A Note on the Green Function Estimates for Symmetric Stable Processes

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Abstract

Suppose D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$ and G_D is the Green function of the killed symmetric α-stable process in D. In this note we establish a sharp upper bound estimate for G_D in such a way that the explicit dependence of the constant in the estimate on α is also given. This sharp estimate allows us to recover the sharp upper bound estimate for the Green function of killed Brownian motion in D by letting $\alpha \uparrow 2$.

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One of the most important family of Markov processes is the family of symmetric stable processes. A symmetric α-stable process X on $\mathbf{R}^n$ is a Lévy process such that for any $\xi \in \mathbf{R}^n$ and $t \geq 0$, $E\left[e^{i\xi\cdot(X_t - X_0)}\right] = e^{-t|\xi|^\alpha}$. Here α must be in the interval $(0, 2]$. When $\alpha = 2$, we get a Brownian motion running with a time clock twice as fast as the standard one. In this note, symmetric stable processes refer to the case when $\alpha \in (0, 2)$. We always assume that $n \geq 2$ in this note.

For any Borel measurable set $A \subset \mathbf{R}^n$, we define $\tau_A = \inf\{t > 0 : X_t \notin A\}$. The killed symmetric stable process X^D in D is defined by

$$X_t^D = \begin{cases} X_t, & \text{if } t < \tau_D, \\ \partial, & \text{if } t \geq \tau_D \end{cases}$$

where ∂ is the cemetery point. The Green function $G_D(x, y)$ of X^D is a function that is continuous on $D \times D$ except along the diagonal such that for every Borel measurable function $f \geq 0$ on D,

$$E_x\left[\int_0^{\tau_D} f(X_s)ds\right] = \int_D G(x, y)f(y)dy, \qquad x \in D.$$

Recall that an open set D in $\mathbf{R}^n$ is said to be $C^{1,1}$ if there is a localization radius $R > 0$ and a constant $\Lambda > 0$ such that for every $Q \in \partial D$, there is a $C^{1,1}$-function $\phi = \phi_Q : \mathbf{R}^{n-1} \to \mathbf{R}$ satisfying $\phi(0) = 0$, $\|\nabla\phi\|_\infty \leq \Lambda$, $|\nabla\phi(x) - \nabla\phi(z)| \leq \Lambda|x - z|$, and an orthonormal coordinate system $y = (y_1, \cdots, y_{n-1}, y_n) = (\tilde{y}, y_n)$ such that $B(Q, R) \cap D = B(Q, R) \cap \{y : y_n > \phi(\tilde{y})\}$. It is well known that for a bounded $C^{1,1}$ domain D, there exists $r_0 > 0$ depending only on D such that for any $z \in \partial D$, $0 < r \leq r_0$, there exist two balls $B_1^z(r)$ and $B_2^z(r)$ of radius r such that $B_1^z(r) \subset D$, $B_2^z(r) \subset \mathbf{R}^n \setminus \overline{D}$ and $\{z\} = \partial B_1^z(r) \cap \partial B_2^z(r)$.

The following sharp estimates on the Green function G_D were obtained independently in Chen and Song [3] and in Kulczycki [5].

Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Let $\delta(x) = d(x, \partial D)$ be the Euclidean distance between x and ∂D. Then there exist constants $C_1 = C_1(D, \alpha) > 0$ and $C_2 = C_2(D, \alpha) > 0$ such that for $x, y \in D$,

$$C_1\left(\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^\alpha} \wedge 1\right)\frac{1}{|x-y|^{n-\alpha}} \leq G_D(x, y) \leq C_2\left(\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^\alpha} \wedge 1\right)\frac{1}{|x-y|^{n-\alpha}}. \tag{1}$$

In Chen [2], the following sharper estimate is given.

Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Then there are constants $c_1 = c_1(D) > 0$ and $c_2 = c_2(D) > 0$, depending on D only, such that for $x, y \in D$,

$$c_1\,\alpha\left(\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^\alpha} \wedge 1\right)\frac{1}{|x-y|^{n-\alpha}} \leq G_D(x, y) \leq \frac{c_2\alpha}{n-\alpha}\left(\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^\alpha} \wedge 1\right)\frac{1}{|x-y|^{n-\alpha}}. \tag{2}$$

The novelty of the estimate (2) over (1) is the more precise and explicit information on how the constants C_1 and C_2 in (1) depend on α, which can be used to recover the Green function estimate for killed Brownian motion in $D \subset \mathbf{R}^n$ when $n \geq 3$. Note that since the symmetric α-stable process converges weakly in $D([0,\infty), \mathbf{R}^n)$ to a Brownian with infinitesimal generator Δ as $\alpha \uparrow 2$ (cf., e.g. Ethier and Kurtz [4]), the Green function G_D converges pointwise to the Green function of the killed Brownian motion in D. Here $D([0,\infty), \mathbf{R}^n)$ is the space of right continuous $\mathbf{R}^n$-valued functions with left limits, equipped with the Skorohod topology. Thus, as is pointed out in Chen [2], by letting $\alpha \uparrow 2$ in estimate (2) we immediately get the following well known sharp estimates on the Green function of the killed Brownian motion, whose upper and lower bound estimates were first obtained by Widman [6] and Zhao [7], respectively.

Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$ with $n \geq 3$ and that G_D is the Green function of the killed Brownian motion in D. Then there exist constants $c_1 = c_1(D) > 0$ and $c_2 = c_2(D) > 0$ such that for all $x, y \in D$,

$$c_1 \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{n-2}} \leq G_D(x,y) \leq c_2 \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right) \frac{1}{|x-y|^{n-2}}.$$

In Chen [2], the estimate (2) is first proved for balls, which is then used to establish the estimates on general bounded $C^{1,1}$ domains. The proof for the lower bound estimate of G_D in (2) for a bounded $C^{1,1}$ domain D is fully given in Chen [2], while for the upper bound estimate it was claimed that it can be obtained by following the proofs of (1.4) and (1.5) in Chen and Song [3], keeping better track of all the constants and using the refined estimates of the Green functions and Poisson kernels for balls obtained in Lemmas 2.2 and 2.3 of Chen [2], However, the original proof of (1.5) given in Chen and Song [3] contains a gap in that it works only for $\alpha \in (0,1)$ rather than for all $\alpha \in (0,2)$ as claimed there. (We thank Piotr Graczyk for pointing out this gap.)

The purpose of this note is to present a proof for the upper bound estimate in (2).

Theorem 1 *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Then there exist a constant $c = c(D)$ depending only on D such that for all $x, y \in D$,*

$$G_D(x,y) \leq \frac{c\alpha}{n-\alpha} \left(\frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^{\alpha}} \wedge 1 \right) \frac{1}{|x-y|^{n-\alpha}}.$$

Lemma 1 *There exists a constant $c = c(n) > 0$ such that for any $a \in \mathbf{R}^n$ and $r > 0$,*

$$G_{B^c(a,r)}(x,y) \leq c\alpha \frac{|y-a|^{\alpha/2}}{r^{\alpha/2}} \frac{\delta_{B^c(a,r)}(x)^{\alpha/2}}{|x-y|^{n-\alpha/2}}, \quad x, y \in B^c(a,r),$$

where $G_{B^c(a,r)}$ is the Green function of $B^c(a,r) = \{x \in \mathbf{R}^n : |x-a| > r\}$ and $\delta_{B^c(a,r)}(x)$ is the Euclidean distance between x and $\partial B(a,r)$.

Proof. By Lemma 2.2 of Chen [2], there is a constant $c = c(n) \geq 1$ such that

$$G_{B(a,r)}(x,y) \leq c\alpha \frac{\delta_{B(a,r)}(x)^{\alpha/2}}{|x-y|^{n-\frac{\alpha}{2}}}, \quad x,y \in B(a,r).$$

The rest of the proof is exactly the same as that of Lemma 2.5 in Chen and Song [3], provided we use the above improved estimate on $B_{B(a,r)}(x,y)$. □

The next result is a strengthened version of (1.4) of Chen and Song [3].

Lemma 2 *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Then there exists a constant $c = c(D)$ depending only on D such that for all $x, y \in D$,*

$$G_D(x,y) \leq c\alpha \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\alpha/2}} \tag{3}$$

Proof. Let $x_0 \in \partial D$ be such that $|x - x_0| = \delta(x)$. Consider the ball $B = B_2^{x_0}(r_0) = B(a, r_0)$. It follows from Lemma 1 that there is a constant $c_1 = c_1(n) > 0$ such that

$$\begin{aligned} G_D(x,y) &\leq G_{B^c}(x,y) \leq c\alpha \frac{|y-a|^{\alpha/2}}{r^{\alpha/2}} \frac{\delta_{B^c(a,r)}(x)^{\alpha/2}}{|x-y|^{n-\alpha/2}} \\ &= c\alpha \frac{|y-a|^{\alpha/2}}{r_0^{\alpha/2}} \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\alpha/2}} \leq c\alpha \frac{\mathrm{diam}(D) + r_0}{r_0} \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\alpha/2}}, \end{aligned}$$

where $\mathrm{diam}(D)$ stands for the diameter of D. □

The following is a refinement of (1.5) in Chen and Song [3].

Lemma 3 *Suppose that D is a bounded $C^{1,1}$ domain in $\mathbf{R}^n$. Then there exists a constant $c = c(D)$ depending only on D such that for all $x, y \in D$,*

$$G_D(x,y) \leq c\alpha \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}. \tag{4}$$

Proof. If $\delta(y) \geq r_0$ or $\delta(y) < r_0$ and $|x - y| \leq 8\delta(y)$, then (4) follows from (3). By the symmetry of the Green function G_D, (4) holds when $\delta(x) \geq r_0$ or $\delta(x) < r_0$ and $|x - y| \leq 8\delta(x)$. So we assume that $\delta(x) < r_0$, $\delta(y) < r_0$ and $|x - y| > 8\max\{\delta(x), \delta(y)\}$. Set $r = \min\{|x - y|/8, r_0\}$. Let $x_0, y_0 \in \partial D$ be such that $|x - x_0| = \delta(x)$ and $|y - y_0| = \delta(y)$, respectively. Let $B(a, r) = B_2^{x_0}(r)$ and $B(b, r) = B_2^{y_0}(r)$. Without loss of generality, we can assume that b is at the origin and that $y_0 = (-r, 0, \cdots, 0)$. In the sequel, c is a positive constant that depends at most on domain D but its value may change from line to line. We remark here that dimension n is a part of information about domain D so c may depend on dimension n as well. Clearly, as $G_{B^c(a,r)}(x)=0$ on $\overline{B(a,r)}$,

$$\begin{aligned} G_D(x,y) &\leq G_{B^c(a,r)\cap B^c(b,r)}(x,y) \\ &= G_{B^c(a,r)}(x,y) - E_y\left[G_{B^c(a,r)}(x, X_{\tau_{B^c(b,r)}}); \tau_{B^c(b,r)}<\infty\right] \\ &= G_{B^c(a,r)}(x,y)P_y(\tau_{B^c(b,r)}=\infty) \\ &\quad + E_y\left[G_{B^c(a,r)}(x,y) - G_{B^c(a,r)}(x, X_{\tau_{B^c(b,r)}}); \tau_{B^c(b,r)}<\infty\right] \\ &= I + II. \end{aligned} \tag{5}$$

Note that, since $|x-y|\geq 8r$, for $u\in B(y_0,4r)$,

$$G_{B^c(a,r)}(x,u) = \int_{B^c(a,r)\cap B^c(y_0,4r)} G_{B^c(a,r)}(x,v)K_{B(y_0,4r)}(u,v)dv, \tag{6}$$

while by Theorem A of Blumenthal, Getoor and Ray [1] and the scaling property for symmetric stable process X,

$$K_{B(y_0,4r)}(u,v) = \Gamma\left(\frac{n}{2}\right)\frac{\sin\pi\frac{\alpha}{2}}{\pi^{\frac{n}{2}+1}}\frac{(16r^2-|u-y_0|^2)^{\alpha/2}}{(|v-y_0|^2-16r^2)^{\alpha/2}|u-v|^n} \quad \text{for } |v-y_0|>4r. \tag{7}$$

Here Γ is the Gamma function defined by $\Gamma(\lambda)=\int_0^\infty t^{\lambda-1}e^{-t}dt$ for $\lambda>0$. We note that there is a constant $c>1$ such that

$$\sin\frac{\pi}{2}\leq c\alpha(2-\alpha) \quad \text{for } 0<\alpha\leq 2, \tag{8}$$

and

$$\frac{1}{\lambda c}\leq\Gamma(\lambda)\leq\frac{c}{\lambda} \quad \text{for } 0<\lambda\leq 1. \tag{9}$$

The latter is due to the property of Γ-function that $\Gamma(\lambda+1)=\lambda\Gamma(\lambda)$ and $\Gamma(1)=1$. It follows from (7) that for $u\in B(y_0,2r)$ and $|v-y_0|>4r$,

$$\left|\nabla_u K_{B(y_0,4r)}(u,v)\right|\leq c\,r^{-1}K_{B(y_0,4r)}(y_0,v),$$

and

$$\left|\frac{\partial^2 K_{B(y_0,4r)}(u,v)}{\partial u_i\partial u_j}\right|\leq c\,r^{-2}K_{B(y_0,4r)}(y_0,v) \quad \text{for } 1\leq i,j\leq n.$$

This implies by (6) that for $u\in B(y_0,2r)$,

$$\left|\nabla_u G_{B^c(a,r)}(x,u)\right|\leq c\,r^{-1}G_{B^c(a,r)}(x,y_0), \tag{10}$$

and

$$\left|\frac{\partial^2 G_{B^c(a,r)}(x,u)}{\partial u_i\partial u_j}\right|\leq c\,r^{-2}G_{B^c(a,r)}(x,y_0) \quad \text{for } 1\leq i,j\leq n. \tag{11}$$

Using Taylor expansion, we have for $y \in B(y_0, 2r)$,

$$\begin{aligned}
& \left|G_{B^c(a,r)}(x,y) - G_{B^c(a,r)}(x,u) - \nabla_y G_{B^c(a,r)}(x,y) \cdot (y-u)\right| \\
\leq\ & \frac{c}{r^2} G_{B^c(a,r)}(x,y_0)|y-u|^2. \qquad (12)
\end{aligned}$$

It is known from Theorem B of Blumenthal, Getoor and Ray [1] that for $|u| < r$,

$$K_{B^c(0,r)}(y,u) = \frac{\Gamma\left(\frac{n}{2}\right)\sin \pi \frac{\alpha}{2}}{\pi^{\frac{n}{2}+1}} \frac{(|y|^2 - r^2)^{\alpha/2}}{(r^2 - |u|^2)^{\alpha/2}|y-u|^n}. \qquad (13)$$

Recall that we've taken b as the origin and $y_0 = (-r, 0, \cdots, 0)$. So $y = (y_1, 0, \cdots, 0)$ is on the negative x_1-axis and by symmetry,

$$\int_{B(0,r)} u_i K_{B^c(0,r)}(y,u)du = 0 \quad \text{for } 2 \leq i \leq n. \qquad (14)$$

Thus by (8), (10), and (13)-(14),

$$\begin{aligned}
& \left|\int_{B(0,r)} \nabla_y G_{B^c(a,r)}(x,y) \cdot (y-u) K_{B^c(0,r)}(y,u)du\right| \\
=\ & \left|\int_{B(0,r)} \frac{\partial G_{B^c(a,r)}(x,y)}{\partial y_1}(y_1 - u_1) K_{B^c(0,r)}(y,u)du\right| \\
\leq\ & \left|\frac{\partial G_{B^c(a,r)}(x,y)}{\partial y_1}\right| \int_{B(0,r)} (u_1 - y_1)\, \frac{c\,\alpha(2-\alpha)(|y|^2 - r^2)^{\alpha/2}}{(r^2 - |u|^2)^{\alpha/2}|y-u|^n}\, du \\
\leq\ & \frac{c\,\alpha(2-\alpha)}{r} G_{B^c(a,r)}(x,y_0)\delta(y)^{\alpha/2} \int_{B(0,r)} \frac{u_1 - y_1}{(r - |u|)^{\alpha/2}|y-u|^n}\, du \\
=\ & \frac{c\,\alpha(2-\alpha)}{r^{\alpha/2}} G_{B^c(a,r)}(x,y_0)\delta(y)^{\alpha/2} \int_{B(0,1)} \frac{\widetilde{u}_1 - \widetilde{y}_1}{(1 - |\widetilde{u}|^2)^{\alpha/2}|\widetilde{y} - \widetilde{u}|^n}\, d\widetilde{u}. \qquad (15)
\end{aligned}$$

In the last equality, we used change of variables $u = r\widetilde{u}$ and $y = r\widetilde{y}$. Now using the spherical coordinates centered at $\widetilde{y} = (\widetilde{y}_1, 0, \cdots, 0)$, where $-2 < \widetilde{y}_1 < -1$, with θ the angle formed by the vector $\overrightarrow{\widetilde{y}\,\widetilde{u}}$ with the positive x_1-axis. Let $p = -\widetilde{y}_1 - 1$, the distance of $\widetilde{y}$ from $\partial B(0,1)$. If l is a ray starting from $\widetilde{y}$ with $\theta \in \left(0, \arccos\left(\frac{\sqrt{p(2+p)}}{1+p}\right)\right)$, it intersects with the unit sphere $\partial B(0,1)$ at two points with distances from $\widetilde{y}$ being $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$, respectively. A little geometry tells us that

$$r_1\, r_2 = p(2+p) \quad \text{and} \quad (1+p)\cos\theta = (r_1 + r_2)/2.$$

Therefore

$$r_1(\theta) = (1+p)\cos\theta - \sqrt{(1+p)^2\cos^2\theta - p(2+p)}$$

and

$$r_2(\theta) = (1+p)\cos\theta + \sqrt{(1+p)^2\cos^2\theta - p(2+p)}.$$

Thus the last integral in (15) becomes

$$\begin{aligned}
& c\int_0^{\arccos\left(\frac{\sqrt{p(2+p)}}{1+p}\right)} \sin^{n-1}\theta \left(\int_{r_1(\theta)}^{r_2(\theta)} \frac{\cos\theta}{(1-r^2-(1+p)^2+2r(1+p)\cos\theta)^{\alpha/2}}dr\right)d\theta \\
&= c\int_0^{\arccos\left(\frac{\sqrt{p(2+p)}}{1+p}\right)} \sin^{n-1}\theta \left(\int_{r_1(\theta)}^{(1+p)\cos\theta} \frac{\cos\theta}{(1-r^2-(1+p)^2+2r(1+p)\cos\theta)^{\alpha/2}}dr\right)d\theta \\
&\le c\int_0^{\arccos\left(\frac{\sqrt{p(2+p)}}{1+p}\right)} \left(\int_0^{\sqrt{(1+p)^2\cos^2\theta-p(2+p)}} \frac{\cos\theta}{(2s(1+p)\cos\theta-2r_1(\theta)s-s^2)^{\alpha/2}}\,ds\right)d\theta \\
&\le c\int_0^{\arccos\left(\frac{\sqrt{p(2+p)}}{1+p}\right)} \left(\int_0^{\sqrt{(1+p)^2\cos^2\theta-p(2+p)}} \frac{\cos\theta}{s^{\alpha/2}\left(\sqrt{(1+p)^2\cos^2\theta-p(2+p)}\right)^{\alpha/2}}\,ds\right)d\theta \\
&= \frac{c}{2-\alpha}\int_0^{\arccos\left(\frac{\sqrt{p(2+p)}}{1+p}\right)} \left((1+p)^2\cos^2\theta-p(2+p)\right)^{(1-\alpha)/2}\cos\theta\,d\theta \\
&\le \frac{c}{2-\alpha}\int_0^{1/(1+p)} (1-(1+p)^2v^2)^{(1-\alpha)/2}\,dv \\
&\le \frac{c}{2-\alpha}.
\end{aligned} \tag{16}$$

In the second to the last inequality, we used substitution $v = \sin\theta$.

By (8) and (13),

$$\begin{aligned}
&\int_{B(0,r)} |y-u|^2\, K_{B^c(0,r)}(y,u)du \\
&\le \int_{B(0,r)} |y-u|^2\, \frac{c\,\alpha(2-\alpha)(|y|^2-r^2)^{\alpha/2}}{(r^2-|u|^2)^{\alpha/2}|y-u|^n}du \\
&\le c\,\alpha(2-\alpha)\delta(y)^{\alpha/2}\int_{B(0,r)} \frac{1}{(r-|u|)^{\alpha/2}|y_0-u|^{n-2}}du \\
&= c\,\alpha(2-\alpha)\delta(y)^{\alpha/2}\,r^{2-\alpha/2}\int_{B(0,1)} \frac{1}{(1-|\widetilde{u}|^2)^{\alpha/2}|\widetilde{y}_0-\widetilde{u}|^{n-2}}d\widetilde{u}.
\end{aligned} \tag{17}$$

In the last equality, we used substitution $u = r\widetilde{u}$ and $y_0 = r\widetilde{y}_0$. Using spherical coordinates centered at $\widetilde{y}_0$ with θ denoting the angle formed by vector $\overrightarrow{\widetilde{y}_0\widetilde{u}}$ with the positive x_1-axis, the integral in (17) becomes

$$
\begin{aligned}
& c\int_0^{\pi/2} \sin^{n-2}\theta \left(\int_0^{2\cos\theta} \frac{r}{(2r\cos\theta - r^2)^{\alpha/2}} dr\right) d\theta \\
& \le c\int_0^{\pi/2} \left(\int_0^{2\cos\theta} \frac{r^{1-\frac{\alpha}{2}}}{(2\cos\theta - r)^{\alpha/2}} dr\right) d\theta \\
& \le \int_0^{\pi/2} \left(\int_0^{2\cos\theta} \frac{c}{(2\cos\theta - r)^{\alpha/2}} dr\right) d\theta \\
& = \frac{c}{2-\alpha}\int_0^{\pi/2} (2\cos\theta)^{1-\frac{\alpha}{2}} d\theta \\
& \le \frac{c}{2-\alpha}.
\end{aligned} \tag{18}
$$

Combining (12), (15)-(18) and Lemma 1, we have

$$
\begin{aligned}
II &= \left|\int_{B(0,r)} \left(G_{B^c(a,r)}(x,y) - G_{B^c(a,r)}(x,u)\right) K_{B^c(0,r)}(y,u)du\right| \\
&\le \frac{c\,\alpha}{r^{\alpha/2}} G_{B^c(a,r)}(x,y_0)\delta(y)^{\alpha/2} \\
&\le c\,\alpha^2 \frac{|y_0-a|^{\alpha/2}}{r^{\alpha/2}} \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{r^{\alpha/2}|x-y_0|^{n-\frac{\alpha}{2}}} \\
&\le c\alpha^2 \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}.
\end{aligned} \tag{19}
$$

Here in the last inequality we used the facts that $|x-y|$ is comparable to $|x-y_0|$, $r \ge |x-y|/8$ and that $|y_0-a|$ and r are comparable by a universal constant multiple that depends only on D. In fact, we have

$$
5r \le |x-y| - 3r \le |y_0-a| \le |x-y| + 3r \le (3 + \max\{8, \mathrm{diam}(D)/r_0\})\ r. \tag{20}
$$

On the other hand, by Lemma 1, inequality (9) above, Corollary 2 of Blumenthal, Getoor and Ray [1] and the scaling property of X,

$$
\begin{aligned}
I &= G_{B^c(a,r)}(x,y)P_y(\tau_{B^c(b,r)} = \infty) \\
&\le c\,\alpha \frac{|y-a|^{\alpha/2}}{r^{\alpha/2}} \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\frac{\alpha}{2}}} \frac{\Gamma(n/2)}{\Gamma((n-\alpha)/2)\Gamma(\alpha/2)} \int_0^{\frac{|y-b|^2}{r^2}-1} (u+1)^{-\frac{n}{2}} u^{\frac{\alpha}{2}-1} du \\
&\le c\,\alpha^2(n-\alpha)\frac{|y-a|^{\alpha/2}}{r^{\alpha/2}} \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\frac{\alpha}{2}}} \int_0^1 \left(\frac{|y-b|^2}{r^2} - 1\right)^{\alpha/2} v^{\frac{\alpha}{2}-1} dv \\
&\le c\,\alpha(n-\alpha)\ \frac{|y-a|^{\alpha/2}}{r^{\alpha/2}} \frac{\delta(x)^{\alpha/2}}{|x-y|^{n-\frac{\alpha}{2}}} \frac{\delta(y)^{\alpha/2}}{r^{\alpha/2}} \\
&\le \frac{c\,\alpha(n-\alpha)\,\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n},
\end{aligned}
$$

where in the last inequality we used the fact $r \geq |x-y|/8$ and that $|y-a|$ is comparable to r by a universal constant multiple that depends only on D. The latter fact can be proved by an argument similar to that in (20). This, combined with (5) and (19), shows

$$G_D(x,y) \leq c\,\alpha \frac{\delta(x)^{\alpha/2}\delta(y)^{\alpha/2}}{|x-y|^n}.$$

This proves the lemma. □

Proof of Theorem 1. Recall that we have the following trivial bound

$$G_D(x,y) \leq G_{\mathbf{R}^n}(x,y) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)} \frac{1}{|x-y|^{n-\alpha}} \leq \frac{c\alpha}{n-\alpha} \frac{1}{|x-y|^{n-\alpha}}.$$

The the conclusion of Theorem 1 now follows from Lemma 3 and the inequality above. □

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