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# Harmonic functions of subordinate killed Brownian motion

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**Abstract**

In this paper we study harmonic functions of subordinate killed Brownian motion in a domain $D$. We first prove that, when the killed Brownian semigroup in $D$ is intrinsic ultracontractive, all nonnegative harmonic functions of the subordinate killed Brownian motion in $D$ are continuous and then we establish a Harnack inequality for these harmonic functions. We then show that, when $D$ is a bounded Lipschitz domain, both the Martin boundary and the minimal Martin boundary of the subordinate killed Brownian motion in $D$ coincide with the Euclidean boundary $\partial D$. We also show that, when $D$ is a bounded Lipschitz domain, a boundary Harnack principle holds for positive harmonic functions of the subordinate killed Brownian motion in $D$.


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*Keywords:* Killed Brownian motions; Subordination; Fractional Laplacian; Harmonic functions; Green function; Martin kernel; Martin boundary; Harnack inequality; Boundary Harnack principle; Intrinsic ultracontractivity

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## 1. Introduction

Suppose that $X_t$ and $T_t$ are two independent processes, where $X_t$ is a Brownian motion in $\mathbb{R}^d$ and $T_t$ is an $\alpha/2$-stable subordinator starting at zero, $0<\alpha<2$. It is well known that $Y^\alpha(t) = X_{T_t}$ is a rotationally invariant $\alpha$-stable process whose generator is $-(-\Delta)^{\alpha/2}$, the fractional power of the negative Laplacian. The potential theory corresponding to the process $Y_\alpha$ is the Riesz potential theory of order $\alpha$.

Suppose that $D$ is a domain in $\mathbb{R}^d$, that is, an open connected subset of $\mathbb{R}^d$. We can kill the process $Y_\alpha(t)$ upon exiting $D$. The killed process $Y_\alpha^D(t)$ has been extensively studied in recent years and various deep properties have been obtained.

Let $\Delta|_D$ be the Dirichlet Laplacian in $D$. The fractional power $-(-\Delta|_D)^{\alpha/2}$ of the negative Dirichlet Laplacian is a very useful object in analysis and partial differential equations, see, for instance, [12,18]. There is a Markov process $Z_\alpha^D$ corresponding to $-(-\Delta|_D)^{\alpha/2}$ which can be obtained as follows: We first kill the Brownian motion $X$ at $\tau_D$, the first exit time of $X$ from the domain $D$, and then we subordinate the killed Brownian motion using the $\alpha/2$-stable subordinator $T_t$. Note that in comparison with $Y_\alpha^D(t)$ the order of killing and subordination has been reversed. For the differences between the processes $Y_\alpha^D(t)$ and $Z_\alpha^D(t)$, please see [17].

Until recently the process $Z_\alpha^D(t)$ had not been studied much. This process was first studied in [10], where, among other things, a relation between the harmonic functions of $Z_\alpha^D(t)$ and the classical harmonic functions in $D$ was established. In [11] (see also [9] and [14]) the domain of the Dirichlet form of $Z_\alpha^D(t)$ was identified when $D$ is a bounded smooth domain and $\alpha \neq 1$. In [17], the process $Z_\alpha^D(t)$ was studied in detail and upper and lower bounds on the jumping function $J_\alpha^D$ and the Green function $G_\alpha^D$ of $Z_\alpha^D(t)$ were established when $D$ is a bounded $C^{1,1}$ domain. However the upper and lower bounds provided in [17] are drastically different near the boundary. In [15], new lower bounds for $J_\alpha^D$ and $G_\alpha^D$ were established when $D$ is a bounded $C^{1,1}$ domain. These lower bounds differ from the upper bounds of [17] only by multiplicative constants and in this sense the bounds are sharp. Sharp bounds for $J_\alpha^D$ and $G_\alpha^D$ were also established in [15] when $D$ is an exterior $C^{1,1}$ domain. In [16] sharp bounds for $J_\alpha^D$ and $G_\alpha^D$ were established when $D$ is the domain above a $C^{1,1}$ function.

Despite the recent progress, there are still a lot of unanswered questions about the potential theory of $Z_\alpha^D$. For instance, are all nonnegative harmonic functions of $Z_\alpha^D$ continuous? Does the Harnack inequality hold for nonnegative harmonic functions of $Z_\alpha^D$? What can one say about the Martin boundary of the process $Z_\alpha^D$? Is there a boundary Harnack principle for positive harmonic functions of $Z_\alpha^D$?

In this paper we will study the potential theory of $Z_\alpha^D$ and we will, among other things, answer the four questions above.

The content of this paper is organized as follows. In Section 2 we introduce the notations and recall the main results from [10]. In Section 3 we improve the results of [10] and establish, under the assumption of intrinsic ultracontractivity, an one-to-one correspondence between the family of positive harmonic functions of killed Brownian motion and the family of positive harmonic functions of subordinate killed Brownian motion $Z_\alpha^D$. We prove that the Harnack inequality holds for $Z_\alpha^D$ in Section 4. Our proof of the Harnack inequality uses the intrinsic ultracontractivity in an essential way and differs from the existing proofs of the Harnack inequality in other settings. In Section 5 we prove that, when $D$ is a bounded Lipschitz domain, the Martin boundary and minimal Martin boundary of $Z_\alpha^D$ both coincide with the Euclidean boundary $\partial D$. In the last section we use the result from Section 5 to show that, when $D$ is a bounded Lipschitz domain, a boundary Harnack principle holds for positive harmonic functions of $Z_\alpha^D$.

## 2. Notation and setting

Let $X_t$ be the Brownian motion in $\mathbb{R}^d$, which runs twice as fast as the standard $d$-dimensional Brownian motion, and let $T_t$ be an $\alpha/2$-stable subordinator starting at zero, $0<\alpha<2$. We assume that $X$ and $T$ are independent. We are going to use $\mathbb{P}_x$ and $\mathbb{E}_x$ to stand for the probability and expectation with respect to the Brownian motion $X$ starting from $x$ respectively, $(P_t)_{t\geqslant 0}$ to stand for the transition semigroup of $X$, and $u_t^{\alpha/2}(s)$ to denote the density of $T_t$.

Let $D\subset\mathbb{R}^d$ be a bounded domain, and let $X_t^D$ be the Brownian motion killed upon exiting $D$. We define now the subordinate killed Brownian motion $Z_t^D$ as the process obtained by subordinating $X^D$ via the $\alpha/2$-stable subordinator $T_t$. More precisely, let $Z_\alpha^D(t) = X^D(T_t)$, $t\geqslant 0$. Then $Z_\alpha^D(t)$ is a symmetric Hunt process on $D$. If we use $(P_t^D)_{t\geqslant 0}$ and $G^D$ to denote the semigroup and potential operator of $X^D$, respectively, then the semigroup $Q_t^\alpha$ and potential operator $G_\alpha^D$ of $Z_\alpha^D$ are as follows:

$$Q_t^\alpha f(x) = \int_0^\infty P_s^D f(x) u_t^{\alpha/2}(s)\, ds$$

$$G_\alpha^D f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} P_t^D f(x)\, dt.$$

Obviously, $Q_t^\alpha$ has a density given by

$$q^\alpha(t,x,y) = \int_0^\infty p^D(s,x,y) u_t^{\alpha/2}(s)\, ds$$

and $G_\alpha^D$ has a density

$$G_\alpha^D(x,y) = \int_0^\infty q^\alpha(t,x,y)\, dt = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty p^D(t,x,y) t^{\alpha/2-1}\, dt,$$

where $p^D(t,x,y)$ is the transition density of $X^D$. We call $G_\alpha^D(x,y)$ the Green function of $Z_\alpha^D$. We are going to use $\tilde{\mathbb{E}}_x$ to stand for the expectation with respect to $Z_\alpha^D$ starting from $x$.

Recall that a Borel function $h$ on $D$ is said to be harmonic with respect to $Z_\alpha^D$ if $h$ is not identically infinite in $D$ and if for every relatively compact open subset $U \subset \bar{U} \subset D$,

$$h(x) = \tilde{\mathbb{E}}_x[h(Z_\alpha^D(\tilde{\tau}_U))], \quad \forall x \in U,$$

where $\tilde{\tau}_U = \inf\{t : Z_\alpha^D(t) \notin U\}$ is the first exit time of $U$. We are going to use $\mathscr{H}_\alpha^+$ to denote the collection of all the functions on $D$ which are harmonic with respect to $Z_\alpha^D$. A nonnegative function which is not identically infinite on $D$ is said to be excessive with respect to $Z_\alpha^D$ if (i) $Q_t^\alpha f(x) \leqslant f(x)$ for every $t>0$ and $x \in D$; and (ii) $\lim_{t\downarrow 0} Q_t^\alpha f(x) = f(x)$ for every $x \in D$. We are going to use $\mathscr{S}_\alpha$ to denote the collection of all the excessive functions with respect to $Z_\alpha^D$. Denote by $\mathscr{H}_+$ and $\mathscr{S}$ the collections of nonnegative harmonic functions and excessive functions with respect to $X^D$, respectively. Recall that $\mathscr{H}_\alpha^+ \subset \mathscr{S}_\alpha$ and $\mathscr{H}^+ \subset \mathscr{S}$.

An important connection between $\mathscr{S}_{2-\alpha}$ and $\mathscr{S}$ was established in [10]. The underlying assumption in that paper was that every excessive function for $X^D$ is purely excessive, i.e., for every $s \in \mathscr{S}$ it holds that $\lim_{t\to\infty} P_t^D s = 0$. The relationship between $\mathscr{S}_{2-\alpha}$ and $\mathscr{S}$, as well as $\mathscr{H}_{2-\alpha}^+$ and $\mathscr{H}^+$, can be summarized as follows (see [10], Theorems 2 and 3, and formula (17)):

**Theorem 2.1.** *If $s \in \mathscr{S}$, there exists a function $g \in \mathscr{S}_{2-\alpha}$ such that $s(x) = G_\alpha^D g(x)$ on $D$. The function $g$ is given by the formula*

$$g(x) = \frac{\alpha}{2\Gamma(1-\alpha/2)} \int_0^\infty t^{-\alpha/2-1}(s(x) - P_t^D s(x))\, dt. \tag{2.1}$$

*If, moreover, $s \in \mathscr{H}^+$,* then $g \in \mathscr{H}_{2-\alpha}^+$.

*Conversely if $g \in \mathscr{S}_{2-\alpha}$, and the function $s := G_\alpha^D g$ is not identically infinite, then $s \in \mathscr{S}$.*

*If, moreover, $g \in \mathscr{H}_{2-\alpha}^+$ and $s$ is not identically infinite, then $s \in \mathscr{H}^+$.*

*Further, if $g \in \mathscr{H}_{2-\alpha}^+$ is such that $G_\alpha^D g$ is not identically infinite, then $g$ is continuous.*

The first goal of this paper is to improve Theorem 2.1 by showing that if $g \in \mathscr{S}_{2-\alpha}$, then $s := G_\alpha^D g$ is not identically infinite, implying the full converse. This will be proved under the condition that *the semigroup $(P_t^D)_{t\geqslant 0}$ is intrinsic ultracontractive*. Intrinsic ultracontractivity plays a fundamental role in this paper: (i) It implies that every excessive function of $X^D$ is purely excessive, thus enabling us to use results from [10], and (ii) It is essentially used in the proof of Harnack inequality.

Let $\phi_0$ denote the positive eigenfunction corresponding to the smallest eigenvalue $\lambda_0$ of the Dirichlet Laplacian $-\Delta|_D$ such that $\int_D \phi_0^2(x)\, dx = 1$. Recall that the semigroup $(P_t^D)_{t\geqslant 0}$ is said to be intrinsic ultracontractive if there exists $C(t)>0$ such that

$$p^D(t,x,y)\leqslant C(t)\phi_0(x)\phi_0(y), \quad \forall t>0, x,y\in D. \tag{2.2}$$

It is well known that (see, for instance, [6]) when $(P_t^D)_{t\geqslant 0}$ is intrinsic ultracontractive, there exists $\tilde{C}(t)>0$ such that

$$p^D(t,x,y)\geqslant \tilde{C}(t)\phi_0(x)\phi_0(y), \quad \forall t>0, x,y\in D. \tag{2.3}$$

The assumption that $(P_t^D)_{t\geqslant 0}$ is intrinsic ultracontractive is a mild geometric assumption on $D$. It is well known that (see, for instance, [1]), when $D$ is a bounded Lipschitz domain, or a Hölder domain of order 0, or a uniformly Hölder domain of order $\beta\in(0,2)$, $(P_t^D)_{t\geqslant 0}$ is intrinsic ultracontractive. We show now that it implies that every excessive function of $X^D$ is purely excessive. To see this, let us recall the concept of $s$-conditioned Brownian motion.

For any $s\in\mathscr{S}$,

$$p^s(t,x,y)=\frac{p^D(t,x,y)s(y)}{s(x)}$$

determines a Markov semigroup. The Markov process corresponding to this semigroup is called the $s$-conditioned Brownian motion and we use $X^s$ to denote this process. We are going to use $\mathbb{E}_x^s$ to denote the expectation with respect to the law of this process starting from $x$ and $\zeta^s$ to denote the lifetime of this process. When $s=G^D(\cdot,y)$ for some $y\in D$, we use $\mathbb{E}_x^y$ to stand for $\mathbb{E}_x^s$ and $\zeta^y$ to stand for $\zeta^s$. As a consequence of the intrinsic ultracontractivity of $(P_t^D)_{t\geqslant 0}$ we get that for any $s\in\mathscr{S}$,

$$\lim_{t\uparrow\infty} e^{\lambda_0 t}\mathbb{P}_x^s(\zeta^s>t)=\frac{\phi_0(x)}{s(x)}\int_D \phi_0(y)s(y)\,dy. \tag{2.4}$$

For the above facts on intrinsic ultracontractivity, one can refer, for instance, to [1,6,7]. Since $s$ is not identically infinite in $D$, we have $s(x)<\infty$ for some $x\in D$, and so

$$\infty>s(x)\geqslant\int_D p^D(t,x,y)s(y)\,dy\geqslant\tilde{C}(t)\phi_0(x)\int_D \phi_0(y)s(y)\,dy,$$

implying

$$\int_D \phi_0(y)s(y)\,dy<\infty.$$

Consequently, it follows from (2.4) that $\mathbb{P}_x^s(\zeta^s=\infty)=0$, which implies that $\lim_{t\uparrow\infty} P_t^D s(x)=0$ for every $x\in D$. Therefore under the assumption that $(P_t^D)$ is intrinsic ultracontractive, every excessive function $s$ of $X^D$ is purely excessive, which is the basic assumption in [10].

## 3. Correspondence between $\mathscr{H}_\alpha^+$ and $\mathscr{H}^+$

In this section we always assume that $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. We start this section with the following improvement of Theorem 2.1.

**Theorem 3.1.** *Suppose that $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. If $s\in\mathscr{S}$, there exists a function $g\in\mathscr{S}_{2-\alpha}$ such that $s(x)=G_\alpha^D g(x)$ on $D$. The function $g$ is given by the formula*

$$g(x)=\frac{\alpha}{2\Gamma(1-\alpha/2)}\int_0^\infty t^{-\alpha/2-1}(s(x)-P_t^D s(x))\,dt.$$

*If, moreover, $s\in\mathscr{H}^+$, then $g\in\mathscr{H}_{2-\alpha}^+$.*

*Conversely if $g\in\mathscr{S}_{2-\alpha}$, then the function $s$ defined by $s=G_\alpha^D g$ is in $\mathscr{S}$. If, moreover, $g\in\mathscr{H}_{2-\alpha}^+$, then $s\in\mathscr{H}^+$.*

**Proof.** In view of Theorem 2.1 we only need to show that, whenever $g\in\mathscr{S}_{2-\alpha}$, the function $s=G_\alpha^D g$ is not identically infinite on $D$.

Since $g$ is in $\mathscr{S}_{2-\alpha}$, there exists $x_0\in D$ such that for every $t>0$,

$$\infty>g(x_0)\geqslant Q_t^{2-\alpha}g(x_0)=\int_0^\infty P_s^D g(x_0)u_t^{1-\alpha/2}(s)\,ds.$$

Thus $P_s^D g(x_0)$ is finite for almost every $s\in(0,\infty)$. Since $(P_t^D)_{t\geqslant 0}$ is intrinsic ultracontractive, by Theorem 4.2.5 of [5] there exists $T>0$ such that

$$\tfrac{1}{2}e^{-\lambda_0 t}\phi_0(x)\phi_0(y)\leqslant p^D(t,x,y)\leqslant\tfrac{3}{2}e^{-\lambda_0 t}\phi_0(x)\phi_0(y),\quad t\geqslant T,x,y\in D. \tag{3.1}$$

Take a $t\geqslant T$ such that $P_t^D g(x_0)<\infty$. Then

$$\infty>P_t^D g(x_0)=\int_D p^D(t,x,y)g(y)\,dy\geqslant\tfrac{1}{2}e^{-\lambda_0 t}\phi_0(x_0)\int_D\phi_0(y)g(y)\,dy,$$

so we have $\int_D\phi_0(y)g(y)dy<\infty$. Consequently,

$$\begin{aligned}\int_D G_\alpha^D g(x)\phi_0(x)\,dx&=\int_D g(x)G_\alpha^D\phi_0(x)\,dx\\&=\frac{1}{\Gamma(\alpha/2)}\int_D g(x)\int_0^\infty t^{\alpha/2-1}P_t^D\phi_0(x)\,dt\,dx\\&=\frac{1}{\Gamma(\alpha/2)}\int_D g(x)\int_0^\infty t^{\alpha/2-1}e^{-\lambda_0 t}\phi_0(x)\,dt\,dx\\&=\frac{1}{\Gamma(\alpha/2)}\int_D\phi_0(y)g(y)\,dy\int_0^\infty t^{\alpha/2-1}e^{-\lambda_0 t}\,dt<\infty.\end{aligned}$$

Therefore $s=G_\alpha^D g$ is not identically infinite in $D$. □

It follows from Theorem 3.1 that, when $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive, $G_\alpha^D$ is a bijection from $\mathscr{S}_{2-\alpha}$ to $\mathscr{S}$, and is also a bijection from $\mathscr{H}_{2-\alpha}^+$ to $\mathscr{H}^+$. We are going to use $(G_\alpha^D)^{-1}$ to denote the inverse map and so we have for any $s \in \mathscr{S}$

$$(G_\alpha^D)^{-1}s(x) = \frac{\alpha}{2\Gamma(1-\alpha/2)} \int_0^\infty t^{-\alpha/2-1}(s(x) - P_t^D s(x))\, dt.$$

Although the map $G_\alpha^D$ is order preserving, we do not know if the inverse map $(G_\alpha^D)^{-1}$ is order preserving on $\mathscr{S}$. But from the formula above we can see that $(G_\alpha^D)^{-1}$ is order preserving on $\mathscr{H}^+$.

**Theorem 3.2.** *If $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive, then every $h \in \mathscr{H}_\alpha^+$ is continuous in $D$.*

**Proof.** From Theorem 3.1 we know that $G_{2-\alpha}^D h$ is not identically infinite in $D$. Now the conclusion follows from Theorem 2.1. □

In the remaining part of this section we prove several results that complement the results from [10]. We start with another form of formula (2.1).

**Proposition 3.3.** *Suppose that $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. If $s \in \mathscr{S}$, then $s = G_\alpha^D g$, where*

$$g(x) = \frac{1}{\Gamma(1-\alpha/2)} s(x)\mathbb{E}_x^s[(\zeta^s)^{-\alpha/2}]. \tag{3.2}$$

**Proof.** From Theorem 2.1, if $s \in \mathscr{S}$, then $s = G_\alpha^D g$, where $g$ is given by (2.1). By the definition of $s$-conditioned Brownian motion we know that

$$\mathbb{P}_x^s(\zeta^s \leqslant t) = 1 - \frac{1}{s(x)} \int_D p^D(t,x,y)s(y)\, dy,$$

thus

$$s(x)P_x^s(\zeta^s \leqslant t) = s(x) - \int_D p^D(t,x,y)s(y)\, dy.$$

Therefore we have

$$
\begin{aligned}
g(x) &= \frac{\alpha}{2\Gamma(1-\alpha/2)} \int_0^\infty t^{-\alpha/2-1} s(x) P_x^s(\zeta^s \leqslant t)\, dt \\
&= \frac{1}{\Gamma(1-\alpha/2)} s(x) \mathbb{E}_x^s[(\zeta^s)^{-\alpha/2}]. \qquad \square
\end{aligned}
$$

As a consequence of this result, we immediately get the following corollaries.

**Corollary 3.4.** *When $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive, the Green function $G_{2-\alpha}^D$ of $Z_{2-\alpha}^D$ can be written in the following form*:

$$
G_{2-\alpha}^D(x,y) = (G_\alpha^D)^{-1}(G^D(\cdot,y))(x) = \frac{1}{\Gamma(1-\alpha/2)} G^D(x,y) \mathbb{E}_x^y[(\zeta^y)^{-\alpha/2}],
$$

*where $G^D(x,y)$ denotes the Green function of $X^D$.*

**Proof.** Using the Markov property and Fubini's theorem we can easily get that for any nonnegative function $f$ on $D$,

$$
G_\alpha^D G_{2-\alpha}^D f(x) = G^D f(x), \quad x \in D.
$$

Now the conclusion of this corollary follows immediately from Proposition 3.3. $\square$

Recall that $\tau_D := \inf\{t : X_t \notin D\}$ is the first exit time of $D$ for the Brownian motion $X$.

**Corollary 3.5.** *Suppose that $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. If $h(x) = \mathbb{E}_x[f(X_{\tau_D})]$ for some bounded function $f$ on $\partial D$, then $h = G_\alpha^D g$, where*

$$
g(x) = \frac{1}{\Gamma(1-\alpha/2)} \mathbb{E}_x[f(X_{\tau_D}) \cdot \tau_D^{-\alpha/2}], \quad x \in D.
$$

**Proof.** When $f$ is nonnegative, the conclusion follows immediately from Proposition 3.3. For the general case we decompose $f$ into its positive and negative parts. $\square$

Repeating the proof of (19) in [10] we immediately get the following corollary of the result above.

**Corollary 3.6.** *Suppose that $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. Then for any nonnegative function $f$ on $\partial D$ and any $\alpha \in (0,2)$*

*we have*

$$\frac{\alpha\Gamma(\alpha/2)\Gamma(1-\alpha/2)}{2}\int_D \mathbb{E}_x[f(X_{\tau_D})]\,dx = \int_D \mathbb{E}_x[f(X_{\tau_D})\tau_D^{-\alpha/2}]\cdot\mathbb{E}_x[\tau_D^{\alpha/2}]\,dx.$$

**Proposition 3.7.** *Suppose that D is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive and that D is regular in the sense that $\mathbb{P}_z(\tau_D = 0) = 1$ for every $z\in\partial D$. If $h = h_1 - h_2$ where $h_i\in\mathscr{H}^+_{2-\alpha}$, $i = 1, 2$, is such that for every $z\in\partial D$,*

$$\lim_{D\ni x\to z}\frac{h(x)}{\mathbb{E}_x[\tau_D^{-\alpha/2}]} = 0,$$

*then $h\equiv 0$.*

**Proof.** Put

$$\kappa_\alpha^D(x) = \frac{1}{\Gamma(1-\alpha/2))}\mathbb{E}_x[\tau_D^{-\alpha/2}],\quad x\in D. \tag{3.3}$$

Then by taking $f = 1$ in Corollary 3.5 we get that $G_\alpha^D\kappa_\alpha^D(x) = 1$ for every $x\in D$. For any $x\in D$, let $\delta(x)$ be the distance between $x$ and $\partial D$. For any $\delta>0$, let $D_\delta := \{x\in D : \delta(x)>\delta\}$. By using the compactness of $\partial D$ it is not difficult to show that, for any $\varepsilon>0$, there exists $\delta>0$ such that

$$|h(x)|\leqslant\varepsilon\kappa_\alpha^D(x),\quad x\in D\backslash D_\delta.$$

For any $x\in D\backslash D_\delta$,

$$G_\alpha^D|h(x)|\leqslant G_\alpha^D(|h|1_{D_\delta})(x) + \varepsilon G_\alpha^D\kappa_\alpha^D(x)\leqslant G_\alpha^D(|h|1_{D_\delta})(x) + \varepsilon.$$

Since $G_\alpha^D(x,y)$ is bounded on $(D\backslash D_{\delta/2})\times D_\delta$ and $\lim_{x\to\partial D} G_\alpha^D(x,y) = 0$ for any $y\in D$, we get by the dominated convergence theorem that $\lim_{x\to\partial D} G_\alpha^D(|h|1_{D_\delta})(x) = 0$. This shows that $\lim_{x\to\partial D} G_\alpha^D h(x) = 0$. Since $G_\alpha^D h(x) = G_\alpha^D h_1(x) - G_\alpha^D h_2(x)$ is a difference of two functions in $\mathscr{H}^+$, it is a harmonic function for $X^D$. Therefore, $G_\alpha^D h = 0$, which implies that $h = 0$ almost everywhere in $D$. Since $h$ is continuous by Theorem 3.2, we get $h\equiv 0$. □

The proposition above implies, in particular, that there are no non-trivial bounded functions in $\mathscr{H}^+_\alpha - \mathscr{H}^+_\alpha$. The following proposition says that there are no nontrivial bounded functions in $\mathscr{H}_\alpha$.

**Proposition 3.8.** *Suppose that D is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. If $h\in\mathscr{H}_\alpha$ is bounded, then $h\equiv 0$.*

**Proof.** It was shown in [17] that the function defined in (3.3) is the killing function of the process $Z_\alpha^D$. Therefore for any compact subset $K$ of $D$ we have

$$\tilde{\mathbb{P}}_x(Z_\alpha^D(\tilde{\zeta}-)\in K) = G_\alpha^D(\kappa_\alpha^D \cdot 1_K)(x), \quad x\in D.$$

Recall that $G_\alpha^D \kappa_\alpha^D(x) = 1$ for every $x\in D$. By taking an increasing sequence of compact sets $K_n$ with $K_n \uparrow D$, we get

$$\tilde{\mathbb{P}}_x(Z_\alpha^D(\tilde{\zeta}-)\in D) = 1, \quad x\in D,$$

where $\tilde{\zeta}$ is the lifetime of $Z_\alpha^D$. Take an increasing sequence of open sets $D_n$ such that $D_n \subset \overline{D_n} \subset D_{n+1} \subset \overline{D_{n+1}} \subset D$ for all $n \geqslant 1$ and $D_n \uparrow D$. Let $\tilde{\tau}_n = \inf\{t : Z_t^D \notin D_n\}$, then $\tilde{\tau}_n \uparrow \tilde{\zeta}$ and from the display above we know that $\tilde{\mathbb{P}}_x(\tilde{\zeta} = \tilde{\tau}_n$ for some $n \geqslant 1) = 1$ for every $x\in D$. Therefore for every $x \in D$,

$$|h(x)| = |\tilde{\mathbb{E}}_x[h(Z_\alpha^D(\tau_n))]| \leqslant ||h||_\infty \tilde{\mathbb{P}}_x(\tilde{\tau}_n < \tilde{\zeta}) \to 0.$$

The proof is now complete. □

The function in $\mathscr{H}_{2-\alpha}^+$ playing the role of a constant function $1 \in \mathscr{H}^+$ is $(G_\alpha^D)^{-1}1$. It is shown in [10] (Example 1) that this function is equal to $\kappa_\alpha^D(x) = 1/(\Gamma(1-\alpha/2))\mathbb{E}_x(\tau_D^{-\alpha/2})$. While the classical formulation of the Dirichlet problem for harmonic functions for $Z_{2-\alpha}^D$ is impossible in view of the last proposition, the following reformulation seems appropriate: for any given bounded function $f$ on $\partial D$, find a function $h \in \mathscr{H}_{2-\alpha}$ such that

$$\lim_{D\ni x\to z} \frac{h(x)}{\mathbb{E}_x[\tau_D^{-\alpha/2}]} = f(z).$$

The following two propositions show that classical conditions for solvability of the Dirichlet problem are sufficient for this reformulated Dirichlet problem as well.

**Proposition 3.9.** *Suppose that $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive and that $z \in \partial D$ is regular for $D^c$, that is, $\mathbb{P}_z(\tau_D = 0) = 1$. Then for any bounded function $f$ on $\partial D$ which is continuous at $z$, we have*

$$\lim_{D\ni x\to z} \frac{\mathbb{E}_x[f(X_{\tau_D}) \cdot \tau_D^{-\alpha/2}]}{\mathbb{E}_x[\tau_D^{-\alpha/2}]} = f(z).$$

**Proof.** It is easy to see that

$$\lim_{D\ni x\to z} \mathbb{E}_x[\tau_D^{-\alpha/2}] = \infty. \tag{3.4}$$

In fact, since the function $x \mapsto \mathbb{P}_x(s > \tau_D)$ is upper semicontinuous in $\mathbb{R}^D$ for any $s > 0$, we have

$$\limsup_{x \to z} \mathbb{P}_x(\tau_D > s) \leqslant \mathbb{P}_z(\tau_D > s) = 0.$$

Hence there exists $\delta > 0$ such that

$$\mathbb{P}_x(\tau_D > s) < \tfrac{1}{2}, \quad x \in B(z, \delta) \cap D,$$

consequently

$$\mathbb{E}_x[\tau_D^{-\alpha/2}] \geqslant \mathbb{E}_x[\tau_D^{-\alpha/2}; s > \tau_D] \geqslant s^{-\alpha/2}\mathbb{P}_x(s > \tau_D) \geqslant \tfrac{1}{2} s^{-\alpha/2}$$

whenever $x \in B(z, \delta) \cap D$. Therefore (3.4) is valid.

Now for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(w) - f(z)| < \frac{\varepsilon}{2}, \quad w \in B(z, \delta) \cap \partial D.$$

For any $x \in B(z, \delta/2)$ we have $\tau_{B(x,\delta/2)} \leqslant \tau_{B(z,\delta)}$. Therefore

$$\begin{aligned}
&\mathbb{E}_x[|f(X(\tau_D)) - f(z)|\tau_D^{-\alpha/2}] \\
&\quad = \mathbb{E}_x[|f(X(\tau_D)) - f(z)|\tau_D^{-\alpha/2}; \tau_D \leqslant \tau_{B(z,\delta)}] \\
&\qquad + \mathbb{E}_x[|f(X(\tau_D)) - f(z)|\tau_D^{-\alpha/2}; \tau_{B(z,\delta)} < \tau_D] \\
&\quad \leqslant \mathbb{E}_x[|f(X(\tau_D)) - f(z)|\tau_D^{-\alpha/2}; \tau_D \leq \tau_{B(z,\delta)}] \\
&\qquad + \mathbb{E}_x[|f(X(\tau_D)) - f(z)|\tau_D^{-\alpha/2}; \tau_{B(x,\delta/2)} < \tau_D] \\
&\quad \leqslant \frac{\varepsilon}{2}\mathbb{E}_x[\tau_D^{-\alpha/2}] + 2||f||_\infty \mathbb{E}_x[\tau_{B(x,\delta/2)}^{-\alpha/2}] \\
&\quad = \frac{\varepsilon}{2}\mathbb{E}_x[\tau_D^{-\alpha/2}] + 2||f||_\infty \mathbb{E}_0[\tau_{B(0,\delta/2)}^{-\alpha/2}]
\end{aligned}$$

By (3.4) we can take $\eta > 0$ such that

$$\mathbb{E}_x[\tau_D^{-\alpha/2}] \geqslant \frac{2}{\varepsilon} 2||f||_\infty \mathbb{E}_0[\tau_{B(0,\delta/2)}^{-\alpha/2}], \quad x \in B(z, \eta) \cap D.$$

Then whenever $x \in B(z, \eta \wedge \frac{\delta}{2}) \cap D$, we have

$$\frac{2||f||_\infty \mathbb{E}_0[\tau_{B(0,\delta)/2}^{-\alpha/2}]}{E_x[\tau_D^{-\alpha/2}]} \leqslant \frac{\varepsilon}{2}.$$

Therefore we have

$$\frac{\mathbb{E}_x[|f(X(\tau_D)) - f(z)|\tau_D^{-\alpha/2}]}{E_x[\tau_D^{-\alpha/2}]} \leqslant \varepsilon, \quad x \in B\left(z, \eta \wedge \frac{\delta}{2}\right) \cap D.$$

The proof is now complete. □

**Proposition 3.10.** *Suppose that D is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive and that D is regular in the sense that $\mathbb{P}_z(\tau_D = 0) = 1$ for every $z \in \partial D$. Let f be a continuous function on $\partial D$. The function $\mathbb{E}_x[f(X_{\tau_D}) \cdot \tau_D^{-\alpha/2}]$ is the unique function $h \in \mathscr{H}_{2-\alpha}$ such that*

$$\lim_{D \ni x \to z} \frac{h(x)}{\mathbb{E}_x[\tau_D^{-\alpha/2}]} = f(z).$$

**Proof.** Without loss of generality, we may assume that $f \geqslant 0$. The last proposition and Corollary 3.5 show that $\mathbb{E}_x[f(X_{\tau_D})\tau_D^{-\alpha/2}]$ is a solution of the problem. To prove uniqueness, suppose that $h_1$ and $h_2$ are two solutions. Then $h_1, h_2 \in \mathscr{H}^+_{2-\alpha}$, and $h = h_1 - h_2$ satisfies

$$\lim_{D \ni x \to z} \frac{h(x)}{\mathbb{E}_x[\tau_D^{-\alpha/2}]} = 0,$$

By Proposition 3.7, $h \equiv 0$. □

Using the fact that $G^D_{2-\alpha}h$ is not identically infinite in $D$ for any $h \in \mathscr{H}^+_\alpha$, we have the following improvement of Theorem 4 in [10].

**Proposition 3.11.** *Suppose that D is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. For $\alpha \in (0, 2)$, if $h \in \mathscr{H}^+_\alpha$, then for any $\phi \in C_0^\infty(D)$,*

$$\int_D f(x)(-\Delta|_D)^{\alpha/2}\phi(x)\,dx = 0.$$

## 4. Harnack inequality

In this section we are going to prove the Harnack inequality for positive harmonic functions for the process $Z^D_\alpha$ under the assumption that $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. The proof we offer uses the intrinsic ultracontractivity in an essential way, and differs from the existing proofs of Harnack inequalities in other settings.

**Lemma 4.1.** *Suppose that $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. There exists a constant $C_3>0$ depending only on $D$ and $\alpha$ such that*

$$G^D_{2-\alpha}s \leqslant C_3 s, \quad \forall s \in \mathscr{S}_\alpha. \tag{4.1}$$

**Proof.** Let $T$ be the constant in (3.1). For any nonnegative function $f$,

$$G^D_\alpha f(x) = \frac{1}{\Gamma(\alpha/2)}\left(\int_0^T t^{\alpha/2-1}P^D_t f(x)\,dt + \int_T^\infty t^{\alpha/2-1}P^D_t f(x)\,dt\right).$$

We obviously have

$$\int_0^T t^{\alpha/2-1}P^D_t f(x)\,dt \geqslant T^{\alpha/2-1}\int_0^T P^D_t f(x)\,dt.$$

By using (3.1) we see that

$$\int_T^\infty t^{\alpha/2-1}P^D_t f(x)\,dt \geqslant \left(\frac{1}{2}\int_T^\infty t^{\alpha/2-1}e^{-\lambda_0 t}dt\right)\int_D \phi_0(x)\phi_0(y)f(y)\,dy.$$

and

$$\int_T^\infty P^D_t f(x)\,dt \leqslant \left(\frac{3}{2}\int_T^\infty e^{-\lambda_0 t}dt\right)\int_D \phi_0(x)\phi_0(y)f(y)\,dy.$$

Combining the three displays above we get that there exists a constant $C>0$ depending only on $D$ and $\alpha$ such that

$$G^D f(x) \leqslant C G^D_\alpha f(x). \tag{4.2}$$

Since $G^D f(x) = G^D_{2-\alpha}G^D_\alpha f(x)$, (4.2) can be rewritten as

$$G^D_{2-\alpha}(G^D_\alpha f)(x) \leqslant C G^D_\alpha f(x).$$

Since any $s \in \mathscr{S}_\alpha$ is the limit of an increasing sequence of functions of the form $G^D_\alpha f$, the lemma follows. $\square$

**Lemma 4.2.** *Suppose $D$ is a bounded domain such that $(P_t^D)$ is intrinsic ultracontractive. If $s \in \mathscr{S}_\alpha$, then for any $x \in D$,*

$$s(x) \geqslant \frac{1}{2C_3}e^{-\lambda_0 T}\lambda_0^{-\alpha/2}\phi_0(x)\int_D s(y)\phi_0(y)\,dy,$$

*where $T$ is the constant in* (3.1) *and $C_3$ is the constant in* (4.1).

**Proof.** From the lemma above we know that, for every $x \in D$, $G^D_{2-\alpha}s(x) \leqslant C_3 s(x)$, where $C_3$ is the constant in (4.1). Since $G^D_{2-\alpha}s$ is in $\mathscr{S}$, we have

$$G^D_{2-\alpha}s(x) \geqslant \int_D p^D(T,x,y) G^D_{2-\alpha}s(y)\,dy \geqslant \frac{1}{2} e^{-\lambda_0 T} \phi_0(x) \int_D \phi_0(y) G^D_{2-\alpha}s(y)\,dy.$$

Hence

$$\begin{aligned} C_3 s(x) \geqslant G^D_{2-\alpha}s(x) &\geqslant \frac{1}{2} e^{-\lambda_0 T} \phi_0(x) \int_D \phi_0(y) G^D_{2-\alpha}s(y)\,dy \\ &= \frac{1}{2} e^{-\lambda_0 T} \phi_0(x) \int_D s(y) G^D_{2-\alpha}\phi_0(y)\,dy \\ &= \frac{1}{2} e^{-\lambda_0 T} \lambda_0^{-\alpha/2} \phi_0(x) \int_D s(y)\phi_0(y)\,dy. \quad \square \end{aligned}$$

**Theorem 4.3.** *Suppose D is a bounded domain such that* $(P^D_t)$ *is intrinsic ultracontractive. For any compact subset K of D, there exists a constant C depending on* $\alpha$*, K and D such that for any* $h \in \mathscr{H}^+_\alpha$,

$$\sup_{x \in K} h(x) \leqslant C \inf_{x \in K} h(x).$$

**Proof.** If the conclusion of the theorem were not true, for any $n \geqslant 1$, there exist $h_n \in \mathscr{H}^+_\alpha$ such that

$$\sup_{x \in K} h_n(x) \geqslant n 2^n \inf_{x \in K} h_n(x). \tag{4.3}$$

By the lemma above, we may assume without loss of generality that

$$\int_D h_n(y)\phi_0(y)\,dy = 1, \quad n \geqslant 1.$$

Define

$$h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x), \quad x \in D.$$

Then

$$\int_D h(y)\phi_0(y)\,dy = 1$$

and so $h \in \mathscr{H}^+_\alpha$. By (4.3) and the lemma above, for every $n \geqslant 1$, there exists $x_n \in K$ such that $h_n(x_n) \geqslant n 2^n c_1$ where

$$c_1 = \frac{1}{2C_3} e^{-\lambda_0 T} \lambda_0^{-\alpha/2} \inf_{x \in K} \phi_0(x)$$

with $T$ as in (3.1) and $C_3$ in (4.1). Therefore we have $h(x_n) \geqslant nc_1$. Since $K$ is compact, there is a convergent subsequence of $x_n$. Let $x_0$ be the limit of this convergent subsequence. Theorem 3.2 implies that $h$ is continuous, and so we have $h(x_0) = \infty$. This is a contradiction. So the conclusion of the theorem is valid. □

## 5. Martin boundary

In this section we are going to assume that $D$ is a bounded Lipschitz domain. Fix a point $x_0 \in D$ and set

$$M^D(x,y) = \frac{G^D(x,y)}{G^D(x_0,y)}, \quad x,y \in D.$$

It is well known that the limit

$$\lim_{D \ni y \to z} M^D(x,y)$$

exists for every $x \in D$ and $z \in \partial D$. The function $M^D(x,z) := \lim_{D \ni y \to z} M^D(x,y)$ on $D \times \partial D$ defined above is called the Martin kernel of $X^D$ based at $x_0$. The Martin boundary and minimal Martin boundary of $X^D$ both coincide with the Euclidean boundary $\partial D$. For these and other results about the Martin boundary of $X^D$, one can see [2]. One of the goals of this section is to determine the Martin boundary of $Z^D_\alpha$.

By using the Harnack inequality, one can easily show that (see, for instance, [8, p. 17], if $h_n$ is a sequence of functions in $\mathscr{H}^+$ converging pointwise to a function $h \in \mathscr{H}^+$, then $(h_n)$ is locally uniformly bounded in $D$ and equicontinuous at every point in $D$. Using this one can get that, if $h_n$ is a sequence of functions in $\mathscr{H}^+$ converging pointwise to a function $h \in \mathscr{H}^+$, then $(h_n)$ converges to $h$ uniformly on compact subsets of $D$. We are going to use this fact below.

**Lemma 5.1.** *Suppose that $D$ is a bounded Lipschitz domain, $x_0 \in D$ is a fixed point.*

(a) *Let $(x_n)$ be a sequence of points in $D$ converging to $x \in D$ and $(h_n)$ be a sequence of functions in $\mathscr{H}^+$ with $h_n(x_0) = 1$ for all $n$. If the sequence $(h_n)$ converges to a function $h \in \mathscr{H}^+$, then for each $t > 0$*

$$\lim_n P^D_t h_n(x_n) = P^D_t h(x).$$

(b) *If $(y_n,\ n \geqslant 1)$ is a sequence of points in $D$ such that $\lim_n y_n = z \in \partial D$, then for each $t > 0$ and for each $x \in D$*

$$\lim_n P^D_t \left( \frac{G^D(\cdot, y_n)}{G^D(x_0, y_n)} \right)(x) = P^D_t (M^D(\cdot, z))(x).$$

**Proof.** (a) For each $n \in \mathbb{N}$, since $h_n(x_0) = 1$, there exists a probability measure $\mu_n$ on $\partial D$ such that

$$h_n(x) = \int_{\partial D} M^D(x,z)\mu_n(dz), \quad x \in D.$$

Similarly, there exists a probability measure $\mu$ on $\partial D$ such that

$$h(x) = \int_{\partial D} M^D(x,z)\mu(dz), \quad x \in D.$$

Let $D_0$ be a relatively compact open subset of $D$, $x_0 \in D_0$, and also $x, x_n \in D_0$. Then

$$\begin{aligned} &|P_t^D h_n(x_n) - P_t^D h(x)| \\ &\quad = \left| \int_D p^D(t,x_n,y)h_n(y)\,dy - \int_D p^D(t,x,y)h(y)\,dy \right| \\ &\quad \leqslant \left| \int_{D_0} p^D(t,x_n,y)h_n(y)\,dy - \int_{D_0} p^D(t,x,y)h(y)\,dy \right| \\ &\qquad + \int_{D \setminus D_0} p^D(t,x_n,y)h_n(y)\,dy + \int_{D \setminus D_0} p^D(t,x,y)h(y)\,dy. \end{aligned}$$

Recall that (see Section 6.2 of [4], for instance) there exists a constant $c > 0$ such that

$$\frac{G^D(x,y)G^D(y,w)}{G^D(x,w)} \leqslant c\left( \frac{1}{|x-y|^{d-2}} + \frac{1}{|y-w|^{d-2}} \right), \quad x,y,w \in D. \tag{5.1}$$

From this and the definition of the Martin kernel we immediately get

$$G^D(x_0,y)M^D(y,z) \leqslant c\left( \frac{1}{|x_0-y|^{d-2}} + \frac{1}{|y-z|^{d-2}} \right), \quad y \in D, z \in \partial D.$$

Now using (2.2), the inequality above and the fact (see [5, Theorem 4.6.11, p. 131]) that

$$\phi_0(y) \leqslant c\frac{G^D(x_0,y)}{\phi_0(x_0)}, \quad y \in D, \tag{5.2}$$

we get that for any $u \in D$,

$$\begin{aligned}
&\int_{D\setminus D_0} p^D(t,u,y)h(y)\,dy \\
&\leqslant C(t)\phi_0(u)\int_{D\setminus D_0} \phi_0(y)h(y)\,dy \\
&= C(t)\phi_0(u)\int_{D\setminus D_0} dy\,\phi_0(y)\int_{\partial D} M^D(y,z)\mu(dz) \\
&= C(t)\phi_0(u)\int_{\partial D} \mu(dz)\int_{D\setminus D_0} \phi_0(y)M^D(y,z)\,dy \\
&\leqslant C(t)c\int_{\partial D} \mu(dz)\int_{D\setminus D_0} G_D(x_0,y)M^D(y,z)\,dy \\
&\leqslant C(t)c\int_{\partial D} \mu(dz)\int_{D\setminus D_0}\left(\frac{1}{|y-z|^{d-2}}+\frac{1}{|x_0-y|^{d-2}}\right)dy \\
&\leqslant C(t)c\int_{\partial D} \mu(dz)\sup_{z\in\partial D}\int_{D\setminus D_0}\left(\frac{1}{|y-z|^{d-2}}+\frac{1}{|x_0-y|^{d-2}}\right)dy \\
&= C(t)c\sup_{z\in\partial D}\int_{D\setminus D_0}\left(\frac{1}{|y-z|^{d-2}}+\frac{1}{|x_0-y|^{d-2}}\right)dy.
\end{aligned}$$

The same estimate holds with $h_n$ instead of $h$. For a given $\varepsilon > 0$ choose $D_0$ large enough so that the last line in the display above is less than $\varepsilon$. Put $A = \sup_{D_0} h$. Take $n_0 \in \mathbb{N}$ large enough so that for all $n \geqslant n_0$ we have

$$|p^D(t,x_n,y) - p^D(t,x,y)| \leqslant \varepsilon \quad \text{and } |h_n(y) - h(y)| < \varepsilon$$

for all $y \in D_0$. Then

$$\begin{aligned}
&\left|\int_{D_0} p^D(t,x_n,y)h_n(y)\,dy - \int_{D_0} p^D(t,x,y)h(y)\,dy\right| \\
&\leqslant \int_{D_0} p^D(t,x_n,y)|h_n(y)-h(y)|dy + \int_{D_0} |p^D(t,x_n,y)-p^D(t,x,y)|h(y)\,dy \\
&\leqslant \varepsilon + A|D_0|\varepsilon,
\end{aligned}$$

where $|D_0|$ stands for the Lebesgue measure of $D_0$. This proves the first part.

(b) We proceed similarly as in the proof of the first part. The only difference is that we use (5.1) to get the following estimate:

$$\begin{aligned}
&\int_{D\setminus D_0} p^D(t,x,y)\,\frac{G^D(y,y_n)}{G^D(x_0,y_n)}\,dy \\
&\quad\leqslant C(t)\phi_0(x)\int_{D\setminus D_0}\phi_0(y)\,\frac{G^D(y,y_n)}{G^D(x_0,y_n)}\,dy \\
&\quad\leqslant cC(t)\int_{D\setminus D_0}\frac{G^D(x_0,y)G^D(y,y_n)}{G^D(x_0,y_n)}\,dy \\
&\quad\leqslant cC(t)\int_{D\setminus D_0}(|x_0-y|^{2-d}+|y-y_n|^{2-d})\,dy \\
&\quad\leqslant cC(t)\sup_n\int_{D\setminus D_0}(|x_0-y|^{2-d}+|y-y_n|^{2-d})\,dy.
\end{aligned}$$

The corresponding estimate for $M^D(\cdot,z)$ is given in part (a) of the lemma. For a given $\varepsilon>0$ find $D_0$ large enough so that the last line in the display above is less than $\varepsilon$. Then find $n_0\in\mathbb{N}$ such that for all $n\geqslant n_0$,

$$\left|\frac{G^D(y,y_n)}{G^D(x_0,y_n)}-M^D(y,z)\right|<\varepsilon,\quad y\in D_0.$$

Then

$$\int_{D_0} p^D(t,x,y)\left|\frac{G^D(y,y_n)}{G^D(x_0,y_n)}-M^D(y,z)\right|dy<\varepsilon\quad\text{for all } n\geqslant n_0.$$

This proves the second part. □

**Theorem 5.2.** *Suppose that $D$ is a bounded Lipschitz domain and $x_0\in D$ is a fixed point.*

(a) *If $(x_n)$ is a sequence of points in $D$ converging to $x\in D$ and $(h_n)$ is a sequence of functions in $\mathscr{H}^+$ converging to a function $h\in\mathscr{H}^+$, then*

$$\lim_n\,(G^D_\alpha)^{-1}h_n(x_n)=(G^D_\alpha)^{-1}h(x).$$

(b) *If $(y_n)$ is a sequence of points in $D$ converging to $z\in\partial D$, then for every $x\in D$,*

$$\lim_n\,(G^D_\alpha)^{-1}\left(\frac{G^D(\cdot,y_n)}{G^D(x_0,y_n)}\right)(x)=\lim_n\frac{(G^D_\alpha)^{-1}(G^D(\cdot,y_n))(x)}{G^D(x_0,y_n)}=(G^D_\alpha)^{-1}M^D(\cdot,z)(x).$$

**Proof.** (a) Normalizing by $h_n(x_0)$ if necessary, we may assume without loss of generality that $h_n(x_0) = 1$ for all $n \geqslant 1$. Let $\varepsilon > 0$. We have

$$
\begin{aligned}
&|(G_\alpha^D)^{-1} h_n(x_n) - (G_\alpha^D)^{-1} h(x)| \\
&\quad = c(\alpha) \left| \int_0^\infty t^{-(\alpha/2+1)} (h_n(x_n) - P_t^D h_n(x_n))\, dt - \int_0^\infty t^{-(\alpha/2+1)} (h(x) - P_t^D h(x))\, dt \right| \\
&\quad \leqslant c(\alpha) \int_0^\varepsilon t^{-(\alpha/2+1)} (h_n(x_n) - P_t^D h_n(x_n))\, dt + c \int_0^\varepsilon t^{-(\alpha/2+1)} (h(x) - P_t^D h(x))\, dt \\
&\qquad + c(\alpha) \left| \int_\varepsilon^\infty t^{-(\alpha/2+1)} (h_n(x_n) - P_t^D h_n(x_n))\, dt - \int_\varepsilon^\infty t^{-(\alpha/2+1)} (h(x) - P_t^D h(x))\, dt \right|,
\end{aligned}
$$

where $c(\alpha) = \alpha/(2\Gamma(1-\alpha/2))$. Let $K$ and $L$ be compact subsets of $D$ such that $(x_n) \subset K \subset L^\circ \subset L$. Since $h_n \to h$ locally uniformly, there exists a constant $M$ such that $h_n, h \leqslant M$ on $L$. The estimate at the end of the proof of Theorem 3 in [10] gives that

$$
\int_0^\varepsilon t^{-(\alpha/2+1)} (h_n - P_t^D h_n)(x_n)\, dt \leqslant \frac{2Md}{(1-\alpha/2)\rho^2}\, \varepsilon^{1-\alpha/2}, \quad n \geqslant 1
$$

and

$$
\int_0^\varepsilon t^{-(\alpha/2+1)} (h - P_t^D h)(x)\, dt \leqslant \frac{2Md}{(1-\alpha/2)\rho^2}\, \varepsilon^{1-\alpha/2}.
$$

Here $\rho = \mathrm{dist}(K, \bar{L}^c)$. Further,

$$
\begin{aligned}
&\left| \int_\varepsilon^\infty t^{-(\alpha/2+1)} (h_n(x_n) - P_t^D h_n(x_n))\, dt - \int_\varepsilon^\infty t^{-(\alpha/2+1)} (h(x) - P_t^D h(x))\, dt \right| \\
&\quad \leqslant \int_\varepsilon^\infty t^{-(\alpha/2+1)} (|h_n(x_n) - h(x_n)| + |h(x_n) - h(x)|)\, dt \\
&\qquad + \int_\varepsilon^\infty t^{-(\alpha/2+1)} |P_t^D h_n(x_n) - P_t^D h(x)| dt.
\end{aligned}
$$

Since $|h_n(x_n) - h(x_n)| + |h(x_n) - h(x)| \leqslant 2M$ and $|P_t^D h_n(x_n) - P_t^D h(x)| \leqslant M$, we can apply Lemma 5.1(a) and the dominated convergence theorem to get

$$
\lim_n \int_\varepsilon^\infty t^{-(\alpha/2+1)} (|h_n(x_n) - h(x_n)| + |h(x_n) - h(x)|)\, dt = 0
$$

and

$$
\lim_n \int_\varepsilon^\infty t^{-(\alpha/2+1)} |P_t^D h_n(x_n) - P_t^D h(x)| dt = 0.
$$

Hence

$$\limsup_n |(G_\alpha^D)^{-1}h_n(x_n) - (G_\alpha^D)^{-1}h(x)| \leqslant \frac{4cMd}{(1-\alpha/2)\rho^2}\varepsilon^{1-\alpha/2}$$

for every $\varepsilon > 0$. The proof of (a) is now complete.

(b) The proof of (b) is similar to (a). The only difference is that we use 5.1(b) in this case. We omit the details. □

Let us define the function $K_{2-\alpha}^D(x,z) := (G_\alpha^D)^{-1}M^D(\cdot,z)(x)$ on $D\times\partial D$. For each fixed $z\in\partial D$, $K_{2-\alpha}^D(\cdot,z)\in\mathscr{H}_{2-\alpha}^+$. By the first part of Theorem 5.2, we know that $K_{2-\alpha}^D(x,z)$ is continuous on $D\times\partial D$. Let $(y_n)$ be a sequence of points in $D$ converging to $z\in\partial D$, then from Theorem 5.2(b) we get that

$$\begin{aligned} K_{2-\alpha}^D(x,z) &= \lim_{n\to\infty} (G_\alpha^D)^{-1}\left(\frac{G^D(\cdot,y_n)}{G^D(x_0,y_n)}\right)(x) \\ &= \lim_{n\to\infty} \frac{(G_\alpha^D)^{-1}(G^D(\cdot,y_n))(x)}{G^D(x_0,y_n)} \\ &= \lim_{n\to\infty} \frac{G_{2-\alpha}^D(x,y_n)}{G^D(x_0,y_n)}, \end{aligned} \tag{5.3}$$

where the last line follows from Proposition 3.4. In particular, there exists the limit

$$\lim_{n\to\infty} \frac{G_{2-\alpha}^D(x_0,y_n)}{G^D(x_0,y_n)} = K_{2-\alpha}^D(x_0,z). \tag{5.4}$$

Now we define a function $M_{2-\alpha}^D$ on $D\times\partial D$ by

$$M_{2-\alpha}^D(x,z) := \frac{K_{2-\alpha}^D(x,z)}{K_{2-\alpha}^D(x_0,z)}, \quad x\in D, z\in\partial D. \tag{5.5}$$

For each $z\in\partial D$, $M_{2-\alpha}^D(\cdot,z)\in\mathscr{H}_{2-\alpha}^+$. Moreover, $M_{2-\alpha}^D$ is jointly continuous on $D\times\partial D$. From the definition above and (5.3) we can easily see that

$$\lim_{D\ni y\to z} \frac{G_{2-\alpha}^D(x,y)}{G_{2-\alpha}^D(x_0,y)} = M_{2-\alpha}^D(x,z), \quad x\in D, z\in\partial D. \tag{5.6}$$

**Theorem 5.3.** *Suppose that $D$ is a bounded Lipschitz domain. The Martin boundary and the minimal Martin boundary of $Z_{2-\alpha}^D$ both coincide with the Euclidean boundary $\partial D$, and the Martin kernel based at $x_0$ is given by the function $M_{2-\alpha}^D$.*

**Proof.** The fact that $M_{2-\alpha}^D$ is the Martin kernel of $Z_{2-\alpha}^D$ based at $x_0$ has been proven in the paragraph above. It follows from Theorem 3.1 that when $z_1$ and $z_2$ are two distinct points on $\partial D$, the functions $M_{2-\alpha}^D(\cdot,z_1)$ and $M_{2-\alpha}^D(\cdot,z_2)$ are not identical.

Therefore the Martin boundary of $Z^D_{2-\alpha}$ coincides with the Euclidean boundary $\partial D$. Since $M^D(\cdot,z)\in\mathscr{H}^+$ is minimal, by the order preserving property of $(G^D_\alpha)^{-1}$ we know that $M^D_{2-\alpha}(\cdot,z)\in\mathscr{H}_{2-\alpha}$ is also minimal. Therefore the minimal Martin boundary of $Z_D^{2-\alpha}$ also coincides with the Euclidean boundary $\partial D$. $\square$

**Corollary 5.4.** *If D is a bounded $C^{1,1}$ domain, then there exists a constant $C>0$ such that*

$$C^{-1}\frac{\delta(x)}{|x-z|^{d+2-\alpha}}\leqslant M^D_\alpha(x,z)\leqslant C\frac{\delta(x)}{|x-z|^{d+2-\alpha}},\quad x\in D, z\in\partial D.$$

**Proof.** The conclusion of this corollary follows immediately from the theorem above and the sharp estimates on the Green function $G^D_\alpha$ in [15] and we omit the details. $\square$

It follows from Theorem 5.3 and the general theory of Martin boundary that for any $u\in\mathscr{H}^+_{2-\alpha}$ there exists a finite measure $v$ on $\partial D$ such that

$$u(x)=\int_{\partial D} M^D_{2-\alpha}(x,z)v(dz),\quad x\in D.$$

The measure $v$ is sometimes called the Martin measure of $u$. The following result gives the relation between the Martin measure of $h\in\mathscr{H}^+$ and the Martin measure of $(G^D_\alpha)^{-1}h\in\mathscr{H}^+_{2-\alpha}$.

**Proposition 5.5.** *Suppose that D is a bounded Lipschitz domain. If $h\in\mathscr{H}^+$ can be written as*

$$h(x)=\int_{\partial D} M^D(x,z)\mu(dz),\quad x\in D,$$

*then*

$$(G^D_\alpha)^{-1}h(x)=\int_{\partial D} M^D_{2-\alpha}(x,z)v(dz),\quad x\in D$$

*with $v(dz)=K^D_{2-\alpha}(x_0,z)\mu(dz)$.*

**Proof.** By assumption we know that

$$h(x)=\int_{\partial D} M^D(x,z)\mu(dz),\quad x\in D.$$

Using (2.1) and Fubini's theorem we get

$$\begin{aligned}(G^D_\alpha)^{-1}h(x) &= \int_{\partial D} (G^D_\alpha)^{-1}(M^D(\cdot,z))(x)\mu(dz)\\ &= \int_{\partial D} M^D_{2-\alpha}(x,z)K^D_{2-\alpha}(x_0,z)\mu(dz)\\ &= \int_{\partial D} M^D_{2-\alpha}(x,z)\nu(dz),\end{aligned}$$

with $\nu(dz) = K^D_{2-\alpha}(x_0,z)\mu(dz)$. The proof is now complete. □

For $z \in \partial D$, $n \in \mathbb{N}$, let $\Delta_n(z) = B(z,2^{-n}) \cap \partial D$, and let $\omega(x,\Delta(z)) = \mathbb{P}_x(X_{\tau_D} \in \Delta_n(z))$. It is well known (see, for instance, [2]) that for a Lipschitz domain $D$,

$$M^D(x,z) = \lim_{n\to\infty} \frac{\omega(x,\Delta_n(z))}{\omega(x_0,\Delta_n(z))}. \tag{5.7}$$

We are going to prove two analogous results for the Martin kernel $M^D_{2-\alpha}$.

**Proposition 5.6.** *Let $D$ be a Lipschitz domain. For $z \in \partial D$, $n \in \mathbb{N}$, let $\Delta_n(z) = B(z,2^{-n}) \cap \partial D$ and $B_n(z) = B(z,2^{-n}) \cap D$. Then*

$$\begin{aligned}M^D_{2-\alpha}(x,z) &= \lim_{n\to\infty} \frac{\mathbb{E}_x[1_{(X_{\tau_D} \in \Delta_n(z))}\tau_D^{-\alpha/2}]}{\mathbb{E}_{x_0}[1_{(X_{\tau_D} \in \Delta_n(z))}\tau_D^{-\alpha/2}]}\\ &= \lim_{n\to\infty} \frac{\tilde{\mathbb{P}}_x[Z^D_{2-\alpha}(\zeta-) \in B_n(z)]}{\tilde{\mathbb{P}}_{x_0}[Z^D_{2-\alpha}(\zeta-) \in B_n(z)]}.\end{aligned}$$

**Proof.** According to Corollary 3.5, we have

$$(G^D_{2-\alpha})^{-1}(\omega(\cdot,\Delta_n(z)))(x) = \frac{1}{\Gamma(1-\alpha/2)} \mathbb{E}_x[1_{(X_{\tau_D} \in \Delta_n(z))}\tau_D^{-\alpha/2}].$$

Hence,

$$\begin{aligned}\frac{\mathbb{E}_x[1_{(X_{\tau_D} \in \Delta_n(z))}\tau_D^{-\alpha/2}]}{\mathbb{E}_{x_0}[1_{(X_{\tau_D} \in \Delta_n(z))}\tau_D^{-\alpha/2}]} &= \frac{(G^D_{2-\alpha})^{-1}(\omega(\cdot,\Delta_n(z)))(x)}{(G^D_{2-\alpha})^{-1}(\omega(\cdot,\Delta_n(z)))(x_0)}\\ &= \frac{(G^D_{2-\alpha})^{-1}\left(\frac{\omega(\cdot,\Delta_n(z))}{\omega(x_0,\Delta_n(z)}\right)(x)}{(G^D_{2-\alpha})^{-1}\left(\frac{\omega(\cdot,\Delta_n(z))}{\omega(x_0,\Delta_n(z)}\right)(x_0)}.\end{aligned}$$

The first equality follows by (5.7), Theorem 5.2 and (5.5).

For the second equation we are going to use the following formula (see (2.11) in [3]): For $x \in D$ and $A \subset D$,

$$\tilde{\mathbb{P}}_x(Z^D_{2-\alpha}(\zeta-) \in A) = \int_A G^D_{2-\alpha}(x,y)\kappa^D_{2-\alpha}(y)\,dy, \tag{5.8}$$

where $\kappa^D_{2-\alpha}$ is the killing function of $Z^D_{2-\alpha}$. For given $\varepsilon > 0$ find $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$

$$M^D_{2-\alpha}(x,z) - \varepsilon \leqslant \frac{G^D_{2-\alpha}(x,y)}{G^D_{2-\alpha}(x_0,y)} \leqslant M^D_{2-\alpha}(x,z) + \varepsilon,$$

for all $y \in B_n(z)$. Then

$$M^D_{2-\alpha}(x,z) - \varepsilon \leqslant \frac{\int_{B_n(z)} G^D_{2-\alpha}(x,y)\kappa^D_{2-\alpha}(y)\,dy}{\int_{B_n(z)} G^D_{2-\alpha}(x_0,y)\kappa^D_{2-\alpha}(y)\,dy} \leqslant M^D_{2-\alpha}(x,z) + \varepsilon,$$

which proves the result. □

From Theorem 5.2 we know that $(G^D_\alpha)^{-1} : \mathscr{H}^+ \to \mathscr{H}^+_{2-\alpha}$ is continuous with respect to topologies of locally uniform convergence. In the next result we show that $G^D_\alpha : \mathscr{H}^+_{2-\alpha} \to \mathscr{H}^+$ is also continuous.

**Proposition 5.7.** *Suppose that $D$ is a bounded Lipschitz domain. Let $(g_n, n \geqslant 0)$ be a sequence of functions in $\mathscr{H}^+_{2-\alpha}$ converging pointwise to the function $g \in \mathscr{H}^+_{2-\alpha}$. Then* $\lim_{n\to\infty} G^D_\alpha g_n(x) = G^D_\alpha g(x)$ *for every $x \in D$.*

**Proof.** Without loss of generality, we may assume that $g_n(x_0) = 1$ for all $n \in \mathbb{N}$. Then there exist probability measures $\nu_n$, $n \in \mathbb{N}$, and $\nu$ on $\partial D$ such that $g_n(x) = \int_{\partial D} M^D_{2-\alpha}(x,z)\nu_n(dz)$, $n \in \mathbb{N}$, and $g(x) = \int_{\partial D} M^D_{2-\alpha}(x,z)\nu(dz)$. It is easy to show that the convergence of the harmonic functions $h_n$ implies that $\nu_n \to \nu$ weakly. Let $G^D_\alpha g_n(x) = \int_{\partial D} M^D(x,z)\mu_n(dz)$ and $G^D_\alpha g(x) = \int_{\partial D} M^D(x,z)\mu(dz)$. Then $\nu_n(dz) = K^D_{2-\alpha}(x_0,z)\mu_n(dz)$ and $\nu(dz) = K^D_{2-\alpha}(x_0,z)\mu(dz)$. Since the density $K^D_{2-\alpha}(x_0,\cdot)$ is bounded away from zero and bounded from above, it follows that $\mu_n \to \mu$ weakly. From this the claim of proposition follows immediately. □

## 6. Boundary Harnack principle

The boundary Harnack principle is a very important result in potential theory and harmonic analysis. For example, it is usually used to prove that, when $D$ is a bounded Lipschitz domain, both the Martin boundary and the minimal Martin boundary of $X^D$ coincide with the Euclidean boundary $\partial D$. We have already proved in Theorem 5.3 that for $Z_\alpha^D$, both the Martin boundary and the minimal Martin boundary coincide with the Euclidean boundary $\partial D$. By using this we are going to prove a boundary Harnack principle for functions in $\mathscr{H}_\alpha^+$.

In this section we will always assume that $D$ is a bounded Lipschitz domain and $x_0 \in D$ is fixed. Recall that $\phi_0$ is the eigenfunction corresponding to the smallest eigenvalue $\lambda_0$ of $-\Delta|_D$.

**Proposition 6.1.** *Suppose that D is a bounded Lipschitz domain. There exist $C > 0$ and $m > d$ such that*

$$G_\alpha^D(x,y) \leqslant C \frac{\phi_0(x)\phi_0(y)}{|x-y|^{m-\alpha}}, \quad x, y \in D.$$

**Proof.** It follows from Theorem 4.6.9 of [5] that the density $p^D$ of the killed Brownian motion on $D$ satisfies the following estimate:

$$p^D(t,x,y) \leqslant c_1 t^{-m/2} \phi_0(x)\phi_0(y) e^{-\frac{|x-y|^2}{6t}}, \quad t > 0, \ x, y \in D,$$

for some $m > d$ and $c > 0$. Now we have

$$\begin{aligned} G_\alpha^D(x,y) &= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} p^D(t,x,y)\, dt \\ &\leqslant \frac{c_1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2-1} t^{-m/2} \phi_0(x)\phi_0(y) e^{-\frac{|x-y|^2}{6t}}\, dt \\ &\leqslant c_2 \frac{\phi_0(x)\phi_0(y)}{|x-y|^{m-\alpha}}. \end{aligned}$$

The proof is now finished. □

**Lemma 6.2.** *Suppose that D is a bounded Lipschitz domain and V is an open subset of $\mathbb{R}^d$ such that $V \cap \partial D$ is non-empty. If $h \in \mathscr{H}_{2-\alpha}^+$ satisfies*

$$\lim_{x \to z} \frac{h(x)}{\kappa_\alpha^D(x)} = 0, \quad \forall z \in V \cap \partial D,$$

*then*

$$\lim_{x \to z} G_\alpha^D h(x) = 0, \quad \forall z \in V \cap \partial D.$$

**Proof.** Fix $z \in V \cap \partial D$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$h(x) \leqslant \varepsilon \kappa_\alpha^D(x), \quad x \in B(z,\delta) \cap D.$$

Thus we have

$$G_\alpha^D h(x) \leqslant G_\alpha^D (h \cdot 1_{D \setminus B(z,\delta)})(x) + \varepsilon G_\alpha^D \kappa_\alpha^D (x) = G_\alpha^D (h \cdot 1_{D \setminus B(z,\delta)})(x) + \varepsilon, \quad x \in D.$$

By Proposition 6.1 we get that there exists $c > 0$ such that for any $x \in B(z, \delta/2) \cap D$,

$$G_\alpha^D h(x) \leqslant c\phi_0(x) \int_D \phi_0(y) h(y)\, dy + \varepsilon.$$

From the proof of Theorem 3.1 we know that $\int_D \phi_0(y) h(y)\, dy < \infty$. Now the conclusion of the lemma follows easily from the fact that $\lim_{x \to z} \phi_0(x) = 0$. □

Now we can prove the main result of this section: the boundary Harnack principle.

**Theorem 6.3.** *Suppose that $D$ is a bounded Lipschitz domain, $V$ is an open subset of $\mathbb{R}^d$ such that $V \cap \partial D$ is nonempty, and that $K$ is a compact subset of $V$. There exists a constant $c > 0$ such that for any two functions $h_1$ and $h_2$ in $\mathscr{H}_{2-\alpha}^+$ satisfying*

$$\lim_{x \to z} \frac{h_i(x)}{\kappa_\alpha^D(x)} = 0, \quad z \in V \cap \partial D, i = 1, 2,$$

*we have*

$$\frac{h_1(x)}{h_2(x)} \leqslant c \frac{h_1(y)}{h_2(y)}, \quad x, y \in K \cap D.$$

**Proof.** It follows from Corollary 4.7 of [17] and Proposition 6.1 that there exist positive constants $c_1$ and $c_2$ such that

$$c_1 \phi_0(x)\phi_0(y) \leqslant G_{2-\alpha}^D(x, y) \leqslant c_2 \frac{\phi_0(x)\phi_0(y)}{|x - y|^{m-2+\alpha}}, \quad x, y \in D,$$

where $m > d$ is given in Proposition 6.1. Therefore by (5.6) we get that there exist positive constants $c_3$ and $c_4$ such that

$$c_3 \phi_0(x) \leqslant M_{2-\alpha}^D(x, z) \leqslant c_4 \phi_0(x), \quad x \in K \cap D, z \in \partial D \setminus V. \tag{6.1}$$

Suppose that $h_1$ and $h_2$ are two functions in $\mathscr{H}_{2-\alpha}^+$ such that

$$\lim_{x \to z} \frac{h_i(x)}{\kappa_\alpha^D(x)} = 0, \quad z \in V \cap \partial D, i = 1, 2,$$

then by Lemma 6.2 we know that

$$\lim_{x \to z} G_\alpha^D h_i(x) = 0, \quad z \in V \cap \partial D, i = 1, 2.$$

Now by Corollary 8.1.6 of [13] we know that the Martin measures $\mu_1$ and $\mu_2$ of $G^D_\alpha h_1$ and $G^D_\alpha h_2$ are supported by $\partial D\backslash V$ and so we have

$$G^D_\alpha h_i(x) = \int_{\partial D\backslash V} M^D(x,z)\mu_i(dz), \quad x\in D, i=1,2.$$

Using Proposition 5.5 we get that

$$h_i(x) = \int_{\partial D\backslash V} M^D_{2-\alpha}(x,z)v_i(dz), \quad x\in D, i=1,2,$$

where $v_i(dz) = K^D_{2-\alpha}(x_0,z)\mu_i(dz), i=1,2$. Now using (6.1) we get that

$$c_3\phi_0(x)v_i(\partial D\backslash V)\leqslant h_i(x)\leqslant c_4\phi_0(x)v_i(\partial D\backslash V), \quad x\in K\cap D, i=1,2.$$

The conclusion of the theorem follows immediately. □

From the proof of Theorem 6.3 we can see that the following result is true.

**Proposition 6.4.** *Suppose that D is a bounded Lipschitz domain and V an open subset of* $\mathbb{R}^d$ *such that* $V\cap\partial D$ *is nonempty. If* $h\in\mathscr{H}^+_{2-\alpha}$ *satisfies*

$$\lim_{x\to z}\frac{h(x)}{\kappa^D_\alpha(x)} = 0, \quad z\in V\cap\partial D,$$

*then*

$$\lim_{x\to z} h(x) = 0, \quad z\in V\cap\partial D.$$

**Proof.** From the proof of Theorem 6.3 we know that the Martin measure $v$ of $h$ is supported by $\partial D\backslash V$ and so we have

$$h(x) = \int_{\partial D\backslash V} M^D_{2-\alpha}(x,z)v(dz), \quad x\in D.$$

For any $z_0\in V\cap\partial D$, take $\delta>0$ small enough so that $B(z_0,\delta)\subset\overline{B(z_0,\delta)}\subset V$. Then by (6.1) we get that

$$c_1\phi_0(x)\leqslant M^D_{2-\alpha}(x,z)\leqslant c_2\phi_0(x), \quad x\in B(z_0,\delta)\cap D, z\in\partial D\backslash V$$

for some positive constants $c_1$ and $c_2$. Thus

$$h(x)\leqslant c_2\phi_0(x)v(\partial D\backslash V), \quad x\in B(z_0,\delta)\cap D,$$

from which the assertion of the proposition follows immediately. □

**Remark.** All the results in this paper remain valid when we replace the Brownian motion $X_t$ by an elliptic diffusion whose generator is given by

$$L = \sum_{i,j=1}^{d} \partial_i (a_{ij} \partial_j),$$

where $(a_{ij})$ satisfies

$$\lambda^{-1} ||\xi||^2 \leqslant \int_{i,j=1}^{d} a_{ij} \xi_i \xi_j \leqslant \lambda ||\xi||^2, \quad x, \xi \in \mathbb{R}^d$$

for some constant $\lambda > 0$.

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