r)}\big)\big] \leqslant \frac{1}{2}, \quad r < r_6,\ Q \in \partial D.$$

Hence by Khasminskii's lemma,

$$\sup_{x,z\in\Delta_Q(r)} u_r^Q(x,y) \leqslant 2, \quad r < r_6,\ Q \in \partial D.$$

By Jensen's inequality, we also have

$$\inf_{x,z\in\Delta_Q(r)} u_r^Q(x,y) > 0, \quad r < r_6,\ Q \in \partial D.$$

Therefore, we have proved the following lemma.

Lemma 6.1. *For $r < r_6$, $\nu|_{\Delta_Q(r)} \in \mathbf{S}_\infty(X^{\Delta_Q(r)})$ and $\nu|_{\Delta_Q(r)}$ is gaugeable. Moreover, there exists a constant c such that $c^{-1} \leqslant u_r^Q(x,y) \leqslant c$ for $x, y \in \Delta_Q(r)$ and $r < r_6$.*

Theorem 6.2 *(Boundary Harnack principle). Suppose D is a bounded Lipschitz domain in $\mathbf{R}^d$ with the Lipschitz characteristic (R_0, Λ_0) and let $M_5 := (\sqrt{1+\Lambda_0^2}+1)$. Then there exists $N > 1$ such that for any $r \in (0, r_6)$ and $Q \in \partial D$, there exists a constant $c > 1$ such that for any non-negative functions u, v which are ν-harmonic in $D \cap B(Q, rM_5)$ with respect to X^D and vanish continuously on $\partial D \cap B(Q, rM_5)$, we have*

$$\frac{u(x)}{v(x)} \leqslant c\frac{u(y)}{v(y)} \quad \text{for any } x, y \in D \cap B\left(Q, \frac{r}{N}\right).$$

Proof. Note that, with $M_5 = (\sqrt{1+\Lambda_0^2}+1)$, $\Delta_Q(r) \subset D \cap B(Q, M_5 r)$. So u, v are ν-harmonic in $\Delta_Q(r)$. For the remainder of the proof, we fix $Q \in \partial D$, $r \in (0, r_5)$ and a point $x_r^Q \in \Delta_Q(r)$. Let

$$M(x,z) := \lim_{U\ni y\to z} \frac{G_U(x,y)}{G_U(x_r^Q, y)}, \qquad K(x,z) := \lim_{U\ni y\to z} \frac{V_U(x,y)}{V_U(x_r^Q, y)}.$$

Since u, v are ν-harmonic with respect to $X^{\Delta_Q(r)}$, by Theorem 7.7 in [6] and our Lemma 6.1, there exist finite measures μ_1 and ν_1 on ∂U such that

$$u(x) = \int_{\partial\Delta_Q(r)} K(x,z)\mu_1(dz) \quad \text{and} \quad v(x) = \int_{\partial\Delta_Q(r)} K(x,z)\nu_1(dz), \quad x \in \Delta_Q(r).$$

Let

$$u_1(x) := \int_{\partial\Delta_Q(r)} M(x,z)\mu_1(dz) \quad \text{and} \quad v_1(x) := \int_{\partial\Delta_Q(r)} M(x,z)\nu_1(dz), \quad x \in \Delta_Q(r).$$

By Theorem 7.3(2) in [6] and our Lemma 6.1, we have for every $x \in U$,

$$\frac{u(x)}{v(x)} = \frac{\int_{\partial\Delta_Q(r)} K(x,z)\mu_1(dz)}{\int_{\partial\Delta_Q(r)} K(x,z)\nu_1(dz)} \leqslant c_1^2 \frac{\int_{\partial\Delta_Q(r)} M(x,z)\mu_1(dz)}{\int_{\partial\Delta_Q(r)} M(x,z)\nu_1(dz)} = c_1^2\frac{u_1(x)}{v_1(x)} \leqslant c_1^4\frac{u(x)}{v(x)}.$$

Since u_1, v_1 are harmonic for X^U and vanish continuously on $\partial\Delta_Q(r) \cap \partial D$, by the boundary Harnack principle (Theorem 4.6 in [12]), there exist N and c_2 such that

$$\frac{u_1(x)}{v_1(x)} \leqslant c_2 \frac{u_1(y)}{v_1(y)}, \quad x, y \in D \cap B\left(Q, \frac{r}{N}\right).$$

Thus for every $x, y \in D \cap B(Q, \frac{r}{N})$

$$\frac{u(x)}{v(x)} \leqslant c_1^2 \frac{u_1(x)}{v_1(x)} \leqslant c_2 c_1^2 \frac{u_1(y)}{v_1(y)} \leqslant c_2 c_1^4 \frac{u(y)}{v(y)}. \qquad \square$$

References

[1] R.F. Bass, Z.-Q. Chen, Brownian motion with singular drift, Ann. Probab. 31 (2) (2003) 791–817.
[2] Ph. Blanchard, Z.M. Ma, Semigroup of Schrödinger operators with potentials given by Radon measures, in: Stochastic Processes, Physics and Geometry, World Sci. Publishing, Teaneck, NJ, 1990, pp. 160–195.
[3] K. Bogdan, Sharp estimates for the Green function in Lipschitz domains, J. Math. Anal. Appl. 243 (2000) 326–337.
[4] Z.-Q. Chen, Gaugeability and conditional gaugeability, Trans. Amer. Math. Soc. 354 (2002) 4639–4679.
[5] Z.-Q. Chen, P. Kim, Green function estimate for censored stable processes, Probab. Theory Related Fields 124 (2002) 595–610.
[6] Z.-Q. Chen, P. Kim, Stability of Martin boundary under non-local Feynman–Kac perturbations, Probab. Theory Related Fields 128 (2004) 525–564.
[7] Z.-Q. Chen, R. Song, General gauge and conditional gauge theorems, Ann. Probab. 30 (2002) 1313–1339.
[8] K.L. Chung, Z.X. Zhao, From Brownian Motion to Schrödinger's Equation, Springer-Verlag, Berlin, 1995.
[9] E.B. Fabes, D.W. Stroock, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash, Arch. Ration. Mech. Anal. 96 (4) (1986) 327–338.
[10] W. Hansen, Uniform boundary Harnack principle and generalized triangle property, J. Funct. Anal. 226 (2) (2005) 452–484.
[11] P. Kim, R. Song, Two-sided estimates on the density of Brownian motion with singular drift, Illinois J. Math. 50 (3) (2006) 635–688.
[12] P. Kim, R. Song, Boundary Harnack principle for Brownian motions with measure-valued drifts in bounded Lipschitz domains, preprint, 2006.
[13] P. Kim, R. Song, On dual processes of non-symmetric diffusions with measure-valued drifts, preprint, 2006.
[14] P. Kim, R. Song, Intrinsic ultracontractivity of non-symmetric diffusions with measure-valued drifts and potentials, preprint, 2006.
[15] L. Riahi, Comparison of Green functions and harmonic measure of parabolic operators, Potential Anal. 23 (4) (2005) 381–402.
[16] R. Song, Sharp bounds on the density, Green function and jumping function of subordinate killed BM, Probab. Theory Related Fields 128 (2004) 606–628.
[17] Q.S. Zhang, A Harnack inequality for the equation $\nabla(a\nabla u) + b\nabla u = 0$, when $|b| \in K_{n+1}$, Manuscripta Math. 89 (1996) 61–77.
[18] Qi S. Zhang, Gaussian bounds for the fundamental solutions of $\nabla(A\nabla u) + B\nabla u - u_t = 0$, Manuscripta Math. 93 (1997) 381–390.
[19] Q.S. Zhang, The boundary behavior of heat kernels of Dirichlet Laplacians, J. Differential Equations 182 (2002) 416–430.