Submitted to the Annals of Probability

DIRICHLET HEAT KERNEL ESTIMATES FOR FRACTIONAL LAPLACIAN WITH GRADIENT PERTURBATION

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Suppose that $d \geq 2$ and $\alpha \in (1, 2)$. Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ and b an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ whose components are in a certain Kato class of the rotationally symmetric α-stable process. In this paper, we derive sharp two-sided heat kernel estimates for $\mathcal{L}^b = \Delta^{\alpha/2} + b \cdot \nabla$ in D with zero exterior condition. We also obtain the boundary Harnack principle for $\mathcal{L}^b$ in D with explicit decay rate.

1. Introduction. Throughout this paper we assume $d \geq 2$, $\alpha \in (1, 2)$ and that X is a (rotationally) symmetric α-stable process on $\mathbb{R}^d$. The infinitesimal generator of X is $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$. We will use $B(x, r)$ to denote the open ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$.

Definition 1.1. *For a function f on $\mathbb{R}^d$, we define for $r > 0$,*

$$M_f^\alpha(r) = \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} \frac{|f|(y)}{|x-y|^{d+1-\alpha}}\, dy.$$

A function f on $\mathbb{R}^d$ is said to belong to the Kato class $\mathbb{K}_{d,\alpha-1}$ if $\lim_{r\downarrow 0} M_f^\alpha(r) = 0$.

Since $1 < \alpha < 2$, using Hölder's inequality, it is easy to see that for every $p > d/(\alpha-1)$, $L^\infty(\mathbb{R}^d; dx) + L^p(\mathbb{R}^d; dx) \subset \mathbb{K}_{d,\alpha-1}$. Throughout this paper we will assume that $b = (b^1, \cdots, b^d)$ is an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ such that $|b| \in \mathbb{K}_{d,\alpha-1}$. Define $\mathcal{L}^b = \Delta^{\alpha/2} + b \cdot \nabla$. Intuitively, the fundamental solution $p^b(t, x, y)$ of $\mathcal{L}^b$ and the fundamental solution $p(t, x, y)$ of $\Delta^{\alpha/2}$, which is also

[*]Research partially supported by NSF Grants DMS-0906743 and DMR-1035196.

[†]Research supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST)(2010-0001984).

AMS 2000 subject classifications: Primary 60J35, 47G20, 60J75; secondary 47D07

Keywords and phrases: symmetric α-stable process, gradient operator, heat kernel, transition density, Green function, exit time, Lévy system, boundary Harnack inequality, Kato class

the transition density of X, should be related by the following Duhamel's formula:

$$p^b(t,x,y) = p(t,x,y) + \int_0^t \int_{\mathbb{R}^d} p^b(s,x,z)\, b(z)\cdot \nabla_z p(t-s,z,y)dzds. \tag{1.1}$$

Applying the above formula repeatedly, one expects that $p^b(t,x,y)$ can be expressed as an infinite series in terms of p and its derivatives. This motivates the following definition. Define $p^b_0(t,x,y) = p(t,x,y)$ and for $k \ge 1$,

$$p^b_k(t,x,y) := \int_0^t \int_{\mathbb{R}^d} p^b_{k-1}(s,x,z)\, b(z)\cdot \nabla_z p(t-s,z,y)dz. \tag{1.2}$$

The following results are shown in [6, Theorem 1, Lemma 15, Lemma 23] and their proofs. Here and in the sequel, we use := as a way of definition. For $a,b\in\mathbb{R}$, $a\wedge b := \min\{a,b\}$ and $a\vee b := \max\{a,b\}$.

Theorem 1.2. (i) *There exist $T_0>0$ and $c_1>1$ depending on b only through the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that $\sum_{k=0}^\infty p^b_k(t,x,y)$ converges locally uniformly on $(0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$ to a positive jointly continuous function $p^b(t,x,y)$ and that on $(0,T_0]\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$c_1^{-1}\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right) \le p^b(t,x,y) \le c_1\left(t^{-d/\alpha}\wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{1.3}$$

Moreover, $\int_{\mathbb{R}^d} p^b(t,x,y)dy = 1$ for every $t\in(0,T_0]$ and $x\in\mathbb{R}^d$.

(ii) *The function $p^b(t,x,y)$ defined in (i) can be extended uniquely to a positive jointly continuous function on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$ so that for all $s,t\in(0,\infty)$ and $(x,y)\in\mathbb{R}^d\times\mathbb{R}^d$, $\int_{\mathbb{R}^d} p^b(t,x,y)dy=1$ and*

$$p^b(s+t,x,y) = \int_{\mathbb{R}^d} p^b(s,x,z)p^b(t,z,y)dz. \tag{1.4}$$

(iii) *If we define*

$$P^b_t f(x) := \int_{\mathbb{R}^d} p^b(t,x,y)f(y)dy, \tag{1.5}$$

then for any $f,g\in C_c^\infty(\mathbb{R}^d)$, the space of smooth functions with compact supports,

$$\lim_{t\downarrow 0}\int_{\mathbb{R}^d} t^{-1}(P^b_t f(x) - f(x))g(x)dx = \int_{\mathbb{R}^d} (\mathcal{L}^b f)(x)\, g(x)dx.$$

Thus $p^b(t,x,y)$ is the fundamental solution of $\mathcal{L}^b$ in distributional sense.

Here and in the rest of this paper, the meaning of the phrase "depending on b only via the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero" is that the statement is true for any $\mathbb{R}^d$-valued function $\widetilde{b}$ on $\mathbb{R}^d$ with

$$M^{\alpha}_{|\widetilde{b}|}(r) \le M^{\alpha}_{|b|}(r) \quad \text{for all } r > 0.$$

Note that the Green function $G(x, y)$ of X is $c/|x-y|^{d-\alpha}$ and so $|\nabla_x G(x, y)| \le c/|x - y|^{d-\alpha+1}$. This indicates that $\mathbb{K}_{d,\alpha-1}$ is the right class of functions for gradient perturbations of fractional Laplacian. The same phenomenon happens for $\Delta + b \cdot \nabla$, see [15].

It is easy to show (see Proposition 2.3 below) that the operators $\{P^b_t; t \ge 0\}$ defined by (1.5) form a Feller semigroup and so there exists a conservative Feller process $X^b = \{X^b_t, t \ge 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$ in $\mathbb{R}^d$ such that $P^b_t f(x) = \mathbb{E}_x[f(X^b_t)]$. The process X^b is in general non-symmetric. We call X^b an α-stable process with drift b, since its infinitesimal generator is $\mathcal{L}^b$.

For any open subset $D \subset \mathbb{R}^d$, we define $\tau^b_D = \inf\{t > 0 : X^b_t \notin D\}$. We will use $X^{b,D}$ to denote the subprocess of X^b in D; that is, $X^{b,D}_t(\omega) = X^b_t(\omega)$ if $t < \tau^b_D(\omega)$ and $X^{b,D}_t(\omega) = \partial$ if $t \ge \tau^b_D(\omega)$, where ∂ is a cemetery state. The subprocess of X in D will be denoted by X^D. Throughout this paper, we use the convention that for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. The infinitesimal generator of $X^{b,D}$ is $\mathcal{L}^b|_D$, that is, $\mathcal{L}^b$ on D with zero exterior condition. The process $X^{b,D}$ has a transition density $p^b_D(t, x, y)$ with respect to the Lebesgue measure. (See (3.4) below.) The transition density $p^b_D(t, x, y)$ of $X^{b,D}$ is the fundamental solution of $\mathcal{L}^b|_D$. The transition density of X^D is denoted by $p_D(t, x, y)$ and it is the fundamental solution of $\mathcal{L}|_D$.

The purpose of this paper is to establish the following sharp two-sided estimates on $p^b_D(t, x, y)$ in Theorem 1.3. To state this theorem, we first recall that an open set D in $\mathbb{R}^d$ is said to be a $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = 0$, $\nabla\phi(0) = (0, \dots, 0)$, $\|\nabla\phi\|_\infty \le \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(z)| \le \Lambda_0|x - z|$, and an orthonormal coordinate system CS_z: $y = (y_1, \cdots, y_{d-1}, y_d) := (\widetilde{y}, y_d)$ with its origin at z such that

$$B(z, R_0) \cap D = \{y \in B(0, R_0) \text{ in } CS_z : y_d > \phi(\widetilde{y})\}.$$

The pair (R_0, Λ_0) is called the characteristics of the $C^{1,1}$ open set D. We remark that in some literatures, the $C^{1,1}$ open set defined above is called a *uniform* $C^{1,1}$ open set as (R_0, Λ_0) is universal for every $z \in \partial D$. For $x \in D$, let $\delta_D(x)$ denote the Euclidean distance between x and ∂D. Note that a bounded $C^{1,1}$ open set may be disconnected.

THEOREM 1.3. *Let D be a bounded $C^{1,1}$ open subset of $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R_0, Λ_0). Define*

$$f_D(t,x,y) = \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right).$$

For each $T > 0$, there are constants $c_1 = c_1(T, R_0, \Lambda_0, d, \alpha, diam(D), b) \geq 1$ and $c_2 = c_2(T, d, \alpha, D, b) \geq 1$ with the dependence on b only through the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero such that

(i) *on $(0, T] \times D \times D$,*

$$c_1^{-1} f_D(t,x,y) \leq p^b_D(t,x,y) \leq c_1 f_D(t,x,y);$$

(ii) *on $[T, \infty) \times D \times D$,*

$$c_2^{-1} e^{-t\lambda_0^{b,D}} \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} \leq p^b_D(t,x,y) \leq c_2 e^{-t\lambda_0^{b,D}} \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2},$$

where $-\lambda_0^{b,D} := \sup \mathrm{Re}(\sigma(\mathcal{L}^b|_D)) < 0.$

Here $\mathrm{diam}(D)$ denotes the diameter of D. At first glance, one might think that the estimates in Theorem 1.3 can be obtained from the estimates for $p_D(t,x,y)$ by using a Duhamel's formula similar to (1.1) with p^b, p and $\mathbb{R}^d$ replaced by p^b_D, p_D and D, respectively. Unfortunately such an approach does not work for $p^b_D(t,x,y)$. This is because unlike the whole space case, we do not have a good control on $\nabla_z p_D(s,z,y)$ when z is near the boundary of D. When $D = \mathbb{R}^d$, $p(t,x,y)$ is the transition density of the symmetric α-stable process and there is a nice bound for $\nabla_z p(t,z,y)$. This is the key reason why the result in Theorem 1.2(i) can be established by using Duhamel's formula. Instead, we establish Theorem 1.3 by using probabilistic means through the Feller process X^b. More specifically, we adapt the road map outlined in our paper [9] that establishes sharp two-sided Dirichlet heat kernel estimates for symmetric α-stable processes in $C^{1,1}$ open sets. Clearly, many new and major difficulties arise when adapting the strategy outlined in [9] to X^b. Symmetric stable processes are Lévy processes that are rotationally symmetric and self-similar. The Feller process X^b here is typically non-symmetric, which is the main difficulty that we have to overcome. In addition, X^b is neither self-similar nor rotationally symmetric. Specifically, our approach consists of the following four ingredients:

(i) determine the Lévy system of X^b that describes how the process jumps;
(ii) derive an approximate stable-scaling property of X^b in bounded $C^{1,1}$ open sets, which will be used to derive heat kernel estimates in bounded $C^{1,1}$ open sets for small time $t \in (0, T]$ from that at time $t = 1$;
(iii) establish sharp two-sided estimates with explicit boundary decay rate on the Green functions of X^b and its suitable dual process in $C^{1,1}$ open sets with sufficiently small diameter;
(iv) prove the intrinsic ultracontractivity of (the non-symmetric process) X^b in bounded open sets, which will give sharp two-sided Dirichlet heat kernel estimates for large time.

In step (ii), we choose a large ball E centered at the origin so that our bounded $C^{1,1}$ open set D is contained in $\frac{1}{4}E$. Then we derive heat kernel estimates in D at time $t = 1$ carefully so that the constants depend on the quantity M defined in (6.5), not on the diameter of D directly. Note that the constant M has the correct scaling property, while the diameter of D does not. In fact, the constant c_1 in Theorem 1.3 depends on the diameter of D only through M.

We also establish the boundary Harnack inequality for X^b and its suitable dual process in $C^{1,1}$ open sets with explicit boundary decay rate (Theorem 6.2). However we like to point out that Theorem 6.2 is not used in the proof of Theorem 1.3.

By integrating the two-sided heat kernel estimates in Theorem 1.3 with respect to t, one can easily get the following estimates on the Green function $G^b_D(x, y) = \int_0^\infty p^b_D(t, x, y)dt$.

Corollary 1.4. *Let D be a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. Then there is a constant $c = c(D, d, \alpha, b) \geq 1$ with the dependence on b only through the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that on $D \times D$,*

$$
\begin{aligned}
& c^{-1} \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{|x-y|^\alpha} \right) \\
& \leq G^b_D(x, y) \leq \frac{c}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}}{|x-y|^\alpha} \right).
\end{aligned}
$$

The above result was obtained independently as the main result in [7]. Clearly the heat kernel $p^b_D(t, x, y)$ contains much more information than the Green function $G^b_D(x, y)$. The estimates on $p^b_D(t, x, y)$ are not studied in [7].

The sharp two-sided estimates for $p_D(t, x, y)$, corresponding to the case $b = 0$ in Theorem 1.3, were first established in [9]. Theorem 1.3 indicates that short time Dirichlet heat kernel estimates for the fractional Laplacian in bounded $C^{1,1}$ open sets are stable under gradient perturbations. Such stability should hold for much more general open sets.

We say that an open set D is κ-fat if there exists an $R_1 > 0$ such that for every $x \in D$ and $r \in (0, R_1]$, there is some y such that $B(y, \kappa r) \subset B(x, r) \cap D$. The pair (R_1, κ) is called the characteristics of the κ-fat open set D.

CONJECTURE 1.5. *Let $T > 0$ and D be a bounded κ-fat open subset of $\mathbb{R}^d$. Then there is a constant $c_1 \geq 1$ depending only on T, D, α and b with the dependence on b only through the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that*

$$c_1^{-1} p_D(t, x, y) \leq p^b(t, x, y) \leq c_1 p_D(t, x, y) \quad \textit{for } t \in (0, T] \textit{ and } x, y \in D$$

and

$$c_1^{-1} G_D(x, y) \leq G^b_D(x, y) \leq c_1 G_D(x, y) \quad \textit{for } x, y \in D.$$

In the remainder of this paper, the constants C_1, C_2, C_3, C_4 will be fixed throughout this paper. The lower case constants $c_0, c_1, c_2, \ldots$ can change from one appearance to another. The dependence of the constants on the dimension d and the stability index α will not be always mentioned explicitly. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure. The space of continuous functions on $\mathbb{R}^d$ will be denoted as $C(\mathbb{R}^d)$, while $C_b(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$ denote the space of bounded continuous functions on $\mathbb{R}^d$ and the space of continuous functions on $\mathbb{R}^d$ that vanish at infinity, respectively. For two non-negative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 so that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ in the common domain of definition for f and g.

2. Feller property and Lévy system. Recall that $d \geq 2$ and $\alpha \in (1, 2)$. A (rotationally) symmetric α-stable process $X = \{X_t, t \geq 0, \mathbb{P}_x, x \in \mathbb{R}^d\}$ in $\mathbb{R}^d$ is a Lévy process such that

$$\mathbb{E}_x \left[e^{i\xi \cdot (X_t - X_0)} \right] = e^{-t|\xi|^\alpha} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

The infinitesimal generator of this process X is the fractional Laplacian $\Delta^{\alpha/2}$, which is a prototype of nonlocal operators. The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2} u(x) = \lim_{\varepsilon \downarrow 0} \int_{\{y \in \mathbb{R}^d :\, |y-x| > \varepsilon\}} (u(y) - u(x)) \frac{\mathcal{A}(d, -\alpha)}{|x - y|^{d+\alpha}} \, dy, \tag{2.1}$$

where $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$.

We will use $p(t,x,y)$ to denote the transition density of X (or equivalently the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$). It is well-known (see, e.g., [2, 12]) that

$$p(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \qquad \text{on } (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d.$$

The next two lemmas will be used later.

LEMMA 2.1. *If f is a function belonging to $\mathbb{K}_{d,\alpha-1}$, then for any compact subset K of $\mathbb{R}^d$,*

$$\sup_{x\in\mathbb{R}^d}\int_K \frac{|f|(y)}{|x-y|^{d-\alpha}}\,dy<\infty.$$

Proof. This follows immediately from the fact that $d-\alpha<d+1-\alpha$. We omit the details. □

Recall that we are assuming that b is an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ such that $|b|\in\mathbb{K}_{d,\alpha-1}$.

LEMMA 2.2. *If f is a function belonging to $\mathbb{K}_{d,\alpha-1}$, then*

$$\lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\int_0^t P^b_s|f|(x)ds=0.$$

Proof. By (1.3),

$$\begin{aligned}&\lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\int_0^t P^b_s|f|(x)ds\\ \le& c_1\lim_{t\to 0}\sup_{x\in\mathbb{R}^d}\int_0^t\left(s\int_{B(x,s^{1/\alpha})^c}\frac{|f(y)|}{|y-x|^{d+\alpha}}dy+s^{-d/\alpha}\int_{B(x,s^{1/\alpha})}|f(y)|dy\right)ds.\end{aligned}$$

So it suffices to show that the right hand side is zero. Clearly, for any $s\le 1$, we have

$$\int_{B(x,s^{1/\alpha})}|f(y)|dy\le (s^{1/\alpha})^{d+1-\alpha}\sup_{x\in\mathbb{R}^d}\int_{B(x,1)}\frac{|f(y)|}{|y-x|^{d+1-\alpha}}dy. \tag{2.2}$$

Applying [35, Lemma 1.1], we get

$$\sup_{x\in\mathbb{R}^d}\int_{B(x,s^{1/\alpha})^c}\frac{|f(y)|}{|y-x|^{d+\alpha}}dy\le c_2(s^{1/\alpha})^{d+1-\alpha}(s^{1/\alpha})^{-(d+\alpha)}=c_2s^{1/\alpha-2}. \tag{2.3}$$

Now the conclusion follows immediately from (2.2)–(2.3). □

By the semigroup property of $p^b(t,x,y)$ and (1.3), there are constants $c_1, c_2 \geq 1$ such that on $(0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$,

(2.4)
$$c_1^{-1}e^{-c_2t}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\leq p^b(t,x,y)\leq c_1e^{c_2t}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right).$$

Proposition 2.3. *The family of operators $\{P_t^b; t\geq 0\}$ defined by (1.5) forms a Feller semigroup. Moreover, it satisfies the strong Feller property; that is, for each $t>0$, $P_t^b f$ maps bounded measurable functions to continuous functions.*

Proof. Since $p^b(t,x,y)$ is continuous, by the bounded convergence theorem, P_t^b enjoys the strong Feller property. Moreover, for every $f\in C_\infty(\mathbb{R}^d)$ and $t>0$,

$$\lim_{x\to\infty}|P_t^b f(x)|\leq\lim_{x\to\infty}c_1e^{c_2t}\int_{\mathbb{R}^d}\left(t^{-d/\alpha}\wedge\frac{t}{|y|^{d+\alpha}}\right)|f(x+y)|dy=0$$

and so $P_t^b f\in C_\infty(\mathbb{R}^d)$. By (2.4), we have

$$\begin{aligned}
&\sup_{t\leq t_0}\sup_{x\in\mathbb{R}^d}\mathbb{P}_x(|X_t^b-X_0^b|\geq\delta)\\
\leq\;& c_1e^{c_2t_0}\sup_{t\leq t_0}\sup_{x\in\mathbb{R}^d}\int_{\{y\in\mathbb{R}^d:|x-y|\geq\delta\}}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)dy\\
=\;& c_3e^{c_2t_0}\sup_{t\leq t_0}\int_\delta^\infty r^{d-1}\left(t^{-d/\alpha}\wedge\frac{t}{r^{d+\alpha}}\right)dr\\
\leq\;& c_4e^{c_2t_0}\int_{\delta t_0^{-1/\alpha}}^\infty u^{d-1}\left(1\wedge\frac{1}{u^{d+\alpha}}\right)du
\end{aligned}$$

for some $c_3=c_3(d)>0$ and $c_4=c_4(d)>0$. Thus

(2.5)
$$\lim_{t_0\downarrow 0}\sup_{t\leq t_0}\sup_{x\in\mathbb{R}^d}\mathbb{P}_x(|X_t^b-X_0^b|\geq\delta)\;=\;0.$$

For every $f\in C_b(\mathbb{R}^d)$, $x\in\mathbb{R}^d$ and $\varepsilon>0$, there is a $\delta>0$ so that $|f(y)-f(x)|<\varepsilon$ for every $y\in B(x,\delta)$. Therefore we have by (2.5),

$$\begin{aligned}
&\lim_{t\downarrow 0}|P_t^b f(x)-f(x)|=\lim_{t\downarrow 0}\left|\int_{\mathbb{R}^d}p^b(t,x,y)(f(y)-f(x))dy\right|\\
\leq\;&\lim_{t\downarrow 0}\int_{\{y\in\mathbb{R}^d:|y-x|<\delta\}}p^b(t,x,y)|f(y)-f(x)|dy\\
&+\lim_{t\downarrow 0}2\|f\|_\infty\,\mathbb{P}_x(|X_t^b-x|\geq\delta)\;<\;\varepsilon.
\end{aligned}$$

Therefore for every $f \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, $\lim_{t\downarrow 0} P_t^b f(x) = f(x)$. This completes the proof of the proposition. □

We will need the next result, which is an extension of Theorem 1.2(iii).

Proposition 2.4. *For any $f \in C_c^\infty(\mathbb{R}^d)$ and $g \in C_\infty(\mathbb{R}^d)$, we have*

$$\lim_{t\downarrow 0} \int_{\mathbb{R}^d} t^{-1}(P_t^b f(x) - f(x))g(x)dx = \int_{\mathbb{R}^d} (\mathcal{L}^b f)(x)g(x)dx.$$

Proof. This proposition can be proved by following the proof of [6, Theorem 1], with some obvious modifications. Indeed, one can follow the same argument of the proof of [6, Theorem 1] until the second display on [6, p. 195] with $f \in C_c^\infty(\mathbb{R}^d)$ and $g \in C_\infty(\mathbb{R}^d)$. Let $\varepsilon > 0$ and use the same notations as in [6, p. 195] except that $K := \{z : \mathrm{dist}(z, K_1) \le 1\}$ and we ignore K_2. Since $h(x, y) = \nabla f(y)g(x)$ is still uniformly continuous, there exists a $\delta > 0$ such that for all x, y, z with $|x - z| < \delta$ and $|y - z| < \delta$, we have that $|h(x, y) - h(z, z)| < \varepsilon$. Thus the third display on [6, p. 195] can be modified as

$$\begin{aligned}
&\left| I_t - \int_{\mathbb{R}^d} b(z) \cdot \nabla f(z)g(z)dz \right| \\
\le &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^t \frac{p(t-s,x,z)p(s,z,y)}{t} ds|b(z)||h(x,y) - h(z,z)|dxdydz \\
\le &2\|h\| \int_{K^c}\int_{K_1} \left(\int_{\mathbb{R}^d} p(t-s,x,z)dx\right) \int_0^t \frac{p(s,z,y)}{t} ds|b(z)|dydz \\
&+ 2\|h\| \int_K \int \int_{(B(z,\delta)\times B(z,\delta))^c} \int_0^t \frac{p(t-s,x,z)p(s,z,y)}{t} ds|b(z)|dxdydz \\
&+ \varepsilon \int_K \int \int_{B(z,\delta)\times B(z,\delta)} \int_0^t \frac{p(t-s,x,z)p(s,z,y)}{t} ds|b(z)|dxdydz.
\end{aligned}$$

The remainder of the proof is the same as that of the proof of [6, Theorem 1]. □

The Feller process X^b possesses a Lévy system (see [33]), which describes how X^b jumps. Intuitively, since the infinitesimal generator of X^b is $\mathcal{L}^b$, X^b should satisfy

$$dX_t^b = dX_t + b(X_t^b)dt.$$

So X^b should have the same Lévy system as that of X, as the drift does not contribute to the jumps. This is indeed true and we are going to give a rigorous proof.

It is well known that the symmetric stable process X has Lévy intensity function

$$J(x,y) = \mathcal{A}(d,-\alpha)|x-y|^{-(d+\alpha)}.$$

The Lévy intensity function gives rise to a Lévy system (N,H) for X, where $N(x,dy)=J(x,y)dy$ and $H_t = t$, which describes the jumps of the process X: for any $x \in \mathbb{R}^d$ and any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(s,x,y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ and stopping time T (with respect to the filtration of X),

$$\mathbb{E}_x\left[\sum_{s\le T} f(s, X_{s-}, X_s)\right] = \mathbb{E}_x\left[\int_0^T\left(\int_{\mathbb{R}^d} f(s,X_s,y)J(X_s,y)dy\right)ds\right].$$

(See, for example, [12, Proof of Lemma 4.7] and [13, Appendix A].)

We first show that X^b is a solution to the martingale problem of $\mathcal{L}^b$.

THEOREM 2.5. *For every $x \in \mathbb{R}^d$ and every $f \in C_c^\infty(\mathbb{R}^d)$,*

$$M_t^f := f(X_t^b) - f(X_0^b) - \int_0^t \mathcal{L}^b f(X_s^b)\,ds$$

is a martingale under $\mathbb{P}_x$.

Proof. Define the adjoint operator $P_t^{b,*}$ of P_t^b with respect to the Lebesgue measure by

$$P_t^{b,*}f(x) := \int_{\mathbb{R}^d} p^b(t,y,x)f(y)dy.$$

It follows immediately from (1.3) and the continuity of $p^b(t,x,y)$ that, for any $g \in C_\infty(\mathbb{R}^d)$ and $s>0$, both $P_s^b g$ and $P_s^{b,*}g$ are in $C_\infty(\mathbb{R}^d)$. Thus, for any $f,g \in C_c^\infty(\mathbb{R}^d)$ and $s>0$, by applying Proposition 2.4 with $h = P_s^{b,*}g$ and (1.4), we get that

$$\begin{aligned}
&\lim_{t\downarrow 0}\int_{\mathbb{R}^d} t^{-1}(P_{t+s}^b f(x) - P_s^b f(x))g(x)dx \\
=\;& \lim_{t\downarrow 0}\int_{\mathbb{R}^d} t^{-1}(P_t^b f(x) - f(x))P_s^{b,*}g(x)dx \\
=\;& \int_{\mathbb{R}^d}\mathcal{L}^b f(x)P_s^{b,*}g(x)dx = \int_{\mathbb{R}^d}\mathbb{E}_x\left[\mathcal{L}^b f(X_s^b)\right]g(x)dx
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^d}(P_t^b f(x) - f(x))g(x)dx = \int_{\mathbb{R}^d}\mathbb{E}_x\left[\int_0^t(\mathcal{L}^b f)(X_s^b)ds\right]g(x)dx. \tag{2.6}$$

Using the strong Feller property of P_t^b, Lemmas 2.1 and 2.2, we can easily see that the function

$$x \mapsto P_t^b f(x) - f(x) - \mathbb{E}_x \left[\int_0^t \mathcal{L}^b f(X_s^b) ds \right] = \mathbb{E}_x \left[M_t^f \right]$$

is continuous, and thus is identically zero on $\mathbb{R}^d$ by (2.6). It follows that for any $f \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, M^f is a martingale with respect to $\mathbb{P}_x$. □

Theorem 2.5 in particular implies that $X_t^b = (X_t^{b,1}, \dots, X_t^{b,d})$ is a semi-martingale. By Ito's formula, we have that, for any $f \in C_c^\infty(\mathbb{R}^d)$,

$$f(X_t^b) - f(X_0^b) = \sum_{i=1}^d \int_0^t \partial_i f(X_{s-}^b) dX_s^{b,i} + \sum_{s \le t} \eta_s(f) + \frac{1}{2} A_t(f), \tag{2.7}$$

where

$$\eta_s(f) = f(X_s^b) - f(X_{s-}^b) - \sum_{i=1}^d \partial_i f(X_{s-}^b)(X_s^{b,i} - X_{s-}^{b,i}) \tag{2.8}$$

and

$$A_t(f) = \sum_{i,j=1}^d \int_0^t \partial_i \partial_j f(X_{s-}^b) d\langle (X^{b,i})^c, (X^{b,j})^c \rangle_s. \tag{2.9}$$

Now suppose that A and B are two bounded closed sets having a positive distance from each other. Let $f \in C_c^\infty(\mathbb{R}^d)$ with $f = 0$ on A and $f = 1$ on B. Then we know that $N_t^f := \int_0^t \mathbf{1}_A(X_{s-}^b) dM_s^f$ is a martingale. Combining Theorem 2.5 and (2.7)–(2.9) with (2.1), we get that

$$\begin{aligned} N_t^f &= \sum_{s \le t} \mathbf{1}_A(X_{s-}^b) f(X_s^b) - \int_0^t \mathbf{1}_A(X_s^b) \left(\Delta^{\alpha/2} f(X_s^b) \right) ds \\ &= \sum_{s \le t} \mathbf{1}_A(X_{s-}^b) f(X_s^b) - \int_0^t \mathbf{1}_A(X_s^b) \int_{\mathbb{R}^d} f(y) J(X_s^b, y) dy ds. \end{aligned}$$

By taking a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ with $f_n = 0$ on A, $f_n = 1$ on B and $f_n \downarrow \mathbf{1}_B$, we get that, for any $x \in \mathbb{R}^d$,

$$\sum_{s \le t} \mathbf{1}_A(X_{s-}^b) \mathbf{1}_B(X_s^b) - \int_0^t \mathbf{1}_A(X_s^b) \int_B J(X_s^b, y) dy ds$$

is a martingale with respect to $\mathbb{P}_x$. Thus,

$$\mathbb{E}_x\left[\sum_{s\le t}\mathbf{1}_A(X^b_{s-})\mathbf{1}_B(X^b_s)\right]=\mathbb{E}_x\left[\int_0^t\int_{\mathbb{R}^d}\mathbf{1}_A(X^b_s)\mathbf{1}_B(y)J(X^b_s,y)dyds\right].$$

Using this and a routine measure theoretic argument, we get

$$\mathbb{E}_x\left[\sum_{s\le t}f(X^b_{s-},X^b_s)\right]=\mathbb{E}_x\left[\int_0^t\int_{\mathbb{R}^d}f(X^b_s,y)J(X^b_s,y)dyds\right]$$

for any non-negative measurable function f on $\mathbb{R}^d\times\mathbb{R}^d$ vanishing on $\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : x=y\}$. Finally following the same arguments as in [12, Lemma 4.7] and [13, Appendix A], we get

THEOREM 2.6. *X^b has the same Lévy system (N,H) as X, that is, for any $x\in\mathbb{R}^d$ and any non-negative measurable function f on $\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d$ vanishing on $\{(s,x,y)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d : x=y\}$ and stopping time T (with respect to the filtration of X^b),*

(2.10)
$$\mathbb{E}_x\left[\sum_{s\le T}f(s,X^b_{s-},X^b_s)\right]=\mathbb{E}_x\left[\int_0^T\left(\int_{\mathbb{R}^d}f(s,X^b_s,y)J(X^b_s,y)dy\right)ds\right].$$

For any open subset E of $\mathbb{R}^d$, let $E_\partial=E\cup\{\partial\}$, where ∂ is the cemetery point. Define for $x,y\in E$,

$$N^E(x,dy):=J(x,y)dy,\qquad N^E(x,\partial):=\int_{E^c}J(x,y)dy$$

and $H^E_t:=t$. Then it follows from the theorem above that (N^E,H^E) is a Lévy system for $X^{b,E}$, that is, for any $x\in E$, any non-negative measurable function f on $\mathbb{R}_+\times E\times E_\partial$ vanishing on $\{(s,x,y)\in\mathbb{R}_+\times E\times E : x=y\}$ and stopping time T (with respect to the filtration of $X^{b,E}$),

(2.11)
$$\mathbb{E}_x\left[\sum_{s\le T}f(s,X^{b,E}_{s-},X^{b,E}_s)\right]=\mathbb{E}_x\left[\int_0^T\left(\int_{E_\partial}f(s,X^{b,E}_s,y)N^E(X^{b,E}_s,dy)\right)dH^E_s\right].$$

3. Subprocess of X^b. In this section we study some basic properties of subprocesses of X^b in open subsets. These properties will be used in later sections.

LEMMA 3.1. *For any $\delta > 0$, we have*

$$\lim_{s \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{P}_x(\tau^b_{B(x,\delta)} \leq s) = 0.$$

Proof. By the strong Markov property of X^b (see, e.g., [3, pp. 43–44]), we have for every $x \in \mathbb{R}^d$,

$$\begin{aligned} (3.1) \qquad & \mathbb{P}_x(\tau^b_{B(x,\delta)} \leq s) \\ \leq\ & \mathbb{P}_x\left(\tau^b_{B(x,\delta)} \leq s,\, X^b_s \in B(x,\delta/2)\right) + \mathbb{P}_x\left(X^b_s \in B(x,\delta/2)^c\right) \\ \leq\ & \mathbb{E}_x\left[\mathbb{P}_{X_{\tau^b_{B(x,\delta)}}}\left(|X^b_{s-\tau^b_{B(x,\delta)}} - X^b_0| \geq \delta/2\right);\, \tau^b_{B(x,\delta)} < s\right] \\ & + \mathbb{P}_x\left(|X^b_s - X^b_0| \geq \delta/2\right) \\ \leq\ & 2 \sup_{t \leq s} \sup_{x \in \mathbb{R}^d} \mathbb{P}_x\left(|X^b_t - X^b_0| \geq \delta/2\right). \end{aligned}$$

Now the conclusion of the lemma follows from (2.5). □

A point z on the boundary ∂G of a Borel set G is said to be a regular boundary point with respect to X^b if $\mathbb{P}_z(\tau^b_G = 0) = 1$. A Borel set G is said to be regular with respect to X^b if every point in ∂G is a regular boundary point with respect to X^b.

PROPOSITION 3.2. *Suppose that G is a Borel set of $\mathbb{R}^d$ and $z \in \partial G$. If there is a cone A with vertex z such that int$(A) \cap B(z,r) \subset G^c$ for some $r > 0$, then z is a regular boundary point of G with respect to X^b.*

Proof. This results follows from (1.3) and Blumenthal's zero-one law by a routine argument. For example, the reader can follow the argument in the proof of [25, Proposition 2.2]. Even though [25, Proposition 2.2] is stated for open sets, the proof there works for Borel sets. We omit the details. □

This result implies that all bounded Lipschitz open sets, and in particular, all bounded $C^{1,1}$ open sets, are regular with respect to X^b. Repeating the argument in the second part of the proof of [17, Theorem 1.23], we immediately get the following result.

PROPOSITION 3.3. *Suppose that D is an open set in $\mathbb{R}^d$ and f is a bounded Borel function on ∂D. If z is a regular boundary point of D with respect to X^b and f is continuous at z, then*

$$\lim_{\overline{D}\ni x\to z} \mathbb{E}_x\left[f(X^b_{\tau^b_D});\tau^b_D<\infty\right]=f(z).$$

Let

$$k^b_D(t,x,y) \,:=\, \mathbb{E}_x\left[p^b(t-\tau^b_D, X^b_{\tau^b_D}, y);\, \tau^b_D<t\right] \tag{3.2}$$

and

$$p^b_D(t,x,y) := p^b(t,x,y) - k^b_D(t,x,y). \tag{3.3}$$

Then $p^b_D(t,x,y)$ is the transition density of $X^{b,D}$. This is because by the strong Markov property of X^b, for every $t>0$ and Borel set $A\subset\mathbb{R}^d$,

$$\mathbb{P}_x(X^{b,D}_t\in A) \,=\, \int_A p^b_D(t,x,y)dy. \tag{3.4}$$

We will use $\{P^{b,D}_t\}$ to denote the semigroup of X^D and $\mathcal{L}^b|_D$ to denote the infinitesimal generator of $\{P^{b,D}_t\}$. Using some standard arguments (for example, [4, 17]), we can show the following.

THEOREM 3.4. *Let D be an open set in $\mathbb{R}^d$. The transition density $p^b_D(t,x,y)$ is jointly continuous on $(0,\infty)\times D\times D$. For every $t>0$ and $s>0$,*

$$p^b_D(t+s,x,y) = \int_D p^b_D(t,x,z)p^b_D(s,z,y)dz. \tag{3.5}$$

If z is a regular boundary point of D with respect to X^b, then for any $t>0$ and $y\in D$, $\lim_{D\ni x\to z} p^b_D(t,x,y)=0$.

Proof. Note that by (2.4), there exist $c_1,c_2>0$ such that for all $t_0>0$ and $\delta>0$,

$$\begin{aligned}\sup_{t\le t_0}\sup_{|x-y|\ge\delta} p^b(t,x,y) &\le c_1e^{c_2t_0}\sup_{t\le t_0}\sup_{|x-y|\ge\delta}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\\ &\le c_1e^{c_2t_0}\frac{t_0}{\delta^{d+\alpha}} \,<\, \infty.\end{aligned} \tag{3.6}$$

We first show that $k^b_D(t,x,\cdot)$ is jointly continuous on $(0,\infty)\times D\times D$. For any $\delta>0$, define $D_\delta=\{x\in D:\ \mathrm{dist}(x,D^c)<\delta\}$. For $0\le s<r$ and $x,y\in D_\delta$, define

$$h(s,r,x,y)=\mathbb{E}_x\left[p^b(r-\tau^b_D,X^b_{\tau^b_D},y);\ s\le\tau^b_D<r\right].$$

Note that

$$\begin{aligned}\mathbb{E}_x[h(s,r,X^b_s,y)] &= \mathbb{E}_x[h(s,r,X^b_s,y);s<\tau^b_D]+\mathbb{E}_x[h(s,r,X^b_s,y);s\ge\tau^b_D]\\ &= h(s,r+s,x,y)+\mathbb{E}_x[h(s,r,X^b_s,y);s\ge\tau^b_D]\end{aligned}$$

and

$$\begin{aligned}&k^b_D(t,x,y)=h(0,t,x,y)\\ =\ &h(s,t,x,y)+\mathbb{E}_x\left[p^b(t-\tau^b_D,X^b_{\tau^b_D},y);\ \tau^b_D<s\right]\\ =\ &\mathbb{E}_x[h(s,t-s,X^b_s,y)]-\mathbb{E}_x[h(s,t-s,X^b_s,y);\tau^b_D\le s]\\ &\quad+\mathbb{E}_x\left[p^b(t-\tau^b_D,X^b_{\tau^b_D},y);\ \tau^b_D<s\right].\end{aligned}$$

For all $t_1,t_2\in(0,\infty)$, by (3.6), $p^b(r,z,y)$ is bounded on $(0,t_2]\times D^c\times D_\delta$ by a constant c_3. Consequently, $h(s,r,x,y)$ is bounded by c_3 for all $x,y\in D_\delta$ and $s,r\in(0,t_2]$ with $s<r\wedge(t_1/3)$. Thus we have from the above display as well as (3.6) that for all $t\in[t_1,t_2]$, $s<t_1/2$ and $x,y\in D_\delta$,

$$\begin{aligned}|k^b_D(t,x,y)-\mathbb{E}_x[h(s,t-s,X^b_s,y)]| &\le 2c_3\mathbb{P}_x(\tau^b_D\le s)\\ &\le 2c_3\sup_{z\in\mathbb{R}^d}\mathbb{P}_z(\tau^b_{B(z,\delta)}\le s),\end{aligned}$$

which by Lemma 3.1 goes to 0 as $s\to0$ (uniformly in $(t,x,y)\in[t_1,t_2]\times D_\delta\times D_\delta$). Since $p^b(t,x,y)$ is jointly continuous, it follows from the bounded convergence theorem that $\mathbb{E}_x[h(s,t-s,X^b_s,y)]$ is jointly continuous in $(s,t,y)\in[0,t_1/3]\times[t_1,t_2]\times D_\delta$. On the other hand, for (s,t,y) in any locally compact subset of $(0,t_1/3)\times[t_1,t_2]\times D_\delta$, $\mathbb{E}_x[h(s,t-s,X^b_s,y)]=\int_{\mathbb{R}^d}p(s,x,z)h(s,t-s,z,y)dy$ is equi-continuous in x. Therefore $\mathbb{E}_x[h(s,t-s,X^b_s,y)]$ is jointly continuous in $(s,t,x,y)\in(0,t_1/3)\times[t_1,t_2]\times D_\delta\times D_\delta$. Consequently, $k^b_D(t,x,y)$ is jointly continuous in $(s,t,y)\in[0,t_1/3]\times[t_1,t_2]\times D_\delta$ and hence on $(0,\infty)\times D\times D$. Since $p^b(t,x,y)$ is jointly continuous, we can now conclude that $p^b_D(t,x,y)$ is jointly continuous on $(0,\infty)\times D\times D$.

By Proposition 3.3, the last assertion of the theorem can be proved using the argument in the last paragraph of the proof of [17, Theorem 2.4]. We omit the details. □

The next result is a short time lower bound estimate for $p^b_D(t,x,y)$ near the diagonal. The technique used in its proof is well-known. We give the proof here to demonstrate that symmetry of the process is not needed.

PROPOSITION 3.5. *For any $a_1 \in (0,1)$, $a_2>0$, $a_3>0$ and $R>0$, there is a constant $c=c(d,\alpha,a_1,a_2,a_3,R,b)>0$ with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that such that for all $x_0\in\mathbb{R}^d$ and $r\in(0,R]$,*

(3.7)
$$p^b_{B(x_0,r)}(t,x,y)\ge c\,t^{-d/\alpha}\qquad \text{for all } x,y\in B(x_0,a_1r) \text{ and } t\in[a_2r^\alpha,\ a_3r^\alpha].$$

Proof. Let $\kappa:=a_2/(2a_3)$ and $B_r:=B(x_0,r)$. We first show that there is a constant $c_1\in(0,1)$ so that (3.7) holds for all $r>0$, $x,y\in B(x_0,a_1r)$ and $t\in[\kappa\, c_1r^\alpha,\ c_1r^\alpha]$.

For $r>0$, $t\in[\kappa\, c_1r^\alpha,\ c_1r^\alpha]$, and $x,y\in B(x_0,a_1r)$, since $|x-y|\le 2a_1r\le 2a_1(\kappa c_1)^{-1/\alpha}t^{-1/\alpha}$ and $t\le c_1r^\alpha\le R^\alpha$, we have by (2.4) (3.2) and (3.3),

$$\begin{aligned}(3.8)\quad & p^b_{B_r}(t,x,y)\\ &\ge c_2c_1^{1+d/\alpha}t^{-d/\alpha}-c_3\mathbb{E}_x\left[\mathbf{1}_{\{\tau^b_{B_r}\le t\}}\left((t-\tau^b_{B_r})^{-d/\alpha}\wedge\frac{t-\tau^b_{B_r}}{|X^b_{\tau^b_{B_r}}-y|^{d+\alpha}}\right)\right],\end{aligned}$$

where the positive constants $c_i=c_i(d,\alpha,a_1,a_2,a_3,R,b), i=2,3$, are independent of $c_1\in(0,1]$. Observe that

$$|X^b_{\tau^b_{B_r}}-y|\ge(1-a_1)r\qquad\text{for } t-\tau^b_{B_r}\le t\le c_1r^\alpha,$$

and so

$$\frac{t-\tau^b_{B_r}}{|X^b_{\tau^b_{B_r}}-y|^{d+\alpha}}\le\frac{t-\tau^b_{B_r}}{((1-a_1)r))^{d+\alpha}}\le\frac{c_1^{1+d/\alpha}}{(1-a_1)^{d+\alpha}}t^{-d/\alpha}. \tag{3.9}$$

Note that if $c_1<((1-a_1)/2)^\alpha$, by (2.4), for $t\le c_1r^\alpha$,

$$\begin{aligned}&\mathbb{P}_x\left(X^b_t\notin B(x,(1-a_1)r/2)\right)=\int_{B(x,(1-a_1)r/2)^c}p^b(t,x,y)dy\\ &\le c_3\int_{B(x,(1-a_1)r/2)^c}\frac{t}{|x-y|^{d+\alpha}}dz\ \le\ c_4\frac{t}{r^\alpha}\ \le\ c_4c_1\end{aligned}$$

where c_4 is independent of c_1. Now by the same argument as in the proof of Lemma 3.1, we have

$$\mathbb{P}_x\left(\tau^b_{B(x,(1-a_1)r)}\le t\right)\le 2c_4c_1. \tag{3.10}$$

Consequently, we have from (3.8)–(3.10),

$$
\begin{aligned}
p^b_{B_r}(t,x,y) &\geq \left(c_2 c_1^{1+d/\alpha} - c_3 \frac{c_1^{1+d/\alpha}}{(1-a_1)^{d+\alpha}} \mathbb{P}_x \left(\tau^b_{B_r} \leq t \right) \right) t^{-d/\alpha} \\
&\geq \left(c_2 c_1^{1+d/\alpha} - c_3 \frac{c_1^{1+d/\alpha}}{(1-a_1)^{d+\alpha}} \mathbb{P}_x \left(\tau^b_{B(x,(1-a_1)r)} \leq t \right) \right) t^{-d/\alpha} \\
&\geq c_1^{1+d/\alpha} \left(c_2 - 2 c_4 c_3 \frac{c_1}{(1-a_1)^{d+\alpha}} \right) t^{-d/\alpha}.
\end{aligned}
$$

Clearly we can choose $c_1 < a_3 \wedge ((1-a_1)/2)^\alpha$ small so that $p^b_{B_r}(t,x,y) \geq c_5 t^{-d/\alpha}$. This establishes (3.7) for any $x_0 \in \mathbb{R}^d$, $r > 0$ and $t \in [\kappa\, c_1 r^\alpha,\, c_1 r^\alpha]$.

Now for $r > 0$ and $t \in [a_2 r^\alpha,\, a_3 r^\alpha]$, define $k_0 = [a_3/c_1] + 1$. Here for $a \geq 1$, $[a]$ denotes the largest integer that does not exceed a. Then, since $c_1 < a_3$, $t/k_0 \in [\kappa\, c_1 r^\alpha,\, c_1 r^\alpha]$. Using the semigroup property (3.5) k_0 times, we conclude that for all $x, y \in B(x_0, a_1 r)$ and $t \in [a_2 r^\alpha,\, a_3 r^\alpha]$,

$$
\begin{aligned}
& p^b_{B(x_0,r)}(t,x,y) \\
= & \int_{B(x_0,r)} \cdots \int_{B(x_0,r)} p^b_{B(x_0,r)}(t/k_0, x, w_1) \ldots \\
& \qquad \ldots p^b_{B(x_0,r)}(t/k_0, w_{n-1}, y) dw_1 \ldots dw_{n-1} \\
\geq & \int_{B(x_0,a_1 r)} \cdots \int_{B(x_0,a_1 r)} p^b_{B(x_0,r)}(t/k_0, x, w_1) \ldots \\
& \qquad \ldots p^b_{B(x_0,r)}(t/k_0, w_{n-1}, y) dw_1 \ldots dw_{n-1} \\
\geq & \; c_5 (t/k_0)^{-d/\alpha} \left(c_5 (t/k_0)^{-d/\alpha} |B(0,1)| (a_1 r)^d \right)^{k_0 - 1} \geq c_6 t^{-d/\alpha}.
\end{aligned}
$$

The proof of (3.7) is now complete. □

Using the domain monotonicity of p^b_D, the semigroup property (3.5) and the Lévy system of X^b, the above proposition yields the following.

Corollary 3.6. *For every open subset $D \subset \mathbb{R}^d$, $p^b_D(t,x,y)$ is strictly positive.*

Proof. For $x \in D$, denote by $D(x)$ the connected component of D that contains x. If $y \in D(x)$, using a chaining argument and Proposition 3.5, we have

$$p^b_D(t,x,y) \geq p^b_{D(x)}(t,x,y) > 0.$$

If $y \notin D(x)$, then by using the strong Markov property and the Lévy system (2.10) of X^b,

$$
\begin{aligned}
& p^b_D(t,x,y) \\
= \; & \mathbb{E}_x \left[p^b_D(t-\tau^b_{D(x)}, X^b_{\tau^b_{D(x)}}, y); \, \tau^b_{D(x)} < t \right] \\
\geq \; & \mathbb{E}_x \left[p^b_D(t-\tau^b_{D(x)}, X^b_{\tau^b_{D(x)}}, y); \tau^b_{D(x)} < t, X^b_{\tau^b_{D(x)}} \in D(y) \right] \\
\geq \; & \int_0^t \int_{D(x)} p^b_{D(x)}(s,x,z) \left(\int_{D(y)} J(z,w) p^b_{D(y)}(t-s,w,y) dw \right) dz ds > 0.
\end{aligned}
$$

The corollary is thus proved. □

In the remainder of this section we assume that D is a bounded open set in $\mathbb{R}^d$. The proof of the next lemma is standard. For example, see [24, Lemma 6.1].

Lemma 3.7. *There exist positive constants C_1 and C_2 depending only on d, α,* diam(D) *and b with the dependence on b only through the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that*

$$
p^b_D(t,x,y) \leq C_1 e^{-C_2 t}, \quad (t,x,y) \in (1,\infty) \times D \times D.
$$

Proof. Put $L := \mathrm{diam}(D)$. By (1.3), for every $x \in D$ we have

$$
\begin{aligned}
& \mathbb{P}_x(\tau^b_D \leq 1) \geq \mathbb{P}_x(X^b_1 \in \mathbb{R}^d \setminus D) = \int_{\mathbb{R}^d \setminus D} p^b(1,x,y) dy \\
& \geq c_1 \int_{\mathbb{R}^d \setminus D} \left(1 \wedge \frac{1}{|x-y|^{d+\alpha}} \right) dy \geq c_1 \int_{\{ |z| \geq L \}} \left(1 \wedge \frac{1}{|z|^{d+\alpha}} \right) dz \; > \; 0.
\end{aligned}
$$

Thus

$$
\sup_{x \in D} \int_D p^b_D(1,x,y) dy \; = \; \sup_{x \in D} \mathbb{P}_x(\tau^b_D > 1) \; < \; 1.
$$

The Markov property of X^b then implies that there exist positive constants c_2 and c_3 such that

$$
\int_D p^b_D(t,x,y) dy \; \leq \; c_2 e^{-c_3 t} \quad \text{for } (t,x) \in (0,\infty) \times D.
$$

It follows from (1.3) that there exists $c_4 > 0$ such that $p_D^b(1, x, y) \le p^b(1, x, y) \le c_4$ for every $(x, y) \in D \times D$. Thus for any $(t, x, y) \in (1, \infty) \times D \times D$, we have

$$\begin{aligned} p_D^b(t, x, y) &= \int_D p_D^b(t-1, x, z) p_D^b(1, z, y) dz \\ &\le c_4 \int_D p_D^b(t-1, x, z) dz \le c_2 c_4 e^{-c_3(t-1)}. \end{aligned}$$

□

Combining the result above with (1.3) we know that there exists a positive constant $c_1 = c_1(d, \alpha, \mathrm{diam}(D), b)$ with the dependence on b only through the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that for any $(t, x, y) \in (0, \infty) \times D \times D$,

$$p_D^b(t, x, y) \le c_1 \left(t^{-\frac{d}{\alpha}} \wedge \frac{t}{|x-y|^{d+\alpha}} \right). \tag{3.11}$$

Therefore the Green function $G_D^b(x, y) = \int_0^\infty p_D^b(t, x, y) dt$ is finite and continuous off the diagonal of $D \times D$ and

$$G_D^b(x, y) \le c_2 \frac{1}{|x-y|^{d-\alpha}} \tag{3.12}$$

for some positive constant $c_2 = c_2(d, \alpha, \mathrm{diam}(D), b)$ with the dependence on b only through the rate at which $M^\alpha_{|b|}(r)$ goes to zero.

4. Uniform estimates on Green functions. Let

$$g_D(x, y) := \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2} \right)^{\alpha/2}.$$

The following lemma is needed in deriving sharp bounds on the Green function G_U^b when U is some small $C^{1,1}$ open set. It can be regarded as a new type of $3G$ estimates.

LEMMA 4.1. *There exists a positive constant $C_3 = C_3(d, \alpha)$ such that for all $x, y, z \in D$,*

(4.1)

$$g_D(x, z) \frac{g_D(z, y)}{|z-y| \wedge \delta_D(z)} \le C_3 g_D(x, y) \left(\frac{1}{|x-z|^{d+1-\alpha}} + \frac{1}{|z-y|^{d+1-\alpha}} \right)$$

and

$$\begin{aligned}(4.2)\qquad &\frac{g_D(x,z)}{|x-z|\wedge\delta_D(x)}\,\frac{g_D(z,y)}{|z-y|\wedge\delta_D(z)}\\ \le\;& C_3\,\frac{g_D(x,y)}{|x-y|\wedge\delta_D(x)}\left(\frac{1}{|x-z|^{d+1-\alpha}}+\frac{1}{|z-y|^{d+1-\alpha}}\right).\end{aligned}$$

Proof. Put $r(x,y)=\delta_D(x)+\delta_D(y)+|x-y|$. Note that for $a,b>0$,

$$(4.3)\qquad \frac{ab}{a+b}\le a\wedge b\le 2\frac{ab}{a+b}.$$

Moreover for $x,y\in D$, since

$$\delta_D(x)^2\le\delta_D(x)(\delta_D(y)+|x-y|)\le\delta_D(x)\delta_D(y)+\delta_D(x)^2/2+|x-y|^2/2,$$

one has

$$\delta_D(x)^2\le 2\delta_D(x)\delta_D(y)+|x-y|^2.$$

It follows from these observations that

$$(4.4)\qquad \frac{\delta_D(x)\delta_D(y)}{(r(x,y))^2}\le\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)\le 24\,\frac{\delta_D(x)\delta_D(y)}{(r(x,y))^2}.$$

Consequently, we have

$$(4.5)\qquad g_D(x,y)\asymp\frac{1}{|x-y|^{d-\alpha}}\frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{(r(x,y))^{\alpha}}.$$

Now

$$\begin{aligned}(4.6)\quad & g_D(x,z)\,\frac{g_D(z,y)}{|z-y|\wedge\delta_D(z)}\\ \asymp\;& g_D(x,y)\,\frac{|z-y|+\delta_D(z)}{|z-y|\,\delta_D(z)}\,\frac{\delta_D(z)^{\alpha}\,r(x,y)^{\alpha}}{r(x,z)^{\alpha}\,r(z,y)^{\alpha}}\left(\frac{|x-y|}{|x-z|\cdot|z-y|}\right)^{d-\alpha}\\ \le\;& g_D(x,y)\,\frac{r(y,z)}{|z-y|}\,\frac{\delta_D(z)^{\alpha-1}\,r(x,y)^{\alpha}}{r(x,z)^{\alpha}\,r(z,y)^{\alpha}}\left(\frac{|x-y|}{|x-z|\cdot|z-y|}\right)^{d-\alpha}\\ =\;& g_D(x,y)\,\frac{r(x,y)}{|z-y|\,r(x,z)}\left(\frac{\delta_D(z)\,r(x,y)}{r(x,z)\,r(z,y)}\right)^{\alpha-1}\left(\frac{|x-y|}{|x-z|\cdot|z-y|}\right)^{d-\alpha}.\end{aligned}$$

Since $r(x,y)\le r(x,z)+r(z,y)$,

$$\frac{\delta_D(z)\,r(x,y)}{r(x,z)\,r(z,y)}\le\frac{\delta_D(z)}{r(x,z)}+\frac{\delta_D(z)}{r(z,y)}\le 2.$$

On the other hand, since $\delta_D(y)\le\delta_D(x)+|x-y|$,

$$\begin{aligned}\frac{r(x,y)}{|z-y|\,r(x,z)} &\le 2\,\frac{|x-y|+\delta_D(x)}{|z-y|\,r(x,z)}\le 2\,\frac{|x-z|+(|z-y|+\delta_D(x))}{|z-y|\,r(x,z)}\\ &\le \frac{2}{r(x,z)}+\frac{2}{|z-y|}\le\frac{2}{|x-z|}+\frac{2}{|z-y|}.\end{aligned}$$

Hence we deduce from (4.6) that

$$\begin{aligned}&g_D(x,z)\,\frac{g_D(z,y)}{|z-y|\wedge\delta_D(z)}\\ \le\;& 2^\alpha g_D(x,y)\left(\frac{1}{|x-z|}+\frac{1}{|z-y|}\right)\left(\frac{|x-y|}{|x-z|\cdot|z-y|}\right)^{d-\alpha}\\ \le\;& c_1 g_D(x,y)\left(\frac{1}{|x-z|}+\frac{1}{|z-y|}\right)\left(\frac{1}{|x-z|^{d-\alpha}}+\frac{1}{|z-y|^{d-\alpha}}\right)\\ \le\;& c_2 g_D(x,y)\left(\frac{1}{|x-z|^{d+1-\alpha}}+\frac{1}{|z-y|^{d+1-\alpha}}\right),\end{aligned}$$

where c_1 and c_2 are positive constants depending only on d and α. This proves (4.1).

Now we show that (4.2) holds. Note that by (4.5),

$$\begin{aligned}&\frac{g_D(x,z)}{|x-z|\wedge\delta_D(x)}\,\frac{g_D(z,y)}{|z-y|\wedge\delta_D(z)} \qquad (4.7)\\ \asymp\;& \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-z|^{d+1-\alpha}|z-y|^{d+1-\alpha}}\,\frac{|x-z|\cdot|z-y|}{(|x-z|\wedge\delta_D(x))(|z-y|\wedge\delta_D(z))}\\ &\cdot\frac{\delta_D(z)^\alpha}{r(x,z)^\alpha\,r(z,y)^\alpha}\\ \asymp\;& \frac{g_D(x,y)}{|x-y|\wedge\delta_D(x)}\cdot\frac{|x-y|^{d+1-\alpha}}{|x-z|^{d+1-\alpha}|z-y|^{d+1-\alpha}}\cdot I,\end{aligned}$$

where

$$I:=\frac{|x-y|\wedge\delta_D(x)}{|x-y|}\cdot\frac{|x-z|\cdot|z-y|}{(|x-z|\wedge\delta_D(x))(|z-y|\wedge\delta_D(z))}\,\frac{\delta_D(z)^\alpha\,r(x,y)^\alpha}{r(x,z)^\alpha\,r(z,y)^\alpha}.$$

It follows from (4.3) and the fact that $|x-z|+\delta_D(z)\asymp r(x,z)$ that

$$\begin{aligned}I\;\asymp\;& \frac{|x-y|\,\delta_D(x)}{|x-y|\,(|x-y|+\delta_D(x))}\\ &\cdot\frac{|x-z|\cdot|z-y|(|x-z|+\delta_D(x))(|z-y|+\delta_D(z))}{(|x-z|\,\delta_D(x))(|z-y|\,\delta_D(z))}\,\frac{\delta_D(z)^\alpha\,r(x,y)^\alpha}{r(x,z)^\alpha\,r(z,y)^\alpha}\\ \asymp\;& \frac{\delta_D(z)^{\alpha-1}\,r(x,y)^{\alpha-1}}{r(x,z)^{\alpha-1}\,r(z,y)^{\alpha-1}}\le\delta_D(z)^{\alpha-1}\left(\frac{1}{r(x,z)^{\alpha-1}}+\frac{1}{r(y,z)^{\alpha-1}}\right)\le 2.\end{aligned}$$

The inequality (4.2) now follows from (4.7). □

Recall that G_D is the Green function of X^D. It is known that

$$|\nabla_z G_D(z,y)| \le \frac{d}{|z-y| \wedge \delta_D(z)} G_D(z,y). \tag{4.8}$$

(See [8, Corollary 3.3].) Recall also that b is an $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ such that $|b| \in \mathbb{K}_{d,\alpha-1}$.

PROPOSITION 4.2. *If D is a bounded open set and $\mathbf{1}_D b$ has compact support in D, then G_D^b satisfies*

$$G_D^b(x,y) = G_D(x,y) + \int_D G_D^b(x,z) b(z) \cdot \nabla_z G_D(z,y) dz. \tag{4.9}$$

Proof. Recall that by Theorem 2.5, for every $f \in C_c^\infty(\mathbb{R}^d)$, $M_t^f := f(X_t^b) - f(X_0^b) - \int_0^t \mathcal{L}^b f(X_s^b)\, ds$ is a martingale with respect to $\mathbb{P}_x$. Since $\mathbf{1}_D b$ has compact support in D, in view of (3.12), (4.8) and the fact that $|b| \in \mathbb{K}_{d,\alpha-1}$, $M^f_{t\wedge\tau_D}$ is a uniformly integrable martingale.

Define $D_j := \{x \in D : \text{dist}(x, D^c) > 1/j\}$. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\phi \ge 1$, $\text{supp}[\phi] \subset B(0,1)$ and $\int_{\mathbb{R}^d} \phi(x)dx = 1$. For any $\psi \in C_c(D)$, define $f = G_D\psi$ and $f_n := \phi_n * f$, where $\phi_n(x) :^d \phi(nx)$. Clearly $f_n \in C_c^\infty(\mathbb{R}^d)$ and f_n converges uniformly to $f = G_D\psi$. Fix $j \ge 1$. Since $\mathbb{E}_x[M_0^{f_n}] = \mathbb{E}_x[M^{f_n}_{\tau_{D_j}}]$, and for every $y \in D_j$ and sufficiently large n,

$$\phi_n * (\Delta^{\alpha/2} f)(y) = \int_{B(0,1/n)} \phi_n(z) \Delta^{\alpha/2}(G_D\psi)(y-z)dz,$$

we have, by Dynkin's formula, that for sufficiently large n,

$$\begin{aligned}
& \mathbb{E}_x\left[f_n\left(X^b_{\tau_{D_j}}\right)\right] - f_n(x) \\
= & \int_{D_j} G^b_{D_j}(x,y)\left(\Delta^{\alpha/2} f_n(y) + b(y)\cdot\nabla f_n(y)\right) dy \\
= & \int_{D_j} G^b_{D_j}(x,y)\left(\phi_n * (\Delta^{\alpha/2} f)(y) + b(y)\cdot \phi_n * (\nabla f)(y)\right) dy \\
= & \int_{D_j} G^b_{D_j}(x,y)\left(-\phi_n * \psi(y) + b(y)\cdot\phi_n * (\nabla(G_D\psi)(y)\right) dy.
\end{aligned}$$

Taking $n \to \infty$, we get, by (3.12), (4.8) and the fact that $|b| \in \mathbb{K}_{d,\alpha-1}$,
(4.10)

$$\mathbb{E}_x\left[f\left(X^b_{\tau_{D_j}}\right)\right] - f(x) = \int_D G^b_{D_j}(x,y)\left(-\psi(y) + b(y)\cdot\nabla(G_D\psi)(y)\right) dy.$$

Now using the fact that $\mathbf{1}_D b$ has compact support in D, taking $j\to\infty$, we have by (3.12), (4.8) and the fact that $|b|\in\mathbb{K}_{d,\alpha-1}$,

$$-f(x)=\int_D G_D^b(x,y)\left(-\psi(y)+b(y)\cdot\nabla(G_D\psi)(y)\right)dy.$$

Hence we have

$$-G_D\psi(x)=-G_D^b\psi+G_D^b(b\cdot\nabla G_D\psi).$$

This shows that for each $x\in D$, (4.9) holds for a.e. $y\in D$. Since G_D^b is continuous off the diagonal of $D\times D$, we get that (4.9) holds for all $x,y\in D$. □

We will derive two-sided estimates on Green function of X^b on certain nice open sets when the diameter of such open sets are less than or equal to some constant depending on b only through the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero.

PROPOSITION 4.3. *There exists a positive constant $r_*=r_*(d,\alpha,b)$ with the dependence on b only via the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero such that for any ball $B=B(x_0,r)$ of radius $r\le r_*$ and any $n\ge 1$,*

$$2^{-1}G_B(x,y)\le G_B^{b_n}(x,y)\le 2G_B(x,y),\quad x,y\in B,$$

where

$$b_n(x)=b(x)\mathbf{1}_{B^c}(x)+b(x)\mathbf{1}_{K_n}(x),\quad x\in\mathbb{R}^d \tag{4.11}$$

with K_n being an increasing sequence of compact subsets of B such that $\cup_n K_n=B$.

Proof. It is well known that there exists a constant $c_1=c_1(d,\alpha)>1$ such that

$$\begin{aligned}&c_1^{-1}\frac{1}{|x-y|^{d-\alpha}}\left(1\wedge\frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right)^{\alpha/2}\\ &\le G_B(x,y)\le c_1\frac{1}{|x-y|^{d-\alpha}}\left(1\wedge\frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right)^{\alpha/2}.\end{aligned}\tag{4.12}$$

Define $\widetilde{I}^n_k(x,y)$ recursively for $n\ge1$, $k\ge0$ and $(x,y)\in B\times B$ by

$$\begin{aligned}\widetilde{I}^n_0(x,y)&:=G_B(x,y),\\ \widetilde{I}^n_{k+1}(x,y)&:=\int_B\widetilde{I}^n_k(x,z)b_n(z)\cdot\nabla_z G_B(z,y)dz.\end{aligned}$$

Iterating (4.9) gives that for each $m \ge 2$ and for every $(x,y) \in B \times B$,

$$G_B^{b_n}(x,y) = \sum_{k=0}^{m} \widetilde{I}_k^n(x,y) + \int_B G_B^{b_n}(x,z) b_n(z) \cdot \nabla_z \widetilde{I}_m^n(z,y) dz. \tag{4.13}$$

Using induction, Lemma 4.1, (4.8) with $D = B$ and (4.12), we see that there exists a positive constant c_2 (in fact, one can take $c_2 = 2dC_3c_1^3$ where C_3 is the constant in Lemma 4.1) depending only on d and α such that for $n, k \ge 1$ and $(x,y) \in B \times B$,

$$|\widetilde{I}_k^n(x,y)| \le c_2 G_B(x,y) \left(c_2 M_{|b|}^\alpha(2r)\right)^k \tag{4.14}$$

and

$$|\nabla_x \widetilde{I}_k^n(x,y)| \le c_2 \frac{G_B(x,y)}{|x-y| \wedge \delta_B(x)} \left(c_2 M_{|b|}^\alpha(2r)\right)^k. \tag{4.15}$$

There exists an $\widehat{r}_1 > 0$ depending on b only via the rate at which $M_{|b|}^\alpha(r)$ goes to zero such that

$$c_2\, M_{|b|}^\alpha(r) < \frac{1}{1+2c_2} \qquad \text{for every } 0 < r \le \widehat{r}_1. \tag{4.16}$$

(3.12) and (4.15)–(4.16) imply that if $r \le \widehat{r}_1/2$, then for $n \ge 1$ and $(x,y) \in B \times B$,

$$\begin{aligned}
&\left| \int_B G_B^{b_n}(x,z) b_n(z) \cdot \nabla_z \widetilde{I}_m^n(z,y) dz \right| \\
\le & c_2 \left(\int_B G_B^{b_n}(x,z) |b_n(z)| \frac{G_B(z,y)}{|z-y| \wedge \delta_B(z)} dz \right) \left(c_2 M_{|b|}^\alpha(2r) \right)^m \\
\le & c_3 \left(\int_B \frac{1}{|x-z|^{d-\alpha}} \frac{G_B(z,y)}{|z-y|} |b(z)| dz \right) \left(\frac{1}{1+2c_2} \right)^m \\
\le & c_4 \left(\int_B \frac{1}{|x-z|^{d+1-\alpha}} \frac{|b(z)|}{|z-y|^{d+1-\alpha}} dz \right) \left(\frac{1}{1+2c_2} \right)^m \\
\le & c_5 \left(1+2c_2\right)^{-m} |x-y|^{-(d+1-\alpha)} \int_B \left(\frac{|b(z)|}{|x-z|^{d+1-\alpha}} + \frac{|b|(z)}{|y-z|^{d+1-\alpha}} \right) dz \\
\le & c_6 \left(1+2c_2\right)^{-(m+1)} |x-y|^{-(d+1-\alpha)},
\end{aligned}$$

which goes to zero as $m \to \infty$. In the second inequality, we have used the fact that b_n is compactly supported in B. Thus, by (4.13), $G_B^{b_n}(x,y) = \sum_{k=0}^\infty \widetilde{I}_k^n(x,y)$. Moreover, by (4.14),

$$\sum_{k=1}^\infty |\widetilde{I}_k^n(x,y)| \;\le\; c_2 G_B(x,y) \sum_{k=1}^\infty (1+2c_2)^{-k} \;\le\; G_B(x,y)/2.$$

It follows that for any $x_0 \in \mathbb{R}^d$ and $B = B(x_0, r)$ of radius $r \le \widehat{r}_1/2$,

$$G_B(x,y)/2 \le G_B^{b_n}(x,y) \le 3G_B(x,y)/2 \quad \text{for all } n \ge 1 \text{ and } x, y \in B.$$

This proves the theorem. □

For any bounded $C^{1,1}$ open set D with characteristic (R_0, Λ_0), it is well known (see, for instance [36, Lemma 2.2]) that there exists $L = L(R_0, \Lambda_0, d) > 0$ such that for every $z \in \partial D$ and $r \le R_0$, one can find a $C^{1,1}$ open set $U_{(z,r)}$ with characteristic $(rR_0/L, \Lambda_0 L/r)$ such that $D \cap B(z, r/2) \subset U_{(z,r)} \subset D \cap B(z,r)$. For the remainder of this paper, given a bounded $C^{1,1}$ open set D, $U_{(z,r)}$ always refers to the $C^{1,1}$ open set above.

For $U_{(z,r)}$, we also have a result similar to Proposition 4.3.

Proposition 4.4. *For every $C^{1,1}$ open set D with the characteristic (R_0, Λ_0), there exists $r_0 = r_0(d, \alpha, R_0, \Lambda_0, b) \in (0, (R_0 \wedge 1)/8]$ with the dependence on b only via the rate at which $M_{|b|}^{\alpha}(r)$ goes to zero such that for all $n \ge 1$, $z \in \partial D$ and $r \le r_0$, we have*

$$2^{-1} G_{U_{(z,r)}}(x,y) \le G_{U_{(z,r)}}^{b_n}(x,y) \le 2 G_{U_{(z,r)}}(x,y), \quad x, y \in U_{(z,r)}, \tag{4.17}$$

where

$$b_n(x) = b(x)\mathbf{1}_{U_{(z,r)}^c}(x) + b(x)\mathbf{1}_{K_n}(x), \quad x \in \mathbb{R}^d \tag{4.18}$$

with K_n being an increasing sequence of compact subsets of $U_{(z,r)}$ such that $\cup_n K_n = U_{(z,r)}$.

Proof. It is well known, (see [23], for instance) that, for any bounded $C^{1,1}$ open set U, there exists $c_1 = c_1(R_0, \Lambda_0, \mathrm{diam}(U)) > 1$ such that

$$\begin{aligned} & c_1^{-1} \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_U(x)\delta_U(y)}{|x-y|^2}\right) \\ & \le G_U(x,y) \le c_1 \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_U(x)\delta_U(y)}{|x-y|^2}\right). \end{aligned} \tag{4.19}$$

It follows from this, the fact that $r^{-1}U_{(z,r)}$ is a $C^{1,1}$ open set with characteristic $(R_0/L, \Lambda_0 L)$ and scaling that, for any bounded $C^{1,1}$ open set D with characteristics (R_0, Λ_0), there exists a constant $c_2 = c_2(R_0, \Lambda_0, d) > 1$ such that for all $z \in \partial D$, $r \le R_0$ and $x, y \in U_{(z,r)}$,

$$\begin{aligned} & c_2^{-1} \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_{U_{(z,r)}}(x) \delta_{U_{(z,r)}}(y)}{|x-y|^2} \right) \\ & \le G_{U_{(z,r)}}(x,y) \le c_2 \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_{U_{(z,r)}}(x) \delta_{U_{(z,r)}}(y)}{|x-y|^2} \right). \end{aligned}$$

Now we can repeat the argument of Theorem 4.3 to complete the proof. □

Now we are going to extend Propositions 4.3–4.4 to G^b_B and $G^b_{U(z,r)}$. For the remainder of this section, we let U be either a ball $B = B(x_0, r)$ with $r \le r_*$ where r_* is the constant in Proposition 4.3 or $U(z,r)$ (for a $C^{1,1}$ open set D with the characteristic (R_0, Λ_0)) with $r \le r_0$ where r_0 is the constant in Proposition 4.4. We also let b_n be defined by either (4.11) or (4.18) and we will take care of the two cases simultaneously.

By [6, Lemma 13] and its proof, there exists a constant $C_4 > 0$ such that

$$\int_{\mathbb{R}^d} \int_0^t p(t-s,x,z) |b(z)| |\nabla_z p(s,z,y)| ds d \le C_4 p(t,x,y) \mathbb{N}_b(t)$$

and so

$$\int_{\mathbb{R}^d} \int_0^t p(t-s,x,z) |b_n(z)| |\nabla_z p(s,z,y)| dsdz \le C_4 p(t,x,y) \mathbb{N}_b(t) \tag{4.20}$$

where

$$\mathbb{N}_b(t) := \sup_{w \in \mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^t |b(z)| \left(|w-z|^{-d-1} \wedge s^{-(d+1)/\alpha} \right) dsdz$$

which is finite and goes to zero as $t \to 0$ by [6, Corollary 12]. We remark that the constant C_4 here is independent of t and is not the same constant C_4 from [6, Lemma 13]. Moreover,

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_0^t p(t-s,x,z) |b(z) - b_n(z)| |\nabla_z p(s,z,y)| dsdz \\ \le \;& C_4 p(t,x,y) \mathbb{N}_{b-b_n}(t) \\ = \;& C_4 p(t,x,y) \sup_{w \in \mathbb{R}^d} \int_{U \setminus K_n} \int_0^t |b(z)| \left(|w-z|^{-d-1} \wedge s^{-(d+1)/\alpha} \right) dsdz. \end{aligned} \tag{4.21}$$

Now, by [6, (27)],

$$|p^b_k(t,x,y)| \vee |p^{b_n}_k(t,x,y)| \;\le\; (C_4 \mathbb{N}_b(t))^k p(t,x,y). \tag{4.22}$$

Choose $T_1>0$ small so that

$$C_4\mathbb{N}_b(t)<\frac{1}{2},\quad t\le T_1. \tag{4.23}$$

We will fix this constant T_1 until the end of this section.

LEMMA 4.5. *For all $k\ge 1$ and $(t,x,y)\in(0,T_1]\times\mathbb{R}^d\times\mathbb{R}^d$,*

$$\begin{aligned}&|p_k^{b_n}(t,x,y)-p_k^b(t,x,y)|\\ \le & kC_4 2^{-(k-1)}p(t,x,y)\sup_{w\in\mathbb{R}^d}\int_{U\setminus K_n}\int_0^t|b(z)|\left(|w-z|^{-d-1}\wedge s^{-(d+1)/\alpha}\right)dsdz.\end{aligned}$$

Proof. We prove the lemma by the induction. For $k=1$, we have

$$|p_1^{b_n}(t,x,y)-p_1^b(t,x,y)|\le\int_0^t\int_{\mathbb{R}^d}p(s,x,z)|\nabla_z p(t-s,z,y)|\,|b-b_n|(z)dzds.$$

Thus by (4.21), the lemma is true for $k=1$.

Next we assume that the lemma holds for $k\ge 1$. We will show that the lemma hods for $k+1$. Let

$$I(n,t,x,y):=\int_0^t\int_{\mathbb{R}^d}|p_k^b(s,x,z)||\nabla_z p(t-s,z,y)|\,|b-b_n|(z)dzds$$

and

$$II(n,t,x,y):=\int_0^t\int_{\mathbb{R}^d}|p_k^{b_n}(s,x,z)-p_k^b(s,x,z)||\nabla_z p(t-s,z,y)||b_n(z)|dzds.$$

Then we have

$$|p_{k+1}^{b_n}(t,x,y)-p_{k+1}^b(t,x,y)|\le I(n,t,x,y)\;+\;II(n,t,x,y).$$

By (4.21)–(4.23),

$$\begin{aligned}&I(n,t,x,y)\\ \le\;& (C_4\mathbb{N}_b(t))^k\int_{\mathbb{R}^d}\int_0^t p(t-s,x,z)|b(z)-b_n(z)||\nabla_z p(s,z,y)|dsdz\\ =\;& C_4 2^{-k}p(t,x,y)\sup_{w\in\mathbb{R}^d}\int_{U\setminus K_n}\int_0^t|b(z)|\left(|w-z|^{-d-1}\wedge s^{-(d+1)/\alpha}\right)dsdz.\end{aligned} \tag{4.24}$$

On the other hand, by the induction assumption, (4.20) and (4.23),

(4.25)

$$\begin{aligned}
&II(n,t,x,y)\\
\le& kC_4 2^{-(k-1)}\left(\sup_{w\in\mathbb{R}^d}\int_{U\setminus K_n}\int_0^t |b(z)|\left(|w-z|^{-d-1}\wedge s^{-(d+1)/\alpha}\right)dsdz\right)\\
&\cdot\int_{\mathbb{R}^d}\int_0^t p(s,x,z)|\nabla_z p(t-s,z,y)||b_n(z)|dzds\\
\le& kC_4 2^{-(k-1)}(C_4\mathbb{N}_b(t))p(t,x,y)\\
&\cdot\sup_{w\in\mathbb{R}^d}\int_{U\setminus K_n}\int_0^t |b(z)|\left(|w-z|^{-d-1}\wedge s^{-(d+1)/\alpha}\right)dsdz.\\
\le& kC_4 2^{-k}p(t,x,y)\sup_{w\in\mathbb{R}^d}\int_{U\setminus K_n}\int_0^t |b(z)|\left(|w-z|^{-d-1}\wedge s^{-(d+1)/\alpha}\right)dsdz.
\end{aligned}$$

Combining (4.24) and (4.25), we see that the lemma holds for $k+1$, and thus by induction, the lemma holds for every $k\ge 1$. □

THEOREM 4.6. *$p^{b_n}(t,x,y)$ converges uniformly to $p^b(t,x,y)$ on any $[t_0,T]\times\mathbb{R}^d\times\mathbb{R}^d$, where $0<t_0<T<\infty$. Moreover,*

$$\lim_{n\to\infty} G_U^{b_n}f = G_U^b f \quad \text{for every } f\in C_b(\overline{U}). \tag{4.26}$$

Proof. Without of loss of generality, we may assume that $0<t_0\le T_1/2$, where T_1 is the constant in (4.23). We first consider the case $(t,x,y)\in[t_0,T_1]\times\mathbb{R}^d\times\mathbb{R}^d$. By Theorem 1.2(i) and Lemma 4.5,

$$\begin{aligned}
&\sup_{(t,x,y)\in[t_0,T_1]\times\mathbb{R}^d\times\mathbb{R}^d}|p^b(t,x,y)-p^{b_n}(t,x,y)|\\
\le&\sup_{(t,x,y)\in[t_0,T_1]\times\mathbb{R}^d\times\mathbb{R}^d}\sum_{k=1}^\infty |p_k^{b_n}(t,x,y)-p_k^b(t,x,y)|\\
\le& C_4\sup_{(t,x,y)\in[t_0,T_1]\times\mathbb{R}^d\times\mathbb{R}^d}\sum_{k=1}^\infty k2^{-(k-1)}p(t,x,y)\\
&\cdot\sup_{w\in\mathbb{R}^d}\int_{U\setminus K_n}\int_0^t |b(z)|\left(|w-z|^{-d-1}\wedge s^{-(d+1)/\alpha}\right)dsdz\\
\le& cC_4 t_0^{-d/\alpha}\sup_{w\in\mathbb{R}^d}\int_{U\setminus K_n}\int_0^{T_1} |b(z)|\left(|w-z|^{-d-1}\wedge s^{-(d+1)/\alpha}\right)dsdz,
\end{aligned}$$

which goes to zero as $n\to\infty$.

If $(t,x,y)\in(T_1,3T_1/2]\times\mathbb{R}^d\times\mathbb{R}^d$, using the semigroup property (1.4) with $t_1=T_1/2$,

$$\begin{aligned}
&\sup_{\substack{(t,x,y)\in\\(T_1,3T_1/2]\times\mathbb{R}^d\times\mathbb{R}^d}}|p^b(t,x,y)-p^{b_n}(t,x,y)|\\
\le&\sup_{\substack{(t,x,y)\in\\(T_1,3T_1/2]\times\mathbb{R}^d\times\mathbb{R}^d}}|\int_{\mathbb{R}^d}p^b(t_1,x,z)p^b(t-t_1,z,y)dz\\
&\qquad\qquad-\int_{\mathbb{R}^d}p^{b_n}(t_1,x,z)p^{b_n}(t-t_1,z,y)dz|\\
\le&\sup_{\substack{(t,x,y)\in\\(T_1,3T_1/2]\times\mathbb{R}^d\times\mathbb{R}^d}}\int_{\mathbb{R}^d}p^b(t_1,x,z)|p^b(t-t_1,z,y)-p^{b_n}(t-t_1,z,y)|dz\\
&+\sup_{\substack{(t,x,y)\in\\(T_1,3T_1/2]\times\mathbb{R}^d\times\mathbb{R}^d}}\int_{\mathbb{R}^d}|p^{b_n}(t_1,x,z)-p^b(t_1,x,z)|p^{b_n}(t-t_1,z,y)|dzds,
\end{aligned}$$

which is, by (1.3), less than or equal to $c_1\,t_1^{-d/\alpha}$ times

$$\begin{aligned}
&\sup_{(t,y)\in(T_1,3T_1/2]\times\mathbb{R}^d}\int_{\mathbb{R}^d}|p^b(t-t_1,z,y)-p^{b_n}(t-t_1,z,y)|dz\\
&+\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|p^{b_n}(t_1,x,z)-p^b(t_1,x,z)|dzds.
\end{aligned}$$

Now, by the first case, we see that the above goes to zero as $n\to\infty$. Iterating the above argument one can deduce that the theorem holds for $L=[t_0,kT_0/2]$ for any integer $k\ge2$. This completes the proof of the first claim of the theorem.

First observe that by (1.3), for each fixed $x\in\mathbb{R}^d$ and for every $0\le t_1<t_2<\cdots<t_k$, the distributions of $\{(X^{b_n}_{t_1},\cdots,X^{b_n}_{t_k}),\mathbb{P}_x\}$ form a tight sequence. Next, by the same argument as that for (3.1),

$$\mathbb{P}_x(X^{b_n}_s\notin B(x,r))\le p\quad\text{for all } n\ge1,\ 0\le s\le t \text{ and } x\in\mathbb{R}^d$$

implies

$$\mathbb{P}_x\left(\sup_{s\le t}|X^{b_n}_t-X^{b_n}_0|\ge2r\right)=\mathbb{P}_x\left(\tau^{b_n}_{B(x,2r)}\le t\right)\le2p\ \text{ for all } n\ge1,x\in\mathbb{R}^d.$$

Hence by (1.3) and the same argument leading to (2.5), we have for every $r>0$,

$$\lim_{t\downarrow0}\sup_{n\ge1,x\in\mathbb{R}}\mathbb{P}_x\left(\sup_{s\le t}|X^{b_n}_t-X^{b_n}_0|\ge2r\right)=0.$$

Thus it follows from the Markov property and [22, Theorem 2] (see also [20, Corollary 3.7.4] and [1, Theorem 3]) that, for each $x \in \mathbb{R}^d$, the laws of $\{X^{b_n}, \mathbb{P}_x\}$ form a tight sequence in the Skorohod space $D([0,\infty), \mathbb{R}^d)$. Combining this and Theorem 4.6 with [20, Corollary 4.8.7] we get that X^{b_n} converges to X^b weakly. It follows directly from the definition of Skorohod topology on $D([0,\infty), \mathbb{R}^d)$ (see, e.g., [20, Section 3.5]) that $\{t < \tau^b_U\}$ and $\{t > \tau^b_{\bar{U}}\}$ are disjoint open subsets in $D([0,\infty), \mathbb{R}^d)$. Thus the boundary of $\{t < \tau^b_U\}$ in $D([0,\infty), \mathbb{R}^d)$ is contained in $\{\tau^b_U \le t \le \tau^b_{\bar{U}}\}$. Note that, by the strong Markov property,

$$\mathbb{P}_x\left(\tau^b_U < \tau^b_{\bar{U}}\right) = \mathbb{P}_x\left(\tau^b_U < \tau^b_U + \tau^b_{\bar{U}} \circ \theta_{\tau^b_U}, X^b_{\tau^b_U} \in \partial U\right)$$
$$= \mathbb{P}_x\left(0 < \tau^b_{\bar{U}} \circ \theta_{\tau^b_U}, X^b_{\tau^b_U} \in \partial U\right) = \mathbb{P}_x\left(\mathbb{P}_{X^b_{\tau^b_U}}\left(0 < \tau^b_{\bar{U}}\right); X^b_{\tau^b_U} \in \partial U\right) = 0.$$

The last equality follows from the regularity of $\bar{U}$; that is, $\mathbb{P}_z(\tau^b_{\bar{U}} = 0) = 1$ for every $z \in \partial U$ (see Proposition 3.2). Therefore, using the Lévy system for X^b,

$$\begin{aligned} & \mathbb{P}_x\left(\tau^b_U \le t \le \tau^b_{\bar{U}}\right) = \mathbb{P}_x\left(\tau^b_U = t = \tau^b_{\bar{U}}\right) \\ \le\ & \mathbb{P}_x(X^b_t \in \partial U) + \mathbb{P}_x\left(t = \tau^b_U \text{ and } X^b_{\tau_U -} \neq X^b_{\tau_U}\right) \\ =\ & \int_{\partial U} p^b(t,x,y)dy + 0 = 0, \end{aligned}$$

which implies that the boundary of $\{t < \tau^b_U\}$ in $D([0,\infty), \mathbb{R}^d)$ is $\mathbb{P}_x$-null for every $x \in U$. For every $f \in C_b(\overline{U})$, $f(X^b_t)\mathbf{1}_{\{t<\tau^b_U\}}$ is a bounded function on $D([0,\infty), \mathbb{R}^d)$ with discontinuity contained in the boundary of $\{t < \tau^b_U\}$. Thus we have (cf. Theorem 2.9.1(vi) in [19])

$$\lim_{n\to\infty} \mathbb{E}_x\left[f(X^{b_n}_t)\mathbf{1}_{\{t<\tau^{b_n}_U\}}\right] = \mathbb{E}_x\left[f(X^b_t)\mathbf{1}_{\{t<\tau^b_U\}}\right]. \tag{4.27}$$

Given $f \in C_b(\overline{U})$ and $\varepsilon > 0$, choose $T > 1$ large such that

$$2C_1C_2^{-1}\|f\|_\infty e^{-C_2T} < \varepsilon$$

where C_1 and C_2 are constants in Lemma 3.7 with $D = U$. By the bounded convergence theorem and Fubini's theorem, from (4.27) we have

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}_x\left[\int_0^T f(X^{b_n}_t)\mathbf{1}_{\{t<\tau^{b_n}_U\}}dt\right] &= \lim_{n\to\infty} \int_0^T \mathbb{E}_x\left[f(X^{b_n}_t)\mathbf{1}_{\{t<\tau^{b_n}_U\}}\right]dt \\ &= \mathbb{E}_x\left[\int_0^T f(X^b_t)\mathbf{1}_{\{t<\tau^b_U\}}dt\right]. \end{aligned}$$

On the other hand, by the choice of T and the fact that C_1 and C_2 depending only on d, α, $\mathrm{diam}(U)$ and b with the dependence on b only through the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero, we have by Lemma 3.7

$$\begin{aligned} & \mathbb{E}_x\left[\int_T^\infty f(X_t^{b_n})\mathbf{1}_{\{t<\tau_U^{b_n}\}}dt\right]+\mathbb{E}_x\left[\int_T^\infty f(X_t^{b})\mathbf{1}_{\{t<\tau_U^{b}\}}dt\right] \\ \le\ & \|f\|_\infty\int_T^\infty\left(\int_D(p_D^{b_n}(t,x,y)+p_D^b(t,x,y))dy\right)dt \\ \le\ & 2C_1\|f\|_\infty\int_T^\infty e^{-C_2t}dt<\varepsilon. \end{aligned}$$

This completes the proof of (4.26). □

As immediate consequences of (4.26) and Propositions 4.3–4.4, we get the following

THEOREM 4.7. *There exists a constant $r_*=r_*(d,\alpha,b)>0$ with the dependence on b only via the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero such that for any ball $B=B(x_0,r)$ of radius $r\le r_*$,*

$$2^{-1}G_B(x,y)\le G_B^b(x,y)\le 2G_B(x,y),\quad x,y\in B.$$

THEOREM 4.8. *For every $C^{1,1}$ open set D with the characteristic (R_0,Λ_0), there exists a constant $r_0=r_0(d,\alpha,R_0,\Lambda_0,b)\in(0,(R_0\wedge 1)/8]$ with the dependence on b only via the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero such that for any for any $z\in\partial D$ and $r\le r_0$, we have*

$$2^{-1}G_{U_{(z,r)}}(x,y)\le G^b_{U_{(z,r)}}(x,y)\le 2G_{U_{(z,r)}}(x,y),\quad x,y\in U_{(z,r)}. \tag{4.28}$$

We will need the above two results later on.

5. Duality. In this section we assume that E is an arbitrary bounded open set in $\mathbb{R}^d$. We will discuss some basic properties of $X^{b,E}$ and its dual process under some reference measure. The results of this section will be used later in this paper.

By Theorem 3.4 and Corollary 3.6, $X^{b,E}$ has a jointly continuous and strictly positive transition density $p_E^b(t,x,y)$. Using the continuity of $p_E^b(t,x,y)$ and the estimate

$$p_E^b(t,x,y)\le p^b(t,x,y)\le c_1e^{c_2t}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right),$$

the proof of the next proposition is easy. We omit the details.

PROPOSITION 5.1. *$X^{b,E}$ is a Hunt process and it satisfies the strong Feller property, i.e, for every $f \in L^\infty(E)$, $P_t^E f(x) := \mathbb{E}_x[f(X_t^{b,E})]$ is bounded and continuous in E.*

Define

$$h_E(x) := \int_E G_E^b(y,x)dy \quad \text{and} \quad \xi_E(dx) := h_E(x)dx.$$

The following result says that ξ_E is a reference measure for $X^{b,E}$.

PROPOSITION 5.2. *ξ_E is an excessive measure for $X^{b,E}$, i.e., for every Borel function $f \geq 0$,*

$$\int_E f(x)\xi_E(dx) \geq \int_E \mathbb{E}_x\left[f(X_t^{b,E})\right]\xi_E(dx).$$

Moreover, h_E is a strictly positive, bounded continuous function on E.

Proof. By the Markov property, we have for any Borel function $f \geq 0$ and $x \in E$,

$$\int_E \mathbb{E}_y\left[f(X_t^{b,E})\right] G_E^b(x,y)dy = \mathbb{E}_x \int_0^\infty \mathbb{E}_{X_s^{b,E}}\left[f(X_t^{b,E})\right] ds$$
$$= \int_0^\infty \mathbb{E}_x\left[f(X_{t+s}^{b,E})\right] ds \leq \int_E f(y)G_E^b(x,y)dy.$$

Integrating with respect to x, we get by Fubini's theorem,

$$\int_E \mathbb{E}_y\left[f(X_t^{b,E})\right] h_E(y)dy \leq \int_E f(y)h_E(y)dy.$$

The second claim follows from (3.12), the continuity of G_E^b and the strict positivity of p_E^b (Corollary 3.6). □

We define a transition density with respect to the reference measure ξ_E by

$$\overline{p}_E^b(t,x,y) := \frac{p_E^b(t,x,y)}{h_E(y)}.$$

Let

$$\overline{G}_E^b(x,y) := \int_0^\infty \overline{p}_E^b(t,x,y)dt = \frac{G_E^b(x,y)}{h_E(y)}.$$

Then $\overline{G}^b_E(x,y)$ is the Green function of $X^{b,E}$ with respect to the reference measure ξ_E.

Before we discuss properties of $\overline{G}^b_E(x,y)$, let us first recall some definitions.

Definition 5.3. *Suppose that U is an open subset of E. A Borel function u on E is said to be*

(i) *harmonic in U with respect to $X^{b,E}$ if*

$$u(x) = \mathbb{E}_x \left[u(X^{b,E}_{\tau^b_B}) \right], \qquad x \in B, \tag{5.1}$$

for every bounded open set B with $\overline{B} \subset U$;

(ii) *excessive with respective to $X^{b,E}$ if u is non-negative and for every $t > 0$ and $x \in E$*

$$u(x) \geq \mathbb{E}_x \left[u(X^{b,E}_t) \right] \quad \textit{and} \quad u(x) = \lim_{t \downarrow 0} \mathbb{E}_x \left[u(X^{b,E}_t) \right];$$

(iii) *a potential with respect to $X^{b,E}$ if it is excessive with respect to $X^{b,E}$ and for every sequence $\{U_n\}_{n\geq 1}$ of open sets with $\overline{U_n} \subset U_{n+1}$ and $\cup_n U_n = E$,*

$$\lim_{n\to\infty} \mathbb{E}_x \left[u(X^{b,E}_{\tau^b_{U_n}}) \right] = 0; \qquad \xi_E\textit{-a.e. } x \in E;$$

(iv) *a pure potential with respect to $X^{b,E}$ if it is a potential with respect to $X^{b,E}$ and*

$$\lim_{t\to\infty} \mathbb{E}_x \left[u(X^{b,E}_t) \right] = 0, \qquad \xi_E\textit{-a.e. } x \in E;$$

(v) *regular harmonic with respect to $X^{b,E}$ in U if u is harmonic with respect to $X^{b,E}$ in U and (5.1) is true for $B = U$.*

We list some properties of the Green function $\overline{G}^b_E(x,y)$ of $X^{b,E}$ that we will need later.

(A1) $\overline{G}^b_E(x,y) > 0$ for all $(x,y) \in E \times E$; $\overline{G}^b_E(x,y) = \infty$ if and only if $x = y \in E$.

(A2) For every $x \in E$, $\overline{G}^b_E(x,\cdot\,)$ and $\overline{G}^b_E(\,\cdot\,,x)$ are extended continuous in E.

(A3) For every compact subset K of E, $\int_K \overline{G}^b_E(x,y)\xi_E(dy) < \infty$.

(A3) follows from (3.12) and Proposition 5.2. Both (A1) and (A2) follow from (3.12), Proposition 5.2, domain monotonicity of Green functions and the lower bound in (4.12).

From (A1)–(A3), we know that the process $X^{b,E}$ satisfies the condition (R) on [18, p. 211] and the conditions (a)–(b) of [18, Theorem 5.4]. It follows from [18, Theorem 5.4] that $X^{b,E}$ satisfies Hunt's Hypothesis (B). Thus by [18, Theorem 13.24] $X^{b,E}$ has a dual process $\widehat{X}^{b,E}$, which is a standard process.

In addition, we have the following.

(A4) For each y, $x \mapsto \overline{G}^b_E(x,y)$ is excessive with respect to $X^{b,E}$ and harmonic with respect to $X^{b,E}$ in $E\setminus\{y\}$. Moreover, for every open subset U of E, we have

$$\mathbb{E}_x\left[\overline{G}^b_E(X^{b,E}_{T^b_U}, y)\right] = \overline{G}^b_E(x,y), \qquad (x,y) \in E \times U \tag{5.2}$$

where $T^b_U := \inf\{t > 0 : X^{b,E}_t \in U\}$. In particular, for every $y \in E$ and $\varepsilon > 0$, $\overline{G}^b_E(\,\cdot\,, y)$ is regular harmonic in $E \setminus B(y,\varepsilon)$ with respect to $X^{b,E}$.

Proof of (A4). It follows from [16, Proposition 3] and [29, Theorem 2 on p. 373] that, to prove (A4), it suffices to show that, for any $x \in E \setminus U$, the function

$$y \mapsto \mathbb{E}_x\left[\overline{G}^b_E(X^{b,E}_{T_U}, y)\right]$$

is continuous on U. (See the proof of [31, Theorem 1].) Fix $x \in E \setminus U$ and $y \in U$. Put $r := \delta_U(y)$. Let $\widehat{y} \in B(y, r/4)$. It follows from (2.11) and (3.12) that, for any $\delta \in (0, \frac{r}{2})$,

$$\begin{aligned}
&\mathbb{E}_x\left[\overline{G}^b_E(X^{b,E}_{T^b_U}, \widehat{y}); X^{b,E}_{T^b_U} \in B(y,\delta)\right] \\
&= \int_{B(y,\delta)} \left(\int_{E\setminus U} G^b_{E\setminus U}(x,w)\frac{\mathcal{A}(d,-\alpha)}{|w-z|^{d+\alpha}}dw\right)\overline{G}^b_E(z,\widehat{y})dz \\
&\le \frac{c_1}{\inf_{\widetilde{y}\in\overline{B(y,r/4)}} h_E(\widetilde{y})}\int_{B(y,\delta)}\left(\int_{E\setminus U}\frac{1}{|x-w|^{d-\alpha}}\frac{1}{|w-z|^{d+\alpha}}dw\right)\frac{dz}{|z-\widehat{y}|^{d-\alpha}}.
\end{aligned}$$

Thus, for any $\epsilon > 0$, there is a $\delta \in (0, \frac{r}{2})$ such that

$$\mathbb{E}_x\left[\overline{G}^b_E(X^{b,E}_{T_U}, y); X^{b,E}_{T_U} \in B(y,\delta)\right] \le \frac{\varepsilon}{4} \quad \text{for every } \widehat{y} \in B(y, r/4). \tag{5.3}$$

Now we fix this δ and let $\{y_n\}$ be a sequence of points in $B(y, r/4)$ converging to y. Since the function $(z,u) \mapsto \overline{G}^b_E(z,u)$ is bounded and continuous in

$(E\setminus B(y,\delta))\times B(y,\frac{\delta}{2})$, we have by the bound convergence theorem that there exists $n_0>0$ such that for all $n\ge n_0$,

(5.4)
$$\left|\mathbb{E}_x\left[\overline{G}^b_E(X^{b,E}_{T_U},y);X^{b,E}_{T_U}\in B(y,\delta)^c\right]-\mathbb{E}_x\left[\overline{G}^b_E(X^{b,E}_{T_U},y_n);X^{b,E}_{T_U}\in B(y,\delta)^c\right]\right| \le\frac{\varepsilon}{2}.$$

Since $\varepsilon>0$ is arbitrary, combining (5.3) and(5.4), the proof of (A4) is now complete. □

Theorem 5.4. *For each $y\in E$, $x\mapsto\overline{G}^b_E(x,y)$ is a pure potential with respect to $X^{b,E}$. In fact, for every sequence $\{U_n\}_{n\ge1}$ of open sets with $\overline{U_n}\subset U_{n+1}$ and $\cup_nU_n=E$, $\lim_{n\to\infty}\mathbb{E}_x[\overline{G}^b_E(X^{b,E}_{\tau^b_{U_n}},y)]=0$ for every $x\neq y$ in E. Moreover, for every $x,y\in E$, we have $\lim_{t\to\infty}\mathbb{E}_x[\overline{G}^b_E(X^{b,E}_t,y)]=0$.*

Proof. For $y\in E$, let $X^{b,E,y}$ denote the h-conditioned process obtained from $X^{b,E}$ with $h(\cdot)=\overline{G}^b_E(\cdot,y)$ and let $\mathbb{E}^y_x$ denote the expectation for $X^{b,E,y}$ starting from $x\in E$.

Let $x\neq y\in E$. Using (A1)-(A2), (A4) and the strict positivity of $\overline{G}^b_E$, and applying [30, Theorem 2], we get that the lifetime $\zeta^{b,E,y}$ of $X^{b,E,y}$ is finite $\mathbb{P}^y_x$-a.s. and

$$\lim_{t\uparrow\zeta^{b,E,y}}X^{b,E,y}_t=y\qquad\mathbb{P}^y_x\text{-a.s.}. \tag{5.5}$$

Let $\{E_k,\ k\ge1\}$ be an increasing sequence of relatively compact open subsets of E such that $E_k\subset\overline{E_k}\subset E$ and $\cup_{k=1}^\infty E_k=E$. Then

$$\mathbb{E}_x\left[\overline{G}^b_E(X^{b,E}_{\tau^b_{E_k}},y)\right]=\overline{G}^b_E(x,y)\mathbb{P}^y_x(\tau^b_{E_k}<\zeta^{b,E,y}).$$

By (5.5), we have $\lim_{k\to\infty}\mathbb{P}^y_x(\tau^b_{E_k}<\zeta^{b,E,y})=0$. Thus

$$\lim_{k\to\infty}\mathbb{E}_x[\overline{G}^b_E(X^{b,E}_{\tau^b_{E_k}},y)]=0.$$

The last claim of the theorem is easy. By (3.11) and (3.12), for every $x,y\in E$, we have

$$\mathbb{E}_x\left[\overline{G}^b_E(X^{b,E}_t,y)\right]\le\frac{c}{t^{\frac{d}{\alpha}}h_E(y)}\int_E\frac{dz}{|z-y|^{d-\alpha}},$$

which converges to zero as t goes to ∞. □

We note that

$$\int_E \overline{G}_E^b(x,y)\xi_E(dx) \le \frac{\|h_E\|_\infty}{h_E(y)} \int_E G_E^b(x,y)dx = \|h_E\|_\infty < \infty.$$

So we have

(A5) for every compact subset K of E, $\int_K \overline{G}_E^b(x,y)\xi_E(dx) < \infty$.

Using (A1)–(A5), (3.12) and Theorem 5.4 we get from [28, 29] that $X^{b,E}$ has a Hunt process as a dual.

Theorem 5.5. *There exists a transient Hunt process $\widehat{X}^{b,E}$ in E such that $\widehat{X}^{b,E}$ is a strong dual of $X^{b,E}$ with respect to the measure ξ_E, that is, the density of the semigroup $\{\widehat{P}_t^E\}_{t\ge 0}$ of $\widehat{X}^{b,E}$ is given by $\overline{p}_E^b(t,y,x)$ and thus*

$$\int_E f(x)P_t^E g(x)\xi_E(dx) = \int_E g(x)\widehat{P}_t^E f(x)\xi_E(dx) \quad \text{for all } f,g \in L^2(E,\xi_E).$$

Proof. The existence of a dual Hunt process $\widehat{X}^{b,E}$ is proved in [28, 29]. To show $\widehat{X}^{b,E}$ is transient, we need to show that for every compact subset K of E, $\int_K \overline{G}_E^b(x,y)\xi_E(dx)$ is bounded. This is just (A5) above. □

In Theorem 2.6, we have determined a Lévy system (N,H) for X^b with respect to the Lebesgue measure dx. To derive a Lévy system for $\widehat{X}^{b,E}$, we need to consider a Lévy system for $X^{b,E}$ with respect to the reference measure $\xi_E(dx)$. One can easily check that, if

$$\begin{aligned}
\overline{N}^E(x,dy) &:= \frac{J(x,y)}{h_E(y)}\xi_E(dy) \qquad \text{for } (x,y)\in E\times E,\\
\overline{N}^E(x,\partial) &:= \int_{E^c} J(x,y)dy \qquad \text{for } x\in E,
\end{aligned}$$

and $\overline{H}_t^E := t$, then $(\overline{N}^E, \overline{H}^E)$ is a Lévy system for $X^{b,E}$ with respect to the reference measure $\xi_E(dx)$. It follows from [21] that a Lévy system $(\widehat{N}^E, \widehat{H}^E)$ for $\widehat{X}^{b,E}$ satisfies $\widehat{H}_t^E = t$ and

$$\widehat{N}^E(y,dx)\xi_E(dy) = \overline{N}^E(x,dy)\xi_E(dx).$$

Therefore, using $J(x,y)=J(y,x)$, we have for every stopping time T with respect to the filtration of $\widehat{X}^{b,E}$,

$$
\begin{aligned}
(5.6)\qquad & \mathbb{E}_x\left[\sum_{s\le T} f(s,\widehat{X}^{b,E}_{s-},\widehat{X}^{b,E}_s)\right] \\
= & \ \mathbb{E}_x\left[\int_0^T\left(\int_E f(s,\widehat{X}^{b,E}_s,y)\frac{J(\widehat{X}^{b,E}_s,y)}{h_E(\widehat{X}^{b,E}_s)}\xi_E(dy)\right)d\widehat{H}^E_s\right] \\
= & \ \mathbb{E}_x\left[\int_0^T\left(\int_E f(s,\widehat{X}^{b,E}_s,y)\frac{J(\widehat{X}^{b,E}_s,y)h_E(y)}{h_E(\widehat{X}^{b,E}_s)}dy\right)ds\right].
\end{aligned}
$$

That is,

$$\widehat{N}^E(x,dy)=\frac{J(x,y)h_E(y)}{h_E(x)}dy.$$

Let

$$P^{b,E}_t f(x):=\int_E \overline{p}^b_E(t,x,y)f(y)\xi_E(dy)$$

and

$$\widehat{P}^{b,E}_t f(x):=\int_E \overline{p}^b_E(t,y,x)f(y)\xi_E(dy).$$

For any open subset U of E, we use $\widehat{X}^{b,E,U}$ to denote the subprocess of $\widehat{X}^{b,E}$ in U, i.e., $\widehat{X}^{b,E,U}_t(\omega)=\widehat{X}^{b,E}_t(\omega)$ if $t<\widehat{\tau}^{b,E}_U(\omega)$ and $\widehat{X}^{b,E,U}_t(\omega)=\partial$ if $t\ge\widehat{\tau}^{b,E}_U(\omega)$, where $\widehat{\tau}^{b,E}_U:=\inf\{t>0:\widehat{X}^{b,E}_t\notin U\}$ and ∂ is the cemetery state. Then by [34, Theorem 2] and Remark 2 following it, $X^{b,U}$ and $\widehat{X}^{b,E,U}$ are dual processes with respect to ξ_E. Now we let

$$\widehat{p}^{b,E}_U(t,x,y):=\frac{p^b_U(t,y,x)h_E(y)}{h_E(x)}. \tag{5.7}$$

By the joint continuity of $p^b_U(t,x,y)$ (Theorem 3.4) and the continuity and positivity of h_E (Proposition 5.2), we know that $\widehat{p}^{b,E}_U(t,\cdot,\cdot)$ is jointly continuous on $U\times U$. Thus we have the following.

THEOREM 5.6. *For every open subset U, $\widehat{p}^{b,E}_U(t,x,y)$ is strictly positive and jointly continuous on $U\times U$ and is the transition density of $\widehat{X}^{b,E,U}$ with respect to the Lebesgue measure. Moreover,*

$$\widehat{G}^{b,E}_U(x,y):=\frac{G^b_U(y,x)h_E(y)}{h_E(x)} \tag{5.8}$$

is the Green function of $\widehat{X}^{b,E,U}$ *with respect to the Lebesgue measure so that for every nonnegative Borel function* f,

$$\mathbb{E}_x\left[\int_0^{\widehat{\tau}_U^{b,E}} f\left(\widehat{X}_t^{b,E}\right)dt\right] = \int_U \widehat{G}_U^{b,E}(x,y)f(y)dy.$$

6. Scaling property and uniform boundary Harnack principle. In this section, we first study the scaling property of X^b, which will be used later in this paper.

For $\lambda > 0$, let $Y_t^{b,\lambda} := \lambda X^b_{\lambda^{-\alpha}t}$. For any function f on $\mathbb{R}^d$, we define $f^\lambda(\,\cdot\,) = f(\lambda\,\cdot\,)$. Then we have

$$\mathbb{E}_x\left[f(Y_t^{b,\lambda})\right] = \int_{\mathbb{R}^d} p^b(\lambda^{-\alpha}t, \lambda^{-1}x, y)f^\lambda(y)dy.$$

It follows from Theorem 1.2(iii) that for any $f, g \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\lim_{t\downarrow 0}\int_{\mathbb{R}^d} t^{-1}(\mathbb{E}_x[f(Y_t^{b,\lambda})] - f(x))g(x)dx\\
=\;& \lim_{t\downarrow 0}\int_{\mathbb{R}^d} \lambda^{-\alpha}(\lambda^\alpha t)^{-1}(P^b_{\lambda^{-\alpha}t}f^\lambda(\lambda^{-1}x) - f^\lambda(\lambda^{-1}x))g^\lambda(\lambda^{-1}x)dx\\
=\;& \lim_{t\downarrow 0}\int_{\mathbb{R}^d} \lambda^{d-\alpha}(\lambda^\alpha t)^{-1}(P^b_{\lambda^{-\alpha}t}f^\lambda(z) - f^\lambda(z))g^\lambda(z)dz\\
=\;& \lambda^{d-\alpha}\int_{\mathbb{R}^d}\left(-(-\Delta)^{\alpha/2}f^\lambda(z) + b(z)\cdot\nabla f^\lambda(z)\right)g^\lambda(z)dz\\
=\;& \lambda^{d-\alpha}\int_{\mathbb{R}^d}\left(-(-\Delta)^{\alpha/2}f^\lambda(z) + \lambda b(z)\cdot\nabla f(\lambda z)\right)g(\lambda z)dz\\
=\;& \int_{\mathbb{R}^d}\left(-(-\Delta)^{\alpha/2}f(x) + \lambda^{1-\alpha}b(\lambda^{-1}x)\cdot\nabla f(x)\right)g(x)dx.
\end{aligned}$$

Thus $\{\lambda X^{b,D}_{\lambda^{-\alpha}t}, t \ge 0\}$ is the subprocess of $X^{\lambda^{1-\alpha}b(\lambda^{-1}\cdot)}$ in λD. So for any $\lambda > 0$, we have

(6.1)
$$p_{\lambda D}^{\lambda^{1-\alpha}b(\lambda^{-1}\cdot)}(t,x,y) = \lambda^{-d}p_D^b(\lambda^{-\alpha}t, \lambda^{-1}x, \lambda^{-1}y) \qquad \text{for } t>0 \text{ and } x,y\in\lambda D,$$

$$G_{\lambda D}^{\lambda^{1-\alpha}b(\lambda^{-1}\cdot)}(x,y) = \lambda^{\alpha-d}G_D^b(\lambda^{-1}x, \lambda^{-1}y) \qquad \text{for } x,y\in\lambda D. \tag{6.2}$$

Define

$$b_\lambda(x) := \lambda^{1-\alpha}b(x/\lambda) \quad \text{for } x\in\mathbb{R}^d. \tag{6.3}$$

Then we have

$$\begin{aligned} M^{\alpha}_{|b_\lambda|}(r) &= \lambda^{1-\alpha}\sum_{i=1}^{d}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le r}\frac{|b^i|(\lambda^{-1}y)dy}{|x-y|^{d+1-\alpha}} \\ &= \sum_{i=1}^{d}\sup_{\widehat{x}\in\mathbb{R}^d}\int_{|\widehat{x}-z|\le\lambda^{-1}r}\frac{|b^i|(z)dz}{|\widehat{x}-z|^{d+1-\alpha}} = M^{\alpha}_{|b|}(\lambda^{-1}r). \end{aligned}$$

Therefore for every $\lambda\ge 1$ and $r>0$,

$$M^{\alpha}_{|b_\lambda|}(r) = M^{\alpha}_{|b|}(\lambda^{-1}r) \le M^{\alpha}_{|b|}(r). \tag{6.4}$$

In the remainder of this paper, we fix a bounded $C^{1,1}$ open set D in $\mathbb{R}^d$ with $C^{1,1}$ characteristics (R_0,Λ_0) and a ball $E\subset\mathbb{R}^d$ centered at the origin so that $D\subset\frac14 E$. Define

$$M := M(b,E) := \sup_{x,y\in\frac34 E}\frac{h_E(x)}{h_E(y)}, \tag{6.5}$$

which is a finite positive constant no less than 1. Note that, in view of the scaling property (6.2), we have

$$M(b,E) = M(b_\lambda,\lambda E). \tag{6.6}$$

Although E and D are fixed, the constants in all the results of this section will depend only on d,α,R_0,Λ_0, b and M (not the diameter of D directly) with the dependence on b only via the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero. In view of (6.4) and (6.6), the results of this section in particular hold for $\mathcal{L}^{b_\lambda}$ (equivalently, for X^{b_λ}) and the pair $(\lambda D,\lambda E)$ for every $\lambda\ge 1$.

In the remainder of this section, we will establish a uniform boundary Harnack principle on D for certain harmonic functions for $X^{b,E}$ and $\widehat{X}^{b,E}$. Since the arguments are mostly similar for $X^{b,E}$ and $\widehat{X}^{b,E}$, we will only give the proof for $\widehat{X}^{b,E}$.

A real-valued function u on E is said to be harmonic in an open set $U\subset E$ with respect to $\widehat{X}^{b,E}$ if for every relatively compact open subset B with $\overline{B}\subset U$,

$$\mathbb{E}_x\left[\left|u(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_B})\right|\right]<\infty \quad\text{and}\quad u(x)=\mathbb{E}_x\left[u(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_B})\right] \qquad \text{for every } x\in B. \tag{6.7}$$

A real-valued function u on E is said to be regular harmonic in an open set $U\subset E$ with respect to $\widehat{X}^{b,E}$ if (6.7) is true with $B=U$. Clearly, a regular harmonic function in U is harmonic in U.

For any bounded open set U, define the Poisson kernel for X^b of U as

$$K_U^b(x,z) := \int_U G_U^b(x,y)J(y,z)dy, \quad (x,z) \in U \times (\mathbb{R}^d \setminus \overline{U}).$$

When $U \subset E$, we define the Poisson kernel for $\widehat{X}^{b,E}$ of $U \subset E$ as

$$\widehat{K}_U^{b,E}(x,z) := \frac{h_E(z)}{h_E(x)} \int_U G_U^b(y,x)J(z,y)dy, \quad (x,z) \in U \times (E \setminus \overline{U}). \tag{6.8}$$

By (2.11) and (5.6), we have

$$\mathbb{E}_x \left[f(X^{b,E}_{\tau^b_U});\, X^b_{\tau^b_U -} \neq X^b_{\tau^b_U} \right] = \int_{\overline{U}^c} K_U^b(x,z)f(z)dz$$

and

$$\begin{aligned}
& \mathbb{E}_x \left[f(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U});\, \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U -} \neq \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U} \right] \\
= \; & \mathbb{E}_x \int_0^{\widehat{\tau}^{b,E}_U} \left(\int_{\overline{U}^c} f(z) \frac{J(\widehat{X}^{b,E}_s, z)h_E(z)}{h_E(\widehat{X}^{b,E}_s)} dz \right) ds \\
= \; & \int_U \frac{G_U^b(y,x)h_E(y)}{h_E(x)} \int_{\overline{U}^c} f(z) \frac{J(y,z)h_E(z)}{h_E(y)} dzdy \qquad (6.9) \\
= \; & \int_{\overline{U}^c} \widehat{K}_U^{b,E}(x,z)f(z)dz.
\end{aligned}$$

Lemma 6.1. *Suppose that U is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$ with $U \subset \frac{1}{2}E$ and* $\mathrm{diam}(U) \le 3r_*$ *where r_* is the constant in Theorem 4.7. Then*

$$\mathbb{P}_x(X^b_{\tau^b_U} \in \partial U) = 0 \qquad \text{for every } x \in U \tag{6.10}$$

and

$$\mathbb{P}_x(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U} \in \partial U) = 0 \qquad \text{for every } x \in U. \tag{6.11}$$

Proof. The proof is similar to that of [4, Lemma 6]. For our readers' convenience, we are going to spell out the details of the proof of (6.11). Let $B_x := B(x, \delta_U(x)/3)$. By (5.6) we have for $x \in U$,

$$\begin{aligned}
& \mathbb{P}_x \left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{B_x}} \in (\frac{3}{4}E) \setminus U \right) \\
= \; & \int_{B_x} \frac{G_{B_x}^b(y,x)h_E(y)}{h_E(x)} \left(\int_{(\frac{3}{4}E)\setminus U} \frac{J(y,z)h_E(z)}{h_E(y)} dz \right) dy.
\end{aligned}$$

Since $\mathrm{diam}(U)\le 3r_*$, $\delta_U(x)/3\le r_*$, thus by Theorem 4.7, for $x\in U$,

$$
\begin{aligned}
(6.12)\qquad & \mathbb{P}_x\left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{B_x}}\in(\frac{3}{4}E)\setminus U\right)\\
\ge\ & c_1\left(\inf_{u,v\in\frac34 E}\frac{h_E(u)}{h_E(v)}\right)\int_{B_x}G_{B_x}(x,y)\left(\int_{(\frac34 E)\setminus U}J(y,z)\,dz\right)dy\\
\ge\ & c_1\,M^{-1}\,\mathbb{P}_x\left(X_{\tau_{B_x}}\in(\frac{3}{4}E)\setminus U\right),
\end{aligned}
$$

where M is the constant defined in (6.5). Let $V_x:=B(\delta_U(x)^{-1}x,1/3)$. By the scaling property of X,

$$
\begin{aligned}
(6.13)\qquad & \mathbb{P}_x\left(X_{\tau_{B_x}}\in(\frac{3}{4}E)\setminus U\right)\\
&=\mathbb{P}_{\delta_U(x)^{-1}x}\left(X_{\tau_{\delta_U(x)^{-1}B_x}}\in\delta_U(x)^{-1}(\frac{3}{4}E)\setminus U\right)\\
&=\int_{V_x}G_{V_x}(\delta_U(x)^{-1}x,a)\left(\int_{\delta_U(x)^{-1}(\frac34 E)\setminus U}J(a,b)db\right)da.
\end{aligned}
$$

Let $z_x\in\partial U$ be such that $\delta_U(x)=|x-z_x|$. Since U is $C^{1,1}$, $\delta_U(x)^{-1}((\frac34 E)\setminus U)\supset\delta_U(x)^{-1}(\frac34 E\setminus\frac12 E)$ and $\delta_U(x)\le 3r_*$, there exists $\eta>0$ such that, under an appropriate coordinate system, we have $z_x+\widehat{C}\subset\delta_U(x)^{-1}((\frac34 E)\setminus U)$ where

$$
\widehat{C}:=\left\{y=(y_1,\cdots,y_d)\in\mathbb{R}^d:0<y_d<(12r_*)^{-1},\sqrt{y_1^2+\cdots+y_{d-1}^2}<\eta y_d\right\}.
$$

Thus there is a constant $c_2>0$ such that

$$
\inf_{a\in V_x}\int_{\delta_U(x)^{-1}((\frac34 E)\setminus U)}J(a,b)db\ \ge\ c_2\ >\ 0\qquad\text{for every}\quad x\in U.
$$

Combining this with (6.12)–(6.13),

$$
(6.14)\qquad \inf_{x\in U}\mathbb{P}_x\left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{B_x}}\in(\frac{3}{4}E)\setminus U\right)\ge c_1c_2\,M^{-1}\,\mathbb{E}_w\left[\tau_{B(0,1/3)}\right]\ge c_3>0.
$$

On the other hand, since by (5.6) $\mathbb{P}_x(\widehat{X}_{\widehat{\tau}^{b,E}_{B_x}}\in\partial U)=0$ for every $x\in U$, we have

$$
\mathbb{P}_x\left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U}\in\partial U\right)=\mathbb{E}_x\left[\mathbb{P}_{\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{B_x}}}(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U}\in\partial U);\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{B_x}}\in U\right].
$$

Thus inductively, $\mathbb{P}_x(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U} \in \partial U) = \lim_{k\to\infty} p_k(x)$, where

$$p_0(x) := \mathbb{P}_x \left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U} \in \partial U \right)$$

and

$$p_k(x) := \mathbb{E}_x \left[p_{k-1}(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{B_x}});\ \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{B_x}} \in U \right] \text{ for } k \geq 1.$$

By (6.14),

$$\sup_{x\in U} p_{k+1}(x) \ \leq\ (1-c_3) \sup_{x\in U} p_k(x) \ \leq\ (1-c_3)^{k+1} \ \to\ 0.$$

Therefore $\mathbb{P}_x(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U} \in \partial U) = 0$ for every $x \in U$. □

Let $z \in \partial D$. We will say that a function $u : \mathbb{R}^d \to \mathbb{R}$ vanishes continuously on $D^c \cap B(z,r)$ if $u = 0$ on $D^c \cap B(z,r)$ and u is continuous at every point of $\partial D \cap B(z,r)$.

Theorem 6.2 (Boundary Harnack principle). *There exist positive constants $c_1 = c_1(d,\alpha,R_0,\Lambda_0,b)$ and $r_1 = r_1(d,\alpha,R_0,\Lambda_0,b)$ with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that for all $z \in \partial D$, $r \in (0,r_1]$ and all function $u \geq 0$ on $\mathbb{R}^d$ that is positive harmonic with respect to X^b (or $\widehat{X}^{b,E}$, respectively) in $D \cap B(z,r)$ and vanishes continuously on $D^c \cap B(z,r)$ (or D^c, respectively) we have*

$$\frac{u(x)}{u(y)} \leq c_1 M^2 \frac{\delta_D(x)^{\alpha/2}}{\delta_D(y)^{\alpha/2}}, \qquad x,y \in D \cap B(z,r/4).$$

Proof. We only give the proof for $\widehat{X}^{b,E}$. Recall that r_* and r_0 are the constants from Theorem 4.7 and Theorem 4.8 respectively. Let $r_1 = r_* \wedge r_0$ and fix $r \in (0,r_1]$ throughout this proof. Recall that there exists $L = L(R_0,\Lambda_0,d)$ such that for every $z \in \partial D$ and $r \leq R_0/2$, one can find a $C^{1,1}$ open set $U = U_{(z,r)}$ with $C^{1,1}$ characteristic $(rR_0/L, \Lambda_0 L/r)$ such that $D \cap B(z,r/2) \subset U \subset D \cap B(z,r)$. Without loss of generality, we assume $z = 0$.

Note that, by the same proof as that of [11, Lemma 4.2], every nonnegative function u in $\mathbb{R}^d$ that is harmonic with respect to $\widehat{X}^{b,E}$ in $D \cap B(0,r)$ and vanishes continuously on D^c is regular harmonic in $D \cap B(0,r)$ with respect to $\widehat{X}^{b,E}$.

For all functions $u\geq 0$ on E that is positive regular harmonic for $\widehat{X}^{b,E}$ in $D\cap B(0,r)$ and vanishing on D^c, by (5.6) and Lemma 6.1, we have

$$
\begin{aligned}
u(x) &= \mathbb{E}_x\left[u(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U}); \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U}\in D\setminus U\right] \\
&= \int_{D\setminus U}\widehat{K}^{b,E}_U(x,w)u(w)dw \qquad (6.15)\\
&= \int_U G^b_U(y,x)\left(\int_{D\setminus U}\frac{h_E(w)}{h_E(x)}J(w,y)u(w)dw\right)dy.
\end{aligned}
$$

Define

$$
\begin{aligned}
h_u(x) &:= \mathbb{E}_x\left[u(X_{\tau_U}); X_{\tau_U}\in D\setminus U\right] \\
&= \int_U G_U(y,x)\left(\int_{D\setminus U}J(w,y)u(w)dw\right)dy,
\end{aligned}
$$

which is positive regular harmonic for X in $D\cap B(0,r/2)$ and vanishing on D^c. Applying Theorem 4.8 to (6.15), we get

$$
c_1^{-1}M^{-1}h_u(x)\ \leq\ u(x)\ \leq\ c_1Mh_u(x)\quad \text{for } x\in D. \tag{6.16}
$$

By the boundary Harnack principle for X in $C^{1,1}$ open sets (see [14, 37]), there is a constant $c_2>1$ that depends only on R_0,Λ_0, d and α so that

$$
\frac{h_u(x)}{h_u(y)}\leq c_2 \quad \text{for } x,y\in D\cap B(0,r/4).
$$

Combining this with (6.16) and the two-sided estimates on $G_U(x,y)$ we arrive at the conclusion of the theorem. □

7. Small time heat kernel estimates.

7. Small time heat kernel estimates. Our strategy is to first establish sharp two-sided estimates on $p^b_D(t,x,y)$ at time $t=1$. We then use a scaling argument to establish estimates for $t\leq T$.

We continue to fix a ball E centered at the origin and a $C^{1,1}$ open set $D\subset\frac14 E$ with characteristics (R_0,Λ_0). Recall that $M>1$ is the constant defined in (6.5).

The next result follows from Proposition 3.5, (5.7) and (6.5)

Proposition 7.1. *For all $a_1\in(0,1)$, $a_2,a_3,R>0$, there is a constant $c_1=c_1(d,\alpha,a_1,a_2,a_3,R,M,b)>0$ with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that for all open ball $B(x_0,r)\subset\frac34 E$ with $r\leq R$,*

$$
\widehat{p}^{b,E}_{B(x_0,r)}(t,x,y)\geq c_1\,t^{-d/\alpha}\qquad \text{for all } x,y\in B(x_0,a_1r) \text{ and } t\in[a_2r^\alpha,\ a_3r^\alpha].
$$

Again, we emphasize that the constants in all the results of the remainder of this section (except Theorem 7.8 where the constant also depends on T for an obvious reason) will depend only on $d, \alpha, R_0, \Lambda_0$, M (not the diameter of D directly) and b with the dependence on b only through the rate at which $M^\alpha_{|b|}(r)$ goes to zero. In view of (6.3), (6.4) and (6.6), in particular all the results of this section are applicable to $\mathcal{L}^{b_\lambda}$ and the pair $(\lambda D, \lambda E)$ for every $\lambda \geq 1$.

Recall that r_* and r_0 are the constants from Theorem 4.7 and Theorem 4.8 respectively, which depend only on d, α, R_0, Λ_0 and b with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero.

LEMMA 7.2. *There is $c_1 = c_1(d, \alpha, R_0, r, M, \Lambda_0, b) > 0$ with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that for all $x \in D$*

$$\mathbb{P}_x(\tau^b_D > 1/4) \leq c_1 \left(1 \wedge \delta_D(x)^{\alpha/2}\right) \tag{7.1}$$

and

$$\mathbb{P}_x(\widehat{\tau}^{b,E}_D > 1/4) \leq c_1 \left(1 \wedge \delta_D(x)^{\alpha/2}\right). \tag{7.2}$$

Proof. We only give the proof of (7.2). The proof of (7.1) is similar. Recall that there exists $L = L(R_0, \Lambda_0, d)$ such that for every $z \in \partial D$ and $r \leq R_0$, one can find a $C^{1,1}$ open set $U_{(z,r)}$ with $C^{1,1}$ characteristic $(rR_0/L, \Lambda_0 L/r)$ such that $D \cap B(z, r/2) \subset U_{(z,r)} \subset D \cap B(z, r)$. Clearly it suffices to prove (7.2) for $x \in D$ with $\delta_D(x) < (r_0 \wedge r_*)/8$.

Choose $Q_x \in \partial D$ such that $\delta_D(x) = |x - Q_x|$ and choose a $C^{1,1}$ open set $U := U_{(Q_x, (r_0 \wedge r_*)/2)}$ with $C^{1,1}$ characteristic $((r_0 \wedge r_*)R_0/(2L), 2\Lambda_0 L/(r_0 \wedge r_*))$ such that $D \cap B(Q_x, (r_0 \wedge r_*)/4) \subset U \subset D \cap B(Q_x, (r_0 \wedge r_*)/2)$.

Note that by (5.8), (6.8) and Lemma 6.1,

$$\begin{aligned}
& \mathbb{P}_x\left(\widehat{\tau}^{b,E}_D > 1/4\right) \\
\leq\ & \mathbb{P}_x\left(\widehat{\tau}^{b,E}_U > 1/4\right) + \mathbb{P}_x\left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U} \in D\right) \\
\leq\ & 4\mathbb{E}_x\left[\widehat{\tau}^{b,E}_U\right] + \mathbb{P}_x\left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_U} \in D\right) \\
=\ & 4\int_U G^b_U(y,x)\frac{h_E(y)}{h_E(x)}dy + \int_{D\setminus U}\int_U G^b_U(y,x)\frac{h_E(z)}{h_E(x)}J(y,z)dydz.
\end{aligned}$$

Now using Theorem 4.8, we get

$$\begin{aligned}
&\mathbb{P}_x\left(\widehat{\tau}^{b,E}_D>1/4\right)\\
\le\ & 4c_1M\int_U G_U(y,x)dy+c_1M\int_{D\setminus U}\int_U G_U(y,x)J(y,z)dydz\\
=\ & 4c_1M\int_U G_U(x,y)dy+c_1M\mathbb{P}_x\left(X_{\tau_U}\in D\setminus\overline{U}\right)\\
\le\ & c_2\delta_U(x)^{\alpha/2}=c_2\delta_D(x)^{\alpha/2}.
\end{aligned}$$

The last inequality is due to (4.19) and the boundary Harnack inequality for X in $C^{1,1}$ open sets. □

LEMMA 7.3. *Suppose that U_1,U_3,U are open subsets of $\mathbb{R}^d$ with $U_1,U_3\subset U\subset\frac34E$ and $\mathrm{dist}(U_1,U_3)>0$. Let $U_2:=U\setminus(U_1\cup U_3)$. If $x\in U_1$ and $y\in U_3$, then for all $t>0$,*

$$\begin{aligned}
p^b_U(t,x,y)\ \le\ & \mathbb{P}_x(X^b_{\tau^b_{U_1}}\in U_2)\cdot\sup_{s<t,\,z\in U_2}p^b_U(s,z,y)\\
&+\left(t\wedge\mathbb{E}_x[\tau^b_{U_1}]\right)\cdot\sup_{u\in U_1,\,z\in U_3}J(u,z),
\end{aligned}\tag{7.3}$$

$$\begin{aligned}
p^b_U(t,y,x)\ \le\ & M\mathbb{P}_x(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_1}}\in U_2)\cdot\sup_{s<t,\,z\in U_2}p^b_U(s,y,z)\\
&+M\left(t\wedge\mathbb{E}_x[\widehat{\tau}^{b,E}_{U_1}]\right)\cdot\sup_{u\in U_1,\,z\in U_3}J(u,z)
\end{aligned}\tag{7.4}$$

and

$$p^b_U(1/3,x,y)\ge\frac{1}{3M}\mathbb{P}_x(\tau^b_{U_1}>1/3)\,\mathbb{P}_y(\widehat{\tau}^{b,E}_{U_3}>1/3)\cdot\inf_{u\in U_1,\,z\in U_3}J(u,z)\,.\tag{7.5}$$

Proof. The proof of (7.3) is similar to the proof of [5, Lemma 2], which is a variation of the proof of [9, Lemma 2.2]. Hence we omit its proof. We will present a proof for (7.4)–(7.5). Using the strong Markov property and (5.7), we have

$$
\begin{aligned}
p^b_U(t,y,x) \;=\; & \frac{h_E(x)}{h_E(y)} \widehat{p}^{b,E}_U(t,x,y) \\
= & \frac{h_E(x)}{h_E(y)} \mathbb{E}_x \left[\widehat{p}^{b,E}_U(t-\widehat{\tau}^{b,E}_{U_1}, \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_1}}, y);\; \widehat{\tau}^{b,E}_{U_1} < t \right] \\
= & \frac{h_E(x)}{h_E(y)} \mathbb{E}_x \left[\widehat{p}^{b,E}_U(t-\widehat{\tau}^{b,E}_{U_1}, \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_1}}, y);\; \widehat{\tau}^{b,E}_{U_1} < t, \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_1}} \in U_2 \right] \\
& + \frac{h_E(x)}{h_E(y)} \mathbb{E}_x \left[\widehat{p}^{b,E}_U(t-\widehat{\tau}^{b,E}_{U_1}, \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_1}}, y);\; \widehat{\tau}^{b,E}_{U_1} < t, \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_1}} \in U_3 \right] =: I + II.
\end{aligned}
$$

Using (5.7) again,

$$
\begin{aligned}
I \;\le\; & \frac{h_E(x)}{h_E(y)} \mathbb{P}_x \left(\widehat{\tau}^{b,E}_{U_1} < t, \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_1}} \in U_2 \right) \left(\sup_{s<t,\, z\in U_2} \widehat{p}^{b,E}_U(s,z,y) \right) \\
= & \frac{h_E(x)}{h_E(y)} \mathbb{P}_x \left(\widehat{\tau}^{b,E}_{U_1} < t, \widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_1}} \in U_2 \right) \left(\sup_{s<t,\, z\in U_2} p^b_U(s,y,z) \frac{h_E(y)}{h_E(z)} \right) \\
\le & \left(\sup_{a,b\in \frac34 E} \frac{h_E(a)}{h_E(b)} \right) \mathbb{P}_x \left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_1}} \in U_2 \right) \left(\sup_{s<t,\, z\in U_2} p^b_U(s,y,z) \right).
\end{aligned}
$$

On the other hand, by (5.6) and (5.7),

$$
\begin{aligned}
& II \\
& = \frac{h_E(x)}{h_E(y)} \int_0^t \int_{U_1} \widehat{p}^{b,E}_{U_1}(s,x,u) \int_{U_3} J(u,z) \frac{h_E(z)}{h_E(u)} p^b_U(t-s,y,z) \frac{h_E(y)}{h_E(z)} dz du ds \\
& \le \left(\sup_{a,b\in \frac34 E} \frac{h_E(a)}{h_E(b)} \right) \int_0^t \int_{U_1} \widehat{p}^{b,E}_{U_1}(s,x,u) \int_{U_3} J(u,z) p^b_U(t-s,y,z) dz du ds \\
& \le M \left(\sup_{u\in U_1,\, z\in U_3} J(u,z) \right) \int_0^t \mathbb{P}_x(\widehat{\tau}^{b,E}_{U_1} > s) \left(\int_{U_3} p^b_U(t-s,y,z) dz \right) ds \\
& \le M \int_0^t \mathbb{P}_x(\widehat{\tau}^{b,E}_{U_1} > s) ds \cdot \sup_{u\in U_1,\, z\in U_3} J(u,z) \\
& \le M (t \wedge \mathbb{E}_x[\widehat{\tau}^{b,E}_{U_1}]) \cdot \sup_{u\in U_1,\, z\in U_3} J(u,z)\,.
\end{aligned}
$$

Now we consider the lower bound. By (2.11) and (5.7),

$$
\begin{aligned}
&p^b_U(1/3,x,y)\\
\ge\;& \mathbb{E}_x\Big[p^b_U\Big(1/3-\tau^b_{U_1},X^b_{\tau^b_{U_1}},y\Big);\tau^b_{U_1}<1/3,X^b_{\tau^b_{U_1}}\in U_3\Big]\\
=\;& \int_0^{1/3}\left(\int_{U_1}p^b_{U_1}(s,x,u)\left(\int_{U_3}J(u,z)p^b_U(1/3-s,z,y)dz\right)du\right)ds\\
\ge\;& \inf_{u\in U_1,\,z\in U_3}J(u,z)\int_0^{1/3}\int_{U_3}p^b_U(1/3-s,z,y)\mathbb{P}_x(\tau^b_{U_1}>s)dzds\\
\ge\;& \mathbb{P}_x(\tau^b_{U_1}>1/3)\inf_{u\in U_1,\,z\in U_3}J(u,z)\int_0^{1/3}\int_{U_3}p^b_{U_3}(1/3-s,z,y)dzds\\
=\;& \mathbb{P}_x(\tau^b_{U_1}>1/3)\inf_{u\in U_1,\,z\in U_3}J(u,z)\int_0^{1/3}\int_{U_3}\widehat{p}^{b,E}_{U_3}(1/3-s,y,z)\frac{h_E(y)}{h_E(z)}dzds\\
\ge\;& M^{-1}\mathbb{P}_x(\tau^b_{U_1}>1/3)\inf_{u\in U_1,\,z\in U_3}J(u,z)\int_0^{1/3}\mathbb{P}_y(\widehat{\tau}^{b,E}_{U_3}>1/3-s)ds\\
\ge\;& \frac{1}{3M}\mathbb{P}_x(\tau^b_{U_1}>1/3)\inf_{u\in U_1,\,z\in U_3}J(u,z)\mathbb{P}_y(\widehat{\tau}^{b,E}_{U_3}>1/3)\,.
\end{aligned}
$$

□

LEMMA 7.4. *There is a positive constant $c_1=c_1(d,\alpha,R_0,\Lambda_0,M,b)$ with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that for all $x,y\in D$,*

$$
p^b_D(1/2,x,y)\le c_1\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\frac{1}{|x-y|^{d+\alpha}}\right) \tag{7.6}
$$

and

$$
p^b_D(1/2,x,y)\le c_1\left(1\wedge\delta_D(y)^{\alpha/2}\right)\left(1\wedge\frac{1}{|x-y|^{d+\alpha}}\right). \tag{7.7}
$$

Proof. We only give the proof of (7.7). Recall that there exists $L=L(R_0,\Lambda_0,d)$ such that for every $z\in\partial D$ and $r\le R_0/2$, one can find a $C^{1,1}$ open set $U_{(z,r)}$ with $C^{1,1}$ characteristic $(rR_0/L,\Lambda_0L/r)$ such that $D\cap B(z,r/2)\subset U_{(z,r)}\subset D\cap B(z,r)$.

It follows from (2.4) that

$$
p^b_D(1/2,x,y)\;\le\;p^b(1/2,x,y)\;\le\;c_1\left(1\wedge\frac{1}{|x-y|^{d+\alpha}}\right),
$$

so it suffices to prove of (7.7) for $y \in D$ with $\delta_D(y) < r_0/(32)$.

When $|x-y| \le r_0$, by the semigroup property (3.5), (1.3) and (5.7),

$$\begin{aligned} p^b_D(1/2,x,y) &= \int_D p^b_D(1/4,x,z)p^b_D(1/4,z,y)dz \\ &\le \int_D p^b(1/4,x,z)\widehat{p}^{b,E}_D(1/4,y,z)\frac{h_E(y)}{h_E(z)}dz \\ &\le c_2 M\int_D \left(1\wedge \frac{1}{|x-z|^{d+\alpha}}\right)\widehat{p}^{b,E}_D(1/4,y,z)dz \\ &\le c_2 M \mathbb{P}_y(\widehat{\tau}^{b,E}_D > 1/4). \end{aligned}$$

Applying (7.2), we get

$$\begin{aligned} p^b_D(1/2,x,y) &\le c_3\left(1\wedge \delta_D(y)^{\alpha/2}\right) \\ &\le c_3\left(1\vee r_0^{d+\alpha}\right)\left(1\wedge \delta_D(y)^{\alpha/2}\right)\left(1\wedge \frac{1}{|x-y|^{d+\alpha}}\right). \end{aligned}$$

Finally we consider the case that $|x-y| > r_0$ (and $\delta_D(y) < r_0/(32)$). Fix $y \in D$ with $\delta_D(y) < r_0/(32)$ and let $Q \in \partial D$ be such that $|y-Q| = \delta_D(y)$. Choose a $C^{1,1}$ open set $U_y := U_{(Q,r_0/8)}$ with $C^{1,1}$ characteristic $(r_0R_0/(8L), 8\Lambda_0 L/r_0)$ such that $D\cap B(Q,r_0/(16)) \subset U_y \subset D\cap B(Q,r_0/8)$.

Let $D_3 := \{z \in D : |z-y| > |x-y|/2\}$ and $D_2 := D\setminus(U_y\cup D_3)$. Note that $|z-y| > r_0/2$ for $z\in D_3$. So, if $u\in U_y$ and $z \in D_3$, then

$$|u-z| \ge |z-y| - |y-u| \ge |z-y| - r_0/4 \ge \frac{1}{2}|z-y| \ge \frac{1}{4}|x-y|.$$

Thus

$$\sup_{u\in U_y,\, z\in D_3} J(u,z) \le \sup_{(u,z):|u-z|\ge\frac14|x-y|} J(u,z) \le c_4\left(1\wedge \frac{1}{|x-y|^{d+\alpha}}\right). \tag{7.8}$$

If $z\in D_2$, then $|z-x| \ge |x-y| - |y-z| \ge |x-y|/2$. Thus by (1.3),

$$\begin{aligned} &\sup_{s<1/2,\, z\in D_2} p^b_D(s,x,z) \le \sup_{s<1/2,\, z\in D_2} p^b(s,x,z) \\ &\le c_5 \sup_{s<1/2,\, z\in D_2}\left(1\wedge \frac{1}{|x-z|^{d+\alpha}}\right) \le c_6\left(1\wedge \frac{1}{|x-y|^{d+\alpha}}\right) \end{aligned} \tag{7.9}$$

for some $c_5, c_6 > 0$. Applying Lemmas 7.3 with (7.8) and (7.9), we obtain,

$$p^b_D(1/2,x,y) \le c_7\left(1\wedge \frac{1}{|x-y|^{d+\alpha}}\right)\left(\mathbb{P}_y\Big(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}} \in D\Big) + \mathbb{E}_y\Big[\widehat{\tau}^{b,E}_{U_y}\Big]\right).$$

On the other hand, by (5.8), (6.8), Lemma 6.1 and Theorem 4.8,

$$\begin{aligned}
&\mathbb{E}_y\left[\widehat{\tau}^{b,E}_{U_y}\right]+\mathbb{P}_y\left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}}\in D\right)\\
=\ &\int_{U_y}G^b_{U_y}(z,y)\frac{h_E(z)}{h_E(y)}dz+\int_{D\setminus U_y}\int_{U_y}G^b_{U_y}(w,y)\frac{h_E(z)}{h_E(y)}J(w,z)dwdz\\
\le\ &c_8M\int_{U_y}G_{U_y}(z,y)dz+c_8M\int_{D\setminus U_y}\int_{U_y}G_{U_y}(w,y)J(w,z)dwdz\\
\le\ &c_9\delta_{U_y}(y)^{\alpha/2}=c_9\delta_D(y)^{\alpha/2}.
\end{aligned}$$

Therefore

$$p^b_D(1/2,x,y)\le c_{10}\delta_D(y)^{\alpha/2}\left(1\wedge\frac{1}{|x-y|^{d+\alpha}}\right).$$

(7.6) can be proved in a similar way. □

LEMMA 7.5. *There is a positive constant $c_1=c_1(d,\alpha,R_0,\Lambda_0,M,b)$ with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that for all $x,y\in D$,*

$$p^b_D(1,x,y)\le c_1\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)\left(1\wedge\frac{1}{|x-y|^{d+\alpha}}\right). \tag{7.10}$$

Proof. Using (7.6)-(7.7), the semigroup property (3.5) and the two-sided estimates of $p(t,x,y)$,

$$\begin{aligned}
p^b_D(1,x,y)\ =\ &\int_{\mathbb{R}^d}p^b_D(1/2,x,z)p^b_D(1/2,z,y)dz\\
\le\ &c\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)\\
&\cdot\int_{\mathbb{R}^d}\left(1\wedge\frac{1}{|x-z|^{d+\alpha}}\right)\left(1\wedge\frac{1}{|z-y|^{d+\alpha}}\right)dz\\
\le\ &c\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)\int_{\mathbb{R}^d}p(1/2,x,z)p(1/2,z,y)dz\\
=\ &c\left(1\wedge\delta_D(x)^{\alpha/2}\right)p(1,x,y)\\
\le\ &c\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)\left(1\wedge\frac{1}{|x-y|^{d+\alpha}}\right).
\end{aligned}$$

□

LEMMA 7.6. *If $r>0$ then there is a constant $c_1=c_1(d,\alpha,r,M,b)>0$ with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that for every $B(u,r), B(v,r)\subset \frac34 E$,*

$$p^b_{B(u,r)\cup B(v,r)}(1/3,u,v)\,\ge\, c_1\left(1\wedge\frac{1}{|u-v|^{d+\alpha}}\right).$$

Proof. If $|u-v|\le r/2$, by Proposition 3.5

$$\begin{aligned} p^b_{B(u,r)\cup B(v,r)}(1/3,u,v) &\ge \inf_{|u-v|<r/2} p^b_{B(u,r)}(1/3,u,v)\\ &\ge c_1\,\ge\, c_2\left(1\wedge\frac{1}{|u-v|^{d+\alpha}}\right). \end{aligned}$$

If $|u-v|\ge r/2$, with $U_1=B(u,r/8)$ and $U_3=B(v,r/8)$, we have by (7.5)

$$\begin{aligned} &p^b_{B(u,r)\cup B(v,r)}(1/3,u,v)\\ \ge\;& \frac13\mathbb{P}_u(\tau^b_{U_1}>1/3)\,\mathbb{P}_v(\widehat{\tau}^{b,E}_{U_3}>1/3)\inf_{w\in U_1,\,z\in U_3} J(w,z)\\ \ge\;& c\int_{B(u,r/16)} p^b_{B(u,r/8)}(1/3,u,z)dz\int_{B(v,r/16)}\widehat{p}^{b,E}_{B(u,r/8)}(1/3,v,z)dz\\ &\cdot\left(1\wedge\frac{1}{|u-v|^{d+\alpha}}\right)\\ \ge\;& c\left(\inf_{z\in B(u,r/16)} p^b_{B(u,r/8)}(1/3,u,z)\right)\left(\inf_{z\in B(v,r/16)}\widehat{p}^{b,E}_{B(u,r/8)}(1/3,v,z)\right)\\ &\cdot\left(1\wedge\frac{1}{|u-v|^{d+\alpha}}\right). \end{aligned}$$

Now applying Propositions 3.5 and 7.1, we conclude that

$$p^b_{B(u,r)\cup B(v,r)}(1/3,u,v)\ge c\left(1\wedge\frac{1}{|u-v|^{d+\alpha}}\right).$$

□

LEMMA 7.7. *There is a positive constant $c_1=c_1(d,\alpha,R_0,\Lambda_0,M,b)$ with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero such that*

$$p^b_D(1,x,y)\ge c_1\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)\left(1\wedge\frac{1}{|x-y|^{d+\alpha}}\right).$$

Proof. Recall that $r_0 \le R_0/8$ is the constant from Theorem 4.8 which depends only on d, α, R_0, Λ_0 and b with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero. Since D is $C^{1,1}$ with $C^{1,1}$ characteristics (R_0, Λ_0), there exist $\delta = \delta(R_0, \Lambda_0) \in (0, r_0/8)$ and $L = L(R_0, \Lambda_0) > 1$ so that for all $x, y \in D$, there are $\xi_x \in D \cap B(x, L\delta)$ and $\xi_y \in D \cap B(y, L\delta)$ with $B(\xi_x, 2\delta) \cap B(x, 2\delta) = \emptyset$, $B(\xi_y, 2\delta) \cap B(y, 2\delta) = \emptyset$ and $B(\xi_x, 8\delta) \cup B(\xi_y, 8\delta) \subset D$.

Note that by the semigroup property (3.5) and Lemma 7.6,

(7.11)

$$
\begin{aligned}
&p^b_D(1, x, y)\\
&\ge \int_{B(\xi_y,\delta)} \int_{B(\xi_x,\delta)} p^b_D(1/3, x, u) p^b_D(1/3, u, v) p^b_D(1/3, v, y) du dv\\
&\ge \int_{B(\xi_y,\delta)} \int_{B(\xi_x,\delta)} p^b_D(1/3, x, u) p^b_{B(u,\delta/2)\cup B(v,\delta/2)}(1/3, u, v) p^b_D(1/3, v, y) du dv\\
&\ge c_1 \int_{B(\xi_y,\delta)} \int_{B(\xi_x,\delta)} p^b_D(1/3, x, u)(J(u, v) \wedge 1) p^b_D(1/3, v, y) du dv\\
&\ge c_1 \left(\inf_{(u,v)\in B(\xi_x,\delta)\times B(\xi_y,\delta)} (J(u, v) \wedge 1) \right)\\
&\qquad \cdot \left(\int_{B(\xi_x,\delta)} p^b_D(1/3, x, u) du \right) \left(\int_{B(\xi_y,\delta)} p^b_D(1/3, v, y) dv \right).
\end{aligned}
$$

If $|x - y| \ge \delta/8$, $|u - v| \le 2(1 + L)\delta + |x - y| \le (17 + 16L)|x - y|$ and we have

$$
\inf_{(u,v)\in B(\xi_x,\delta)\times B(\xi_y,\delta)} (J(u, v) \wedge 1) \ge c_2 \left(1 \wedge \frac{1}{|x - y|^{d+\alpha}} \right). \tag{7.12}
$$

If $|x - y| \le \delta/8$, $|u - v| \le 2(2 + L)\delta$ and

$$
\inf_{(u,v)\in B(\xi_x,\delta)\times B(\xi_y,\delta)} (J(u, v) \wedge 1) \ge c_3 \ge c_4 \left(1 \wedge \frac{1}{|x - y|^{d+\alpha}} \right). \tag{7.13}
$$

We claim that

$$
\int_{B(\xi_x,\delta)} p^b_D(1/3, x, u) du \ge c_5 \left(1 \wedge \delta_D(x)^{\alpha/2} \right) \tag{7.14}
$$

and

$$
\int_{B(\xi_y,\delta)} p^b_D(1/3, v, y) dv \ge c_5 \left(1 \wedge \delta_D(y)^{\alpha/2} \right), \tag{7.15}
$$

which, combined with (7.11)–(7.13), proves the theorem.

We only give the proof of (7.15). If $\delta_D(y) > \delta$, since $\mathrm{dist}(B(\xi_y, \delta), B(y, \delta)) > 0$, by (7.5),

$$\begin{aligned} &\int_{B(\xi_y,\delta)} p^b_D(1/3, v, y)dv \\ &\geq \frac{1}{3M}\left(\int_{B(\xi_y,\delta)} \mathbb{P}_v(\tau^b_{B(\xi_y,\delta)} > 1/3)dv\right) \mathbb{P}_y(\widehat{\tau}^{b,E}_{B(y,\delta)} > 1/3) \\ &\quad \cdot \inf_{w\in B(\xi_y,\delta),\, z\in B(y,\delta)} J(w, y) \end{aligned} \tag{7.16}$$

which is greater than or equal to some positive constant depending only on $d, \alpha, R_0, \Lambda_0$, M and b with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero by Propositions 3.5 and 7.1.

If $\delta_D(y) \leq \delta$, choose a $Q \in \partial D$ be such that $|y - Q| = \delta_D(y)$ and choose a $C^{1,1}$ open set $U_y := U_{(Q,4\delta)}$ with $C^{1,1}$ characteristic $(4\delta R_0/L, \Lambda_0 L/(4\delta))$ such that

$$D \cap B(Q, 2\delta) \subset U_y \subset D \cap B(Q, 4\delta) \subset D \cap B(Q, 6\delta) =: V_y.$$

Then, since $\mathrm{dist}(B(\xi_y, \delta), V_y) > 0$, by (7.5),

$$\begin{aligned} &\int_{B(\xi_y,\delta)} p^b_D(1/3, v, y)dv \\ &\geq \frac{1}{3M}\left(\int_{B(\xi_y,\delta)} \mathbb{P}_v\left(\tau^b_{B(\xi_y,\delta)} > 1/3\right) dv\right) \mathbb{P}_y\left(\widehat{\tau}^{b,E}_{V_y} > 1/3\right) \\ &\quad \cdot \inf_{w\in B(\xi_y,\delta),\, z\in V_y} J(w, y) \end{aligned} \tag{7.17}$$

which is greater than or equal to $c_6\, \mathbb{P}_y(\widehat{\tau}^{b,E}_{V_y} > 1/3)$ for some positive constant c_6 depending only on $d, \alpha, R_0, \Lambda_0$, M and b with the dependence on b only via the rate at which $M^\alpha_{|b|}(r)$ goes to zero by Propositions 3.5 and 7.1.

Let $B(y_0, 2c_7\delta)$ be a ball in $D\cap(B(Q, 6\delta)\setminus B(Q, 4\delta))$ where $c_7 = c_7(\Lambda_0, d) >$

0. By the strong Markov property,

$$
\begin{aligned}
&\left(\inf_{w\in B(y_0,c_7\delta/2)}\mathbb{P}_w\left(\widehat{\tau}^{b,E}_{B(w,c_7\delta)}>1/3\right)\right)\mathbb{P}_y\left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}}\in B(y_0,c_7\delta/2)\right)\\
\le\ & \mathbb{E}_y\left[\mathbb{P}_{\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}}}\left(\widehat{\tau}^{b,E}_{B(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}},c_7\delta)}>1/3\right);\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}}\in B(y_0,c_7\delta/2)\right]\\
\le\ & \mathbb{E}_y\left[\mathbb{P}_{\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}}}\left(\widehat{\tau}^{b,E}_{V_y}>1/3\right);\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}}\in B(y_0,c_7\delta/2)\right]\\
\le\ & \mathbb{P}_y\left(\widehat{\tau}^{b,E}_{V_y}>1/3,\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}}\in B(y_0,c_7\delta/2)\right)\le\mathbb{P}_y\left(\widehat{\tau}^{b,E}_{V_y}>1/3\right).
\end{aligned}
$$

Using Propositions 7.1, we get

$$
\mathbb{P}_y\left(\widehat{\tau}^{b,E}_{V_y}>1/3\right)\ge c_8\mathbb{P}_y\left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}}\in B(y_0,c_7\delta/2)\right). \tag{7.18}
$$

Now applying (5.8), (6.8) and Theorem 4.8,

$$
\begin{aligned}
&\mathbb{P}_y\left(\widehat{X}^{b,E}_{\widehat{\tau}^{b,E}_{U_y}}\in B(y_0,c_7\delta/2)\right) \qquad (7.19)\\
=\ & \int_{B(y_0,c_7\delta/2)}\int_{U_y}G^b_{U_y}(w,y)\frac{h_E(z)}{h_E(y)}J(w,z)dwdz\\
\ge\ & c_9M^{-1}\int_{B(y_0,c_7\delta/2)}\int_{U_y}G_{U_y}(w,y)J(w,z)dwdz\\
\ge\ & c_{10}\delta_{U_y}(y)^{\alpha/2}=c_{10}\delta_D(y)^{\alpha/2}.
\end{aligned}
$$

Combining (7.16)–(7.19), we have proved (7.15). □

THEOREM 7.8. *There exists $c=c(d,\alpha,R_0,\Lambda_0,T,M,b)>0$ with the dependence on b only via the rate at which $M^{\alpha}_{|b|}(r)$ goes to zero such that for $0<t\le T$, $x,y\in D$,*

(7.20)
$$
\begin{aligned}
&c^{-1}\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\\
\le\ & p^b_D(t,x,y)\ \le\ c\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right).
\end{aligned}
$$

Proof. Let $D_t := t^{-1/\alpha} D$ and $E_t := t^{-1/\alpha} E$. By the scaling property in (6.1), (7.20) is equivalent to

$$\begin{aligned} & c^{-1}(1\wedge \delta_{D_t}(x)^{\alpha/2})(1\wedge \delta_{D_t}(y)^{\alpha/2}) \left(1\wedge \frac{1}{|x-y|^{d+\alpha}}\right) \\ \le\ & p_{D_t}^{t^{(\alpha-1)/\alpha} b(t^{1/\alpha}\cdot)}(1,x,y) \\ \le\ & c(1\wedge \delta_{D_t}(x)^{\alpha/2})(1\wedge \delta_{D_t}(y)^{\alpha/2}) \left(1\wedge \frac{1}{|x-y|^{d+\alpha}}\right). \end{aligned}$$

The above holds in view of (6.3), (6.4), (6.6) and the fact that for $t \le T$, the D_t's are $C^{1,1}$ open sets in $\mathbb{R}^d$ with the same $C^{1,1}$ characteristics $(R_0(T)^{-1/\alpha}, \Lambda_0(T)^{-1/\alpha})$. The theorem is thus proved. □

8. Large time heat kernel estimates. Recall that we have fixed a ball E centered at the origin and $M > 1$ is the constant in (6.5). Let U be an arbitrary open set $U \subset \frac{1}{4}E$ and we let

$$\overline{p}_U^{b,E}(t,x,y) := \frac{p_U^b(t,x,y)}{h_E(y)},$$

which is strictly positive, bounded and continuous on $(t,x,y) \in (0,\infty)\times U\times U$ because $p_U^b(t,x,y)$ is strictly positive, bounded and continuous on $(t,x,y) \in (0,\infty)\times U\times U$ and $h_E(y)$ is strictly positive and continuous on E. For each $x \in U$, $(t,y) \mapsto \overline{p}_U^{b,E}(t,x,y)$ is the transition density of $(X^{b,U}, \mathbb{P}_x)$ with respect to the reference measure ξ_E and, for each $y \in U$, $(t,x) \mapsto \overline{p}_U^{b,E}(t,x,y)$ is the transition density of $(\widehat{X}^{b,E,U}, \mathbb{P}_y)$, the dual process of $X^{b,U}$ with respect to the reference measure ξ_E.

Let

$$P_t^{b,E,U} f(x) := \int_U \overline{p}_U^{b,E}(t,x,y) f(y) \xi_E(dy)$$

and

$$\widehat{P}_t^{b,E,U} f(x) := \int_U \overline{p}_U^{b,E}(t,y,x) f(y) \xi_E(dy).$$

Let $\mathcal{L}_U^{b,E}$ and $\widehat{\mathcal{L}}_U^{b,E}$ be the infinitesimal generators of the semigroups $\{P_t^{b,E,U}\}$ and $\{\hat{P}_t^{b,E,U}\}$ on $L^2(U,\xi_E)$, respectively.

Note that, since for each $t > 0$, $\overline{p}_U^{b,E}(t,x,y)$ is bounded in $U\times U$, it follows from Jentzsch's Theorem ([32, Theorem V.6.6 on page 337]) that the common value $-\lambda_0^{b,E,U} := \sup \mathrm{Re}(\sigma(\mathcal{L}_U^{b,E})) = \sup \mathrm{Re}(\sigma(\widehat{\mathcal{L}}_U^{b,E}))$ is an eigenvalue of multiplicity 1 for both $\mathcal{L}_U^{b,E}$ and $\widehat{\mathcal{L}}_U^{b,E}$, and that an eigenfunction

$\phi^{b,E}_U$ of $\mathcal{L}^{b,E}_U$ associated with $\lambda^{b,E,U}_0$ can be chosen to be strictly positive with $\|\phi^{b,E}_U\|_{L^2(U,\xi_E(dx))} = 1$ and an eigenfunction $\psi^{b,E}_U$ of $\widehat{\mathcal{L}}^{b,E}_U$ associated with $\lambda^{b,E,U}_0$ can be chosen to be strictly positive with $\|\psi^{b,E}_U\|_{L^2(U,\xi_E(dx))} = 1$.

It is clear from the definition that, for any Borel function f,

$$P^{b,E,U}_t f(x) = P^{b,U}_t f(x) \quad \text{for every } x \in U \text{ and } t > 0.$$

Thus the operators $\mathcal{L}^b|_U$ and $\mathcal{L}^{b,E}_U$ have the same eigenvalues. In particular, the eigenvalue $\lambda^{b,E,U}_0$ does not depend on E and so from from now on we will denote it by $\lambda^{b,U}_0$.

Definition 8.1. *The semigroups* $\{P^{b,E,U}_t\}$ *and* $\{\widehat{P}^{b,E,U}_t\}$ *are said to be intrinsically ultracontractive if, for any* $t > 0$*, there exists a constant* $c_t > 0$ *such that*

$$\overline{p}^{b,E}_U(t,x,y) \le c_t \phi^{b,E}_U(x)\psi^{b,E}_U(y) \qquad \text{for } x, y \in U.$$

It follows from [26, Theorem 2.5] that if $\{P^{b,E,U}_t\}$ and $\{\widehat{P}^{b,E,U}_t\}$ are intrinsically ultracontractive then for any $t > 0$ there exists a positive constant $c_t > 1$ such that

$$\overline{p}^{b,E}_U(t,x,y) \ge c_t^{-1} \phi^{b,E}_U(x)\psi^{b,E}_U(y) \qquad \text{for } x, y \in U. \tag{8.1}$$

Theorem 8.2. *For every* $B(x_0, 2r) \subset U$ *there exists a constant* $c = c(d, \alpha, r, \mathrm{diam}(U), M) > 0$ *such that for every* $x \in D$*,*

$$\mathbb{E}_x \left[\int_0^{\tau^b_U} \mathbf{1}_{B(x_0,r)}(X^{b,U}_t)dt \right] \ge c\,\mathbb{E}_x \left[\tau^b_U\right] \tag{8.2}$$

and

$$\mathbb{E}_x \left[\int_0^{\widehat{\tau}^{b,E}_U} \mathbf{1}_{B(x_0,r)}(\widehat{X}^{b,E,U}_t)dt \right] \ge c\,\mathbb{E}_x \left[\widehat{\tau}^{b,E}_U\right]. \tag{8.3}$$

Proof. The method of the proof to be given below is now well-known. (See [10, 27]). For the reader's convenience, we present the details here. We give the proof of (8.3) only. The proof for (8.2) is similar. Fix a ball $B(x_0, 2r) \subset U$ and put

$$B_0 := B(x_0, r/4), \quad K_1 := \overline{B(x_0, r/2)} \quad \text{and} \quad B_2 := B(x_0, r).$$

Let $\{\theta_t, t > 0\}$ be the shift operators of $\widehat{X}^{b,E}$ and we define stopping times S_n and T_n recursively by

$$\begin{aligned} S_1(\omega) &:= 0, \\ T_n(\omega) &:= S_n(\omega) + \widehat{\tau}^{b,E}_{U\setminus K_1} \circ \theta_{S_n}(\omega) \quad \text{for } S_n(\omega) < \widehat{\tau}^{b,E}_U \\ \text{and} \quad S_{n+1}(\omega) &:= T_n(\omega) + \widehat{\tau}^{b,E}_{B_2} \circ \theta_{T_n}(\omega) \quad \text{for } T_n(\omega) < \widehat{\tau}^{b,E}_U. \end{aligned}$$

Clearly $S_n \le \widehat{\tau}^{b,E}_U$. Let $S := \lim_{n\to\infty} S_n \le \widehat{\tau}^{b,E}_U$. On $\{S < \widehat{\tau}^{b,E}_U\}$, we must have $S_n < T_n < S_{n+1}$ for every $n \ge 0$. Using the fact that $\mathbb{P}_x(\widehat{\tau}^{b,E}_U < \infty) = 1$ for every $x \in U$ and the quasi-left continuity of $\widehat{X}^{b,E,U}$, we have $\mathbb{P}_x(S < \widehat{\tau}^{b,E}_U) = 0$. Therefore, for every $x \in U$,

$$\mathbb{P}_x\left(\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n = \widehat{\tau}^{b,E}_U\right) = 1. \tag{8.4}$$

For any $x \in K_1$, by Proposition (7.1) we have

$$\mathbb{E}_x[\widehat{\tau}^{b,E}_{B_2}] \ge c_0 \int_{B(x_0,r/2)} \int_{r^\alpha}^{2r^\alpha} \widehat{p}^{b,E}_{B_2}(t,x,y)dtdy \ge c_1 \qquad \text{for every } x \in K_1.$$

Now it follows from the strong Markov property that

$$\begin{aligned} &\mathbb{E}_x\left[S_{n+1} - T_n\right] = \mathbb{E}_x\left[\mathbb{E}_{\widehat{X}^{b,E,U}_{T_n}}[\widehat{\tau}^{b,E}_{B_2}];\, T_n < \widehat{\tau}^{b,E}_U\right] \\ &\ge c_1 \mathbb{P}_x\left(\widehat{X}^{b,E,U}_{T_n} \in B_0\right) = c_1 \mathbb{E}_x\left[\mathbb{P}_{\widehat{X}^{b,E,U}_{S_n}}\left(\widehat{X}^{b,E,U}_{\widehat{\tau}^{b,E}_{U\setminus K_1}} \in B_0\right)\right]. \end{aligned}$$

Note that for any $x \in U \setminus B_2$, by (6.9), we have

$$\begin{aligned} &\mathbb{P}_x\left(\widehat{X}^{b,E,U}_{\widehat{\tau}^{b,E}_{U\setminus K_1}} \in B_0\right) \\ =\;& \int_{U\setminus K_1} \frac{G^b_{U\setminus K_1}(y,x)}{h_E(x)} \int_{B_0} \left(\frac{J(y,z)h_E(z)}{h_E(y)} dz\right) \xi_E(dy) \\ \ge\;& M^{-1}\mathcal{A}(d,-\alpha) \int_{U\setminus K_1} \frac{G^b_{U\setminus K_1}(y,x)}{h_E(x)} \int_{B_0} \left(\frac{dz}{(\operatorname{diam}(U))^{d+\alpha}}\right) \xi_E(dy) \\ =\;& c_2 \mathbb{E}_x[\widehat{\tau}^{b,E}_{U\setminus K_1}] \end{aligned}$$

for some constant $c_2 = c_2(\alpha, r, \operatorname{diam}(U), M) > 0$. It follows then

$$\mathbb{E}_x\left[S_{n+1} - T_n\right] \ge c_1 c_2 \mathbb{E}_x\left[\mathbb{E}_{\widehat{X}^{b,E,U}_{S_n}}[\widehat{\tau}^{b,E}_{U\setminus K_1}]\right] = c_1 c_2 \mathbb{E}_x[T_n - S_n]. \tag{8.5}$$

Since $\widehat{X}_t^{b,E,U} \in B_2$ for $T_n < t < S_{n+1}$, we have by (8.4)

$$
\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\widehat{\tau}_U^{b,E}} \mathbf{1}_{B_2}(\widehat{X}_t^{b,E,U})dt\right] \\
= \;& \mathbb{E}_x\left[\sum_{n=1}^{\infty}\left(\int_{S_n}^{T_n} \mathbf{1}_{B_2}(\widehat{X}_t^{b,E,U})dt + \int_{T_n}^{S_{n+1}} \mathbf{1}_{B_2}(\widehat{X}_t^{b,E,U})dt\right)\right] \\
\geq \;& \mathbb{E}_x\left[\sum_{n=1}^{\infty}\left(\int_{T_n}^{S_{n+1}} \mathbf{1}_{B_2}(\widehat{X}_t^{b,E,U})dt\right)\right] = \mathbb{E}_x\left[\sum_{n=1}^{\infty}(S_{n+1} - T_n)\right].
\end{aligned}
$$

Using (8.4) and (8.5) and noting that $\widehat{X}_t^{b,E,U} \notin U \setminus B_2$ for $t \in [T_n, S_{n+1})$, we get

$$
\begin{aligned}
&\mathbb{E}_x\left[\int_0^{\widehat{\tau}_U^{b,E}} \mathbf{1}_{B_2}(\widehat{X}_t^{b,E,U})dt\right] \geq c_1 c_2 \mathbb{E}_x\left[\sum_{n=1}^{\infty}(T_n - S_n)\right] \\
\geq \;& c_1 c_2 \mathbb{E}_x\left[\sum_{n=1}^{\infty}\left(\int_{S_n}^{T_n} \mathbf{1}_{U\setminus B_2}(\widehat{X}_t^{b,E,U})dt + \int_{T_n}^{S_{n+1}} \mathbf{1}_{U\setminus B_2}(\widehat{X}_t^{b,E,U})dt\right)\right] \\
= \;& c_1 c_2 \mathbb{E}_x\left[\int_0^{\widehat{\tau}_U^{b,E}} \mathbf{1}_{U\setminus B_2}(\widehat{X}_t^{b,E,U})dt\right].
\end{aligned}
$$

Thus

$$
\mathbb{E}_x\left[\int_0^{\widehat{\tau}_U^{b,E}} \mathbf{1}_{B_2}(\widehat{X}_t^{b,E,U})dt\right] \geq \frac{c_1\, c_2}{1 + c_1\, c_2}\mathbb{E}_x\left[\widehat{\tau}_U^{b,E}\right].
$$

□

THEOREM 8.3. *$\{P_t^{b,E,U}\}$ and $\{\widehat{P}_t^{b,E,U}\}$ are intrinsically ultracontractive.*

Proof. Since $\psi_U^{b,E} = e^{\lambda_0^{b,U}} \widehat{P}_1^{b,E,U} \psi_U^{b,E}$, it follows that $\psi_U^{b,E}$ is strictly positive, bounded and continuous in U. Theorem 8.2 implies that

$$
\begin{aligned}
\text{(8.6)} \qquad \mathbb{E}_x[\widehat{\tau}_U^{b,E}] &\leq c_1 \int_{B_2} \frac{G_U^{b,E}(z,y)}{h_E(y)}\psi_U^{b,E}(z)\xi_E(dz) \\
&\leq c_1 \int_U \frac{G_U^{b,E}(z,y)}{h_E(y)}\psi_U^{b,E}(z)\xi_E(dz) = \frac{c_1}{\lambda_0^{b,U}}\psi_U^{b,E}(y).
\end{aligned}
$$

Similarly,

$$
\text{(8.7)} \qquad \mathbb{E}_x[\tau_U^b] \leq \frac{c_2}{\lambda_0^{b,U}}\phi_U^{b,E}(x).
$$

By the semigroup property and (1.3),

$$\begin{aligned}
&\overline{p}_U^{b,E}(t,x,y) \\
= \;& \int_U \overline{p}_U^{b,E}(t/3,x,z)\int_U \overline{p}_U^{b,E}(t/3,z,w)\overline{p}_U^{b,E}(t/3,w,y)\xi_E(dw)\xi_E(dz) \\
\le \;& c_3 t^{-d/\alpha}\int_U \overline{p}_U^{b,E}(t/3,x,z)\xi_E(dz)\int_U \overline{p}_U^{b,E}(t/3,w,y)\xi_E(dw) \\
= \;& c_3 t^{-d/\alpha}\,\mathbb{P}_x\big(\tau_U^{b,E}>t/3\big)\,\mathbb{P}_y\big(\widehat{\tau}_U^{b,E}>t/3\big) \\
\le \;& (9c_3/t^2)\,t^{-d/\alpha}\,\mathbb{E}_x\big[\tau_U^b\big]\,\mathbb{E}_y\big[\widehat{\tau}_U^{b,E}\big].
\end{aligned}$$

This together with (8.6)–(8.7) establishes the intrinsic ultracontractivity of $\{P_t^{b,E,U}\}$ and $\{\widehat{P}_t^{b,E,U}\}$. □

Applying [26, Theorem 2.7], we obtain

THEOREM 8.4. *There exist positive constants c and ν such that*

$$\left|\frac{M_U^{b,E}e^{t\lambda_0^{b,U}}\,\overline{p}_U^{b,E}(t,x,y)}{\phi_U^{b,E}(x)\psi_U^{b,E}(y)}-1\right|\le ce^{-\nu t},\qquad (t,x,y)\in(1,\infty)\times U\times U \tag{8.8}$$

where $M_U^{b,E}:=\int_U \phi_U^{b,E}(y)\psi_U^{b,E}(y)\xi_E(dy)\le 1$.

Now we can present the

Proof of Theorem 1.3(ii). Assume that the ball E is large enough so that $D\subset \frac14 E$. Since $\phi_D^{b,E}=e^{\lambda_0^{b,D}}P_1^{b,D}\phi_D^{b,E}$ and $\psi_D^{b,E}=e^{\lambda_0^{b,D}}\widehat{P}_1^{b,E,D}\psi_D^{b,E}$, we have from Theorem 1.3(i) that on D,

(8.9)

$$\begin{aligned}
\phi_D^{b,E}(x) &\asymp \left(1\wedge\delta_D(x)^{\alpha/2}\right)\int_D\left(1\wedge\delta_D(y)^{\alpha/2}\right)\left(1\wedge\frac{1}{|x-y|^{d+\alpha}}\right)\phi_D^{b,E}(y)dy \\
&\asymp \delta_D(x)^{\alpha/2}
\end{aligned}$$

and

(8.10)

$$\begin{aligned}
&\psi_D^{b,E}(x) \\
&\asymp \left(1\wedge\delta_D(x)^{\alpha/2}\right)\int_D\left(1\wedge\delta_D(y)^{\alpha/2}\right)\left(1\wedge\frac{1}{|x-y|^{d+\alpha}}\right)\frac{h_E(y)}{h_E(x)}\psi_D^{b,E}(y)dy \\
&\asymp \delta_D(x)^{\alpha/2}.
\end{aligned}$$

Theorem 8.3 and (8.9)–(8.10) imply that

$$c_t^{-1}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}\le \overline{p}_D^{b,E}(t,x,y)\le c_t\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}$$

for every $(t,x,y)\in(0,\infty)\times D\times D$, and so

$$c_1^{-1}c_t^{-1}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}\le p_D^b(t,x,y)\le c_1c_t\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}$$

for every $(t,x,y)\in(0,\infty)\times D\times D$.

Furthermore, by Theorem 8.4 and (8.9), there exist $c_2>1$ and $T_1>0$ such that for all $(t,x,y)\in[T_1,\infty)\times D\times D$,

$$c_2^{-1}e^{-t\lambda_0^{b,D}}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}\le \overline{p}_D^{b,E}(t,x,y)\le c_2e^{-t\lambda_0^{b,D}}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2},$$

which implies that

$$c_3^{-1}e^{-t\lambda_0^{b,D}}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}\le p_D^b(t,x,y)\le c_3e^{-t\lambda_0^{b,D}}\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}.$$

If $T<T_1$, by Theorem 1.3(i), there is a constant $c_2\ge 1$ such that

$$c_2^{-1}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2}\ \le\ p_D^b(t,x,y)\ \le\ c_2\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2}$$

for every $t\in[T,T_1)$ and $x,y\in D$. This establishes Theorem 1.3(ii). □

Remark 8.5. (i) Using Corollary 1.4 and the argument of the proof of Lemma 6.1, (6.10) is, in fact, true for all bounded open set U with exterior cone condition.

(ii) In view of Corollary 1.4, the estimate (4.8) and Lemma 4.1, we can deduce from (4.10) by the dominated convergence theorem that Proposition 4.2 holds for general b with $|b|\in\mathbb{K}_{d,\alpha-1}$. □

Acknowledgements. The main results of this paper were presented by the authors at the *Sixth International Conference on Lévy Processes: Theory and Applications* held in Dresden, Germany from July 26 to 30, 2010 and at the *34th Stochastic Processes and their Applications* conference held in Osaka, Japan from September 6 to 10, 2010.

K. Bogdan announced at the *Sixth International Conference on Lévy Processes: Theory and Applications* in Dresden that he and T. Jakubowski have also obtained the same sharp estimates on G_D^b in bounded $C^{1,1}$ domains as given in Corollary 1.4 of this paper. They have also obtained one part (for harmonic functions of X^b only) of the boundary Harnack principles established in Theorem 6.2 of our paper. Their preprint [7] containing these two results (Theorem 1 and Lemma 18 there) appeared in the arXiv on September 14, 2010.

We thank the referees for their helpful comments on the first version of this paper.

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