

INTRINSIC ULTRA CONTRACTIVITY OF NON-SYMMETRIC DIFFUSIONS WITH MEASURE-VALUED DRIFTS AND POTENTIALS

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Recently in [18], we extended the concept of intrinsic ultracontractivity to non-symmetric semigroups. In this paper, we study the intrinsic ultracontractivity of non-symmetric diffusions with measure-valued drifts and measure-valued potentials in bounded domains. Our process Y is a diffusion process whose generator can be formally written as $L + \mu \cdot \nabla - \nu$ with Dirichlet boundary conditions, where L is a uniformly elliptic second order differential operator and $\mu = (\mu^1, \dots, \mu^d)$ is such that each component μ^i , $i = 1, \dots, d$, is a signed measure belonging to the Kato class $\mathbf{K}_{d,1}$ and ν is a (non-negative) measure belonging to the Kato class $\mathbf{K}_{d,2}$. We show that scale invariant parabolic and elliptic Harnack inequalities are valid for Y .

In this paper, we prove the parabolic boundary Harnack principle and the intrinsic ultracontractivity for the killed diffusion Y^D with measure-valued drift and potential when D is one of the following types of bounded domains: twisted Hölder domains of order $\alpha \in (1/3, 1]$, uniformly Hölder domains of order $\alpha \in (0, 2)$ and domains which can be locally represented as the region above the graph of a function. This extends the results in [1] and [2]. As a consequence of the intrinsic ultracontractivity, we get that the supremum of the expected conditional lifetimes of Y^D is finite.

1. Introduction. In this paper, we study the intrinsic ultracontractivity of a non-symmetric diffusion process Y with measure-valued drift and measure-valued potential in bounded domains $D \subset \mathbf{R}^d$ for $d \geq 3$. The generator of Y can be formally written as $L + \mu \cdot \nabla - \nu$ with Dirichlet boundary conditions, where L is a uniformly elliptic second order differential operator and $\mu = (\mu^1, \dots, \mu^d)$ is such that each component μ^i , $i = 1, \dots, d$, is a signed measure belonging to the Kato class $\mathbf{K}_{d,1}$ and ν is a (non-negative) measure

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belonging to the Kato class $\mathbf{K}_{d,2}$ (see below for the definitions of $\mathbf{K}_{d,1}$ and $\mathbf{K}_{d,2}$). The existence and uniqueness of this process Y were proved in Bass and Chen [3]. In [15, 16, 17, 19], we have studied properties of diffusions with measure-value drifts in bounded domains. Using results in [15, 16, 17, 19], we will prove that, with respect to a certain reference measure, Y has a dual process which is a continuous Hunt process satisfying the strong Feller property.

The notion of intrinsic ultracontractivity, introduced in [11] for symmetric semigroups, is a very important concept and has been studied extensively. In [18] the concept of intrinsic ultracontractivity was extended to non-symmetric semigroups and it was proved there that the semigroup of the killed diffusion process in a bounded Lipschitz domain is intrinsic ultracontractive if the coefficients of the generator of the diffusion process are smooth.

In this paper, using the duality of our processes we prove that the semigroups of the killed diffusion Y^D and its dual are intrinsic ultracontractive if D is one of the following types of bounded domains:

- (a) a twisted Hölder domain of order $\alpha \in (1/3, 1]$ or
- (b) a uniformly Hölder domain of order $\alpha \in (0, 2)$ or
- (c) a domain which can be locally represented as the region above the graph of a function.

In fact, we first prove parabolic boundary Harnack principles for Y^D and its dual process (see Theorem 5.6 and Corollary 5.7). Then we show that the parabolic boundary Harnack principles imply that the semigroups of Y^D and its dual are intrinsic ultracontractive. The fact that the parabolic boundary Harnack principle implies the intrinsic ultracontractivity in the symmetric diffusion case was used and discussed in [2] and [12]. As a consequence of the intrinsic ultracontractivity, we have that the supremum of the expected conditional lifetimes of Y^D is finite if D is one of the domains above.

Many results in this paper are stated for both the diffusion process Y and its dual. In these cases, the proofs for the dual process are usually harder. Once the proofs for the dual process are done, it is very easy to see that the results for the diffusion process Y can be proved through similar and simpler arguments. For this reason, we only present the proof for the dual process.

The content of this paper is organized as follows. In Section 2, we present some preliminary properties of Y and the existence of the dual process of Y ; Section 3 contains the proof of parabolic Harnack inequalities for Y and its dual process. In section 4, we discuss some properties of Y and its dual

in twisted Hölder domains, uniformly Hölder domains and domains which can be locally represented as the region above the graph of a function. In the last section we prove the parabolic boundary Harnack principles and show that the parabolic boundary Harnack principles imply the intrinsic ultracontractivity of the non-symmetric semigroups. Finally, we get that the supremum of the expected conditional lifetime is finite.

In this paper we always assume that $d \geq 3$. Throughout this paper, we use the notations $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. The distance between x and ∂D , the boundary of D , is denote by $\rho_D(x)$. We use the convention $f(\partial) = 0$. In this paper we will also use the following convention: the values of the constants r_1, t_0, t_1 will remain the same throughout this paper, while the values of the constants c_1, c_2, \dots might change from one appearance to another. The labeling of the constants c_1, c_2, \dots starts anew in the statement of each result.

In this paper, we use “:=” to denote a definition, which is read as “is defined to be”.

2. Dual process for Diffusion processes with measure-valued drifts and potentials. First we recall the definition of the Kato class $\mathbf{K}_{d,j}$ for $j = 1, 2$. For any function f on \mathbf{R}^d and $r > 0$, we define

$$M_f^j(r) = \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leq r} \frac{|f|(y) dy}{|x-y|^{d-j}}, \quad j = 1, 2.$$

For any signed measure ν on \mathbf{R}^d , we use ν^+ and ν^- to denote its positive and negative parts, and $|\nu| := \nu^+ + \nu^-$. For any signed measure ν on \mathbf{R}^d and any $r > 0$, we define

$$M_\nu^j(r) = \sup_{x \in \mathbf{R}^d} \int_{|x-y| \leq r} \frac{|\nu|(dy)}{|x-y|^{d-j}}, \quad j = 1, 2.$$

DEFINITION 2.1. *Let $j = 1, 2$. We say that a function f on \mathbf{R}^d belongs to the Kato class $\mathbf{K}_{d,j}$ if $\lim_{r \downarrow 0} M_f^j(r) = 0$. We say that a signed Radon measure ν on \mathbf{R}^d belongs to the Kato class $\mathbf{K}_{d,j}$ if $\lim_{r \downarrow 0} M_\nu^j(r) = 0$.*

Throughout this paper we assume that $\mu = (\mu^1, \dots, \mu^d)$ and ν are fixed with each μ^i being a signed measure on \mathbf{R}^d belonging to $\mathbf{K}_{d,1}$ and ν being a (non-negative) measure on \mathbf{R}^d belonging to $\mathbf{K}_{d,2}$.

We also assume that the operator L is either L_1 or L_2 where

$$L_1 := \frac{1}{2} \sum_{i,j=1}^d \partial_i (a_{ij} \partial_j) \quad \text{and} \quad L_2 := \frac{1}{2} \sum_{i,j=1}^d a_{ij} \partial_i \partial_j$$

with $\mathbf{A}(x) := (a_{ij}(x))$ being C^1 and uniformly elliptic. Since $\mathbf{A}(x) = (a_{ij}(x))$ is C^1 , without loss of generality, one can assume that the matrix $A(x)$ is symmetric (for example, see Section 6 of [16]).

We will use X to denote the diffusion process in \mathbf{R}^d whose generator can be formally written as $L + \mu \cdot \nabla$. When each μ^i is given by $U^i(x)dx$ for some function U^i , X is a diffusion in \mathbf{R}^d with generator $L + U \cdot \nabla$ and it is a solution to the stochastic differential equation $dX_t = dX_t^0 + U(X_t) \cdot dt$, where X^0 is a diffusion in \mathbf{R}^d with generator L . For a precise definition of a (non-symmetric) diffusion X with drift μ in $\mathbf{K}_{d,1}$, we refer to Section 6 in [16] and Section 1 in [19]. The existence and uniqueness of X were established in [3] (see Remark 6.1 in [3]).

For any open set U , we use τ_U^X to denote the first exit time of U for X , i.e., $\tau_U^X = \inf\{t > 0 : X_t \notin U\}$. We define $X_t^U(\omega) = X_t(\omega)$ if $t < \tau_U^X(\omega)$ and $X_t^U(\omega) = \partial$ if $t \geq \tau_U^X(\omega)$, where ∂ is a cemetery state. The process X^U is called a killed diffusion with drift μ in U . X^U is a Hunt process with the strong Feller property, i.e, for every $f \in L^\infty(U)$, $\mathbf{E}_x[f(X_t^U)]$ is in $C(U)$, the space of continuous functions in U (Proposition 2.1 [19]). Moreover, X^U has a jointly continuous density $q^U(t, x, y)$ with respect to the Lebesgue measure (Theorem 2.4 in [16]).

From Section 3 in [17] and Proposition 7.1 in [19], we know that for every bounded domain U , there exists a positive continuous additive functional A^U of X^U with Revuz measure $\nu|_U$, i.e., for any $x \in U$, $t > 0$ and bounded nonnegative function f on U ,

$$\mathbf{E}_x \int_0^t f(X_s^U) dA_s^U = \int_0^t \int_U q^U(s, x, y) f(y) \nu(dy) ds.$$

Throughout this paper, we assume that V is a bounded smooth domain in \mathbf{R}^d and consider the transient diffusion process Y such that

$$\mathbf{E}_x[f(Y_t)] = \mathbf{E}_x[\exp(-A_t^V) f(X_t^V)].$$

(See III.3 of [4] for the construction of such a killed process.) We will use ζ to denote the lifetime of Y . Note that the process Y might have killing inside V , i.e., $\mathbf{P}_x(Y_{\zeta-} \in V)$ might be positive.

A simple example for Y is a diffusion whose infinitesimal generator is a second order differential operator $L - b \cdot \nabla - c$, where (b_1, \dots, b_d) and $c \geq 0$ belong to the Kato classes $\mathbf{K}_{d,1}$ and $\mathbf{K}_{d,2}$ respectively. If (b_1, \dots, b_d) is differentiable and $L = L_1$, then the formal adjoint of the above operator is $L_1 + b \cdot \nabla - (c - \nabla b)$. If one further assumes that $c - \nabla b \geq 0$, then there is a diffusion process with generator $L_1 + b \cdot \nabla - (c - \nabla b)$. We can not and do not make such assumptions in this paper. Instead, we will introduce a new

reference measure and consider a dual process with respect to this reference measure.

Recall that, for any domain $D \subset \mathbf{R}^d$, $\rho_D(x)$ is the distance between x and ∂D . It is shown in [17] that the process Y has a jointly continuous and strictly positive transition density function $r(t, x, y)$ with respect to the Lebesgue measure and for each $T > 0$, there exist positive constants $c_j, 1 \leq j \leq 4$, depending on V such that for $t \leq T$,

$$(2.1) \quad \begin{aligned} c_1 t^{-\frac{d}{2}} (1 \wedge \frac{\rho_V(x)}{\sqrt{t}}) (1 \wedge \frac{\rho_V(y)}{\sqrt{t}}) e^{-\frac{c_2|x-y|^2}{2t}} &\leq r(t, x, y) \\ &\leq c_3 t^{-\frac{d}{2}} (1 \wedge \frac{\rho_V(x)}{\sqrt{t}}) (1 \wedge \frac{\rho_V(y)}{\sqrt{t}}) e^{-\frac{c_4|x-y|^2}{2t}}. \end{aligned}$$

Moreover, for every smooth subset U of V , the killed process Y^U has a jointly continuous and strictly positive transition density function $r^U(t, x, y)$ with respect to the Lebesgue measure and for each $T > 0$, there exist positive constants $c_j, 5 \leq j \leq 8$, depending on U such that for $t \leq T$,

$$(2.2) \quad \begin{aligned} c_5 t^{-\frac{d}{2}} (1 \wedge \frac{\rho_U(x)}{\sqrt{t}}) (1 \wedge \frac{\rho_U(y)}{\sqrt{t}}) e^{-\frac{c_6|x-y|^2}{2t}} &\leq r^U(t, x, y) \\ &\leq c_7 t^{-\frac{d}{2}} (1 \wedge \frac{\rho_U(x)}{\sqrt{t}}) (1 \wedge \frac{\rho_U(y)}{\sqrt{t}}) e^{-\frac{c_8|x-y|^2}{2t}}. \end{aligned}$$

(See Theorem 4.4(1) in [17].)

Let $C_0(V)$ be the class of bounded continuous functions on V vanishing continuously near the boundary of V . We will use $\|\cdot\|_\infty$ to denote the L^∞ -norm in $C_0(V)$. Using the joint continuity of $r(t, x, y)$ and $r^U(t, x, y)$ and the estimates above, it is easy to show the following result and we omit the proof.

PROPOSITION 2.2. *Y is a doubly Feller process (a Feller process satisfying the strong Feller property), i.e., for every $g \in C_0(V)$, $\mathbf{E}_x[g(Y_t)] = \mathbf{E}_x[g(Y_t); t < \zeta]$ is in $C_0(V)$ and $\|\mathbf{E}_x[g(Y_t)] - g(x)\|_\infty \rightarrow 0$ as $t \rightarrow 0$, and for every $f \in L^\infty(V)$, $\mathbf{E}_x[f(Y_t)]$ is bounded and continuous in V .*

In particular, the proposition above implies that for any domain $U \subset V$, Y^U is Hunt process with the strong Feller property (for example, see [7]).

We will use $G(x, y)$ to denote the Green function of Y . For any domain $U \subset V$, we will use $G_U(x, y)$ to denote the Green function of Y^U . Thus

$$\mathbf{E}_x \int_0^\infty f(Y_t) dt = \mathbf{E}_x \int_0^\zeta f(Y_t) dt = \int_V G(x, y) f(y) dy$$

and

$$\mathbf{E}_x \int_0^\infty f(Y_t^U) dt = \mathbf{E}_x \int_0^{\tau_U} f(Y_t^U) dt = \int_U G_U(x, y) f(y) dy,$$

where τ_U is the first exit time of U for Y , i.e., $\tau_U = \inf\{t > 0 : Y_t \notin U\}$. We will use $G_U^X(x, y)$ to denote the Green function of X^U and $G_V^0(x, y)$ the Green function of the killed Brownian motion in V . Since Y is transient, combining Theorem 6.2 in [15] and the result in Section 3 of [17], we have that there exists constant $c = c(V)$ such that

$$(2.3) \quad c^{-1} G_V^0(x, y) \leq G(x, y) \leq c G_V^0(x, y), \quad V \times V \setminus \{x = y\}.$$

Thus for every $U \subset V$,

$$(2.4) \quad G_U(x, y) \leq G(x, y) \leq \frac{c}{|x - y|^{d-2}}, \quad \text{for every } x, y \in D$$

for some constant $c > 0$.

Let

$$H(x) := \int_V G(y, x) dy \quad \text{and} \quad \xi(dx) := H(x) dx.$$

Then it is easy to check (see the proof of Proposition 2.2 in [19]) that ξ is an excessive measure with respect to Y , i.e., for every Borel function $f \geq 0$,

$$\int_V f(x) \xi(dx) \geq \int_V \mathbf{E}_x [f(Y_t)] \xi(dx).$$

We define a new transition density function with respect to the reference measure ξ by

$$\bar{r}(t, x, y) := \frac{r(t, x, y)}{H(y)}.$$

Then

$$\bar{G}(x, y) := \int_0^\infty \bar{r}(t, x, y) dt = \frac{G(x, y)}{H(y)}$$

is the Green function of Y with respect to the reference measure $\xi(dy)$.

Before we discuss properties of Y any further, let's recall some definitions. Recall that $\tau_A = \inf\{t > 0 : Y_t \notin A\}$.

DEFINITION 2.3. *Suppose U is an open subset of V . A non-negative Borel function u defined on U is said to be*

(1) *harmonic with respect to Y in U if*

$$(2.5) \quad u(x) = \mathbf{E}_x [u(Y_{\tau_B})] = \mathbf{E}_x [u(Y_{\tau_B}); \tau_B < \zeta], \quad x \in B,$$

for every bounded open set B with $\bar{B} \subset U$;

(2) *superharmonic with respect to Y^U if*

$$u(x) \geq \mathbf{E}_x [u(Y_{\tau_B}^U)], \quad x \in B,$$

for every bounded open set B with $\overline{B} \subset U$;

(3) *excessive for Y^U if*

$$u(x) \geq \mathbf{E}_x [u(Y_t^U)] = \mathbf{E}_x [u(Y_t^U); t < \zeta], \quad t > 0, x \in U$$

and

$$u(x) = \lim_{t \downarrow 0} \mathbf{E}_x [u(Y_t^U)], \quad x \in U;$$

(4) *a potential for Y^U if it is excessive for Y^U and for every sequence $\{U_n\}_{n \geq 1}$ of open sets with $\overline{U_n} \subset U_{n+1}$ and $\cup_n U_n = U$,*

$$\lim_{n \rightarrow \infty} \mathbf{E}_x [u(Y_{\tau_{U_n}}^U)] = 0; \quad \xi\text{-a.e. } x \in U.$$

A Borel function u defined on \overline{U} is said to be regular harmonic with respect to Y in U if u is harmonic with respect to Y in U and (2.5) is true for $B = U$.

Since Y^U is a Hunt processes with the strong Feller property, it is easy to check that u is excessive for Y^U if and only if f is lower-semicontinuous in U and superharmonic with respect to Y^U . (See Theorem 4.5.3 in [10] for the Brownian motion case, and the proof there can adapted easily to the present case.)

Using (2.1)-(2.2) and the joint continuity of $r(t, x, y)$ and $r^U(t, x, y)$, one can easily check that $G_U(x, y)$ is strictly positive and jointly continuous on $(U \times U) \setminus \{(x, y) : x = y\}$. $G_U(x, y)$ is infinite if and only if $x = y$ (see the proof of Theorem 2.6 in [16]). Thus by (2.3), we see that H is a strictly positive, bounded continuous function on V . Moreover, using the estimates for $G_V^0(x, y)$, one can check that there exists a constant $c = c(V)$ such that

$$(2.6) \quad c^{-1} \rho_V(x) \leq H(x) \leq c \rho_V(x).$$

(See Lemma 6.4 in [19] and its proof.) Now using the above properties and (2.4), we see that Y is a transient diffusion with its Green function $\overline{G}(x, y)$ with respect to ξ satisfying the conditions in [9] and [23] (see (A1)-(A4) in [19]). In fact, one can follow the arguments in [19] and check that all the results in Sections 2-3 of [19] are true for Y . In particular, using the same arguments in the proofs of Theorems 2.4-2.5 in [19], it is easy to check that the conditions (i)-(vii) and (70)-(71) in [20] (also see Remark on page 391 in [21]) are satisfied. Thus with respect to the reference measure ξ , Y has a nice dual process. For more detail arguments, we refer our readers to [19].

THEOREM 2.4. *There exists a continuous transient Hunt process \widehat{Y} in V such that \widehat{Y} is a strong dual of Y with respect to the measure ξ , that is, the density of the semigroup $\{\widehat{P}_t\}_{t \geq 0}$ of \widehat{Y} is $\widehat{r}(t, x, y) := \bar{r}(t, y, x)$ and thus*

$$\int_V f(x) P_t g(x) \xi(dx) = \int_V g(x) \widehat{P}_t f(x) \xi(dx) \quad \text{for all } f, g \in L^2(V, \xi)$$

We will use $\widehat{\zeta}$ to denote the lifetime of \widehat{Y} . Note that \widehat{Y} also might have killing inside V , i.e., $\mathbf{P}_x(\widehat{Y}_{\widehat{\zeta}_-} \in V)$ might be positive.

By Theorem 2 and Remark 2 after it in [25], for any domain $U \subset V$, Y^U and \widehat{Y}^U are duals of each other with respect to ξ . For any domain $U \subset V$, we define

$$\widehat{r}^U(t, x, y) := \frac{r^U(t, y, x) H(y)}{H(x)}.$$

Since H is strictly positive and continuous, by the joint continuity of $r^U(t, x, y)$ (see Section 4 of [17] and the references therein) $\widehat{r}^U(t, x, y)$ is jointly continuous on $U \times U$. Thus $\widehat{r}^U(t, x, y)$ is the transition density of \widehat{Y}^U with respect to the Lebesgue measure and

$$(2.7) \quad \widehat{G}_U(x, y) := \frac{G_U(y, x) H(y)}{H(x)}$$

is the Green function for \widehat{Y}^U with respect to the Lebesgue measure so that for every nonnegative Borel function f ,

$$\mathbf{E}_x \left[\int_0^{\widehat{\tau}_U} f(\widehat{Y}_t) dt \right] = \int_U \widehat{G}_U(x, y) f(y) dy$$

where $\widehat{\tau}_U := \inf\{t > 0 : \widehat{Y}_t \notin U\}$.

We will use $\{\widehat{G}_\lambda^U, \lambda \geq 0\}$ to denote the resolvent of \widehat{Y}^U with respect to ξ . Following the argument in Proposition 3.4 in [19], one can check that \widehat{Y}^U has the strong Feller property. We include the proof here for the reader's convenience.

PROPOSITION 2.5. *For any $U \subset V$, \widehat{Y}^U has the strong Feller property in the resolvent sense; that is, for every bounded Borel function f on D and $\lambda \geq 0$, $\widehat{G}_\lambda^U f(x)$ is bounded continuous function on U .*

PROOF. By the resolvent equation $\widehat{G}_0^U = \widehat{G}_\lambda^U + \lambda \widehat{G}_0^U \widehat{G}_\lambda^U$, it is enough to show the strong Feller property for \widehat{G}_0^U . Fix a bounded Borel function f on U and a sequence $\{y_n\}_{n \geq 1}$ converges to y in U . Let $M := \|fH\|_{L^\infty(U)} < \infty$. We

assume $\{y_n\}_{n \geq 1} \subset K$ for a compact subset K of U . Let $A := \inf_{y \in K} H(y)$. By (2.6), we know that A is strictly positive. Note that there exists a constant c_1 such that for every $\delta > 0$

$$\left(\int_{B(y, \delta)} \frac{dx}{|x-y|^{d-2}} + \int_{B(y_n, 2\delta)} \frac{dx}{|x-y_n|^{d-2}} \right) \leq c_1 \delta^2.$$

Thus by (2.4), there exist a constant c_2 such that for every $\delta > 0$ and y_n with $y_n \in B(y, \frac{\delta}{2}) \subset B(y, 2\delta) \in K$,

$$\begin{aligned} & \int_{B(y, \delta)} \frac{G_U(x, y)H(x)f(x)}{H(y)} dx + \int_{B(y_n, 2\delta)} \frac{G_U(x, y_n)H(x)f(x)}{H(y_n)} dx \\ & \leq \frac{M}{A} \left(\int_{B(y, \delta)} G_U(x, y) dx + \int_{B(y_n, 2\delta)} G_U(x, y_n) dx \right) \\ & \leq \frac{c_2 M}{A} \left(\int_{B(y, \delta)} \frac{dx}{|x-y|^{d-2}} + \int_{B(y_n, 2\delta)} \frac{dx}{|x-y_n|^{d-2}} \right) \leq \frac{1}{A} c_1 c_2 M \delta^2. \end{aligned}$$

Given ε , choose δ small enough such that $\frac{1}{A} c_1 c_2 M \delta^2 < \frac{\varepsilon}{2}$. Then

$$|\widehat{G}_0^U f(y) - \widehat{G}_0^U f(y_n)| \leq M \int_{U \setminus B(y, \delta)} \left| \frac{G_U(x, y)}{H(y)} - \frac{G_U(x, y_n)}{H(y_n)} \right| dx + \frac{\varepsilon}{2}.$$

Note that $G_U(x, y_n)/H(y_n)$ converges to $G_U(x, y)/H(y)$ for every $x \neq y$ and that $\{G_U(x, y_n)/H(y_n)\}$ are uniformly bounded on $x \in U \setminus B(y, \delta)$ and $y_n \in B(y, \frac{\delta}{2})$. So the first term on the right hand side of the inequality above goes to zero as $n \rightarrow \infty$ by the bounded convergence theorem. \square

Applying the results in [23] and [24], we have the following.

PROPOSITION 2.6. *Suppose $D \subset V$. Any function which is harmonic for Y (\widehat{Y} , respectively) in D is continuous. For each y , $x \rightarrow G_D(x, y)$ is excessive for Y^D and harmonic for Y in $D \setminus \{y\}$, and $x \rightarrow \widehat{G}_D(x, y)$ is excessive for \widehat{Y}^D and harmonic for \widehat{Y} in $D \setminus \{y\}$. Moreover, for every open subset U of D , we have*

$$(2.8) \quad \mathbf{E}_x[G_D(Y_{T_U}^D, y)] = G_D(x, y) \text{ and } \mathbf{E}_x[\widehat{G}_D(\widehat{Y}_{\widehat{T}_U}^D, y)] = \widehat{G}_D(x, y), \quad (x, y) \in D \times U$$

where $T_U := \inf\{t > 0 : Y_t^D \in U\}$ and $\widehat{T}_U := \inf\{t > 0 : \widehat{Y}_t^D \in U\}$. In particular, for every $y \in D$ and $\varepsilon > 0$, $G_D(\cdot, y)$ is regular harmonic with respect to Y^D in $D \setminus B(y, \varepsilon)$ and $\widehat{G}_D(\cdot, y)$ is regular harmonic with respect to \widehat{Y}^D in $D \setminus B(y, \varepsilon)$.

By Theorem 3.7 in [16], there exist constants $r_1 = r_1(d, \mu) > 0$ and $c = c(d, \mu) > 1$ depending on μ only via the rate at which $\max_{1 \leq i \leq d} M_{\mu^i}^1(r)$ goes to zero such that for $r \leq r_1$, $w \in \mathbf{R}^d$, $x, y \in B(w, r)$,

$$(2.9) \quad c^{-1} G_{B(w,r)}^0(x, y) \leq G_{B(w,r)}^X(x, y) \leq c G_{B(w,r)}^0(x, y).$$

Thus there exists a positive constant c independent of $r \leq r_1$ such that for every $x, y, z \in B(w, r)$ and $w \in \mathbf{R}^d$

$$(2.10) \quad \frac{G_{B(w,r)}^X(x, y) G_{B(w,r)}^X(y, z)}{G_{B(w,r)}^X(x, z)} \leq c(|x - y|^{2-d} + |y - z|^{2-d}).$$

For any $z \in B(w, r)$, let $(\mathbf{P}_x^z, X_t^{B(w,r)})$ be the $G_{B(w,r)}^X(\cdot, z)$ -transform of $(\mathbf{P}_x, X_t^{B(w,r)})$, that is, for any nonnegative Borel function f ,

$$\mathbf{E}_x^z \left[f(X_t^{B(w,r)}) \right] = \mathbf{E}_x \left[\frac{G_{B(w,r)}^X(X_t^{B(w,r)}, z)}{G_{B(w,r)}^X(x, z)} f(X_t^{B(w,r)}) \right].$$

Recall that A^V is the positive continuous additive functionals of X^V with Revuz measures $\nu|_V$. (2.10) implies that there exists a positive constant $c_1 < \infty$ such that for every $r \in (0, r_1]$, $w \in \mathbf{R}^d$ and $x, z \in B(w, r)$,

$$(2.11) \quad \mathbf{E}_x^z \left[A_{\tau_{B(w,r)}^X}^V \right] \leq \int_{B(w,r)} \frac{G_{B(w,r)}^X(x, y) G_{B(w,r)}^X(y, z)}{G_{B(w,r)}^X(x, z)} \nu(dy) < c_1.$$

Hence by Jensen's inequality, for $x, z \in B(w, r)$ we have

$$\mathbf{E}_x^z \left[\exp \left(-A_{\tau_{B(w,r)}^X}^V \right) \right] \geq \exp \left(-\mathbf{E}_x^z \left[A_{\tau_{B(w,r)}^X}^V \right] \right) \geq e^{-c_1} > 0.$$

Combining the identity

$$G_{B(w,r)}(x, z) = G_{B(w,r)}^X(x, z) \mathbf{E}_x^z \left[\exp \left(-A_{\tau_{B(w,r)}^X}^V \right) \right], \quad x, z \in B(w, r),$$

(Lemma 3.5 (1) of [5]) with (2.9), we arrive at the following result.

PROPOSITION 2.7. *There exist positive constants c and $r_1 := r_1(d, \mu, \nu)$ such that for all $r \in (0, r_1]$ and $B(w, r) \in V$, we have*

$$c^{-1} G_{B(w,r)}^0(x, y) \leq G_{B(w,r)}(x, y) \leq c G_{B(w,r)}^0(x, y), \quad x, y \in B(w, r).$$

In the remainder of this paper, we will always assume D is a bounded domain with $\bar{D} \subset V$. Let $\gamma_1 := \frac{1}{2} \text{dist}(\partial V, \bar{D})$ and $\check{V} := \{z \in V; \rho_V(z) > \gamma_1\}$. We fix D , \check{V} and γ_1 throughout this paper. For any subdomain $U \subset V$ and any subset A of U , we define

$$(2.12) \quad \text{Cap}^U(A) := \sup\{\eta(A) : \eta \text{ is a measure supported on } A \\ \text{with } \int_U G_U^0(x, y)\eta(dy) \leq 1\}.$$

The next lemma is a non-symmetric version of Lemma 2.1 in [2] for small balls. For any set A , we define $A_r^z := z + rA = \{w \in \mathbf{R}^d : w = z + ra, a \in A\}$, $A_r := A_r^0$ and $A^z := A_1^z$.

LEMMA 2.8. *There exists $c = c(V, d, \mu, \nu) > 0$ such that for any compact subset K of $B(0, 1)$, $r \in (0, r_1]$, $B(z, r) \subset \check{V}$ and compact set $A \subset K_r$, we have for any $x \in B(z, r)$,*

$$c^{-1}r^{2-d} \left(\inf_{y \in K} G_{B(0,1)}^0((x-z)/r, y) \right) \text{Cap}^{B(0,r)}(A) \leq \mathbf{P}_x(T_{A^z} < \tau_{B(z,r)}) \\ \leq cr^{2-d} \left(\sup_{y \in K} G_{B(0,1)}^0((x-z)/r, y) \right) \text{Cap}^{B(0,r)}(A)$$

and

$$c^{-1}r^{2-d} \left(\inf_{y \in K} G_{B(0,1)}^0((x-z)/r, y) \right) \text{Cap}^{B(0,r)}(A) \leq \mathbf{P}_x(\widehat{T}_{A^z} < \widehat{\tau}_{B(z,r)}) \\ \leq cr^{2-d} \left(\sup_{y \in K} G_{B(0,1)}^0((x-z)/r, y) \right) \text{Cap}^{B(0,r)}(A).$$

PROOF. For $B(z, r) \subset \check{V}$ and $U \subset B(z, r)$, define

$$(2.13) \quad \text{Cap}_{\widehat{Y}}^{B(z,r)}(U) := \sup\{\eta(U) : \eta \text{ is a measure supported on } U \\ \text{with } \int_{B(z,r)} \widehat{G}_{B(z,r)}(x, y)\eta(dy) \leq 1\}.$$

From (2.6), (2.7) and Proposition 2.7, we see that there is a constant $c > 0$ such that for every $r < r_1$ and $B(z, r) \subset \check{V}$, we have

$$(2.14) \quad c^{-1} \text{Cap}_{\widehat{Y}}^{B(z,r)}(U) \leq \text{Cap}^{B(z,r)}(U) \leq c \text{Cap}_{\widehat{Y}}^{B(z,r)}(U), \quad U \subset B(z, r).$$

Note that $Y^{B(z,r)}$ and $\widehat{Y}^{B(z,r)}$ are Hunt processes with the strong Feller property and they are in the strong duality with respect to ξ (Propositions

2.4 and 2.5). Since A^z is a compact subset of $B(z, r)$, there exist capacity measures μ_{A^z} for A^z with respect to $Y^{B(z,r)}$ and $\widehat{\mu}_{A^z}$ for A^z with respect to $\widehat{Y}^{B(z,r)}$ such that $\text{Cap}_{\widehat{Y}}^{B(z,r)}(A^z) = \mu_{A^z}(A^z) = \widehat{\mu}_{A^z}(A^z)$. (For example, see VI.4 of [4] and sections 5.1-5.2 of [10] for details.)

Using Proposition 2.7 and (2.6), we have for every $x \in B(z, r)$,

$$\begin{aligned}
\int_{A^z} \widehat{G}_{B(z,r)}(x, y) \widehat{\mu}_{A^z}(dy) &= \int_{A^z} \frac{G_{B(z,r)}(y, x) H(y)}{H(x)} \widehat{\mu}_{A^z}(dy) \\
&\geq c_1^{-1} \int_{A^z} G_{B(z,r)}^0(x, y) \widehat{\mu}_{A^z}(dy) \\
&\geq c_1^{-1} \left(\inf_{y \in K_r^z} G_{B(z,r)}^0(x, y) \right) \widehat{\mu}_{A^z}(A^z) \\
(2.15) \qquad &= c_1^{-1} \left(\inf_{y \in K_r^z} G_{B(z,r)}^0(x, y) \right) \text{Cap}_{\widehat{Y}}^{B(z,r)}(A^z)
\end{aligned}$$

for some constant $c_1 > 0$. Applying (2.14) to the above equation and using the scaling property of Brownian motion, we get that for every $x \in B(z, r)$,

$$\begin{aligned}
(2.16) \qquad &\left(\inf_{y \in K_r^z} G_{B(z,r)}^0(x, y) \right) \text{Cap}_{\widehat{Y}}^{B(z,r)}(A^z) \\
&\geq c^{-1} r^{2-d} \left(\inf_{y \in K} G_{B(0,1)}^0((x-z)/r, y) \right) \text{Cap}^{B(0,r)}(A).
\end{aligned}$$

On the other hand, by (2.8) we have for every $x \in B(z, r)$,

$$\begin{aligned}
&\int_{A^z} \widehat{G}_{B(z,r)}(x, y) \widehat{\mu}_{A^z}(dy) \\
&= \int_{A^z} \mathbf{E}_x \left[\widehat{G}_{B(z,r)}(\widehat{Y}_{\widehat{T}_{A^z}}^{B(z,r)}, y) \right] \widehat{\mu}_{A^z}(dy) \\
&\leq \left(\sup_{w \in A^z} \int_{A^z} \widehat{G}_{B(z,r)}(w, y) \mu_{A^z}(dy) \right) \mathbf{P}_x \left(\widehat{T}_{A^z} < \widehat{\tau}_{B(z,r)} \right) \\
(2.17) \qquad &\leq c_2 \mathbf{P}_x \left(\widehat{T}_{A^z} < \widehat{\tau}_{B(z,r)} \right)
\end{aligned}$$

for some constant $c_2 > 0$. In the last inequality above, we have used (2.6) and (2.13).

Combining (2.15)-(2.17), we have for every $x \in B(z, r)$,

$$\mathbf{P}_x \left(\widehat{T}_{A^z} < \widehat{\tau}_{B(z,r)} \right) \geq c_3 r^{2-d} \left(\inf_{y \in K} G_{B(0,1)}^0((x-z)/r, y) \right) \text{Cap}^{B(0,r)}(A)$$

for some constant $c_3 > 0$. Thus we have shown the first inequality in (2).

By Corollary 1 to Theorem 2 in [9], the function $x \mapsto \mathbf{P}_x(\widehat{T}_{A^z} < \widehat{\tau}_{B(z,r)})$ is a potential for $\widehat{Y}^{B(z,r)}$, thus there exists a Radon measure $\widehat{\nu}_1$ on A^z such that

$$\mathbf{P}_x(\widehat{T}_{A^z} < \widehat{\tau}_{B(z,r)}) = \int_{A^z} \widehat{G}_{B(z,r)}(x, y) \widehat{\nu}_1(dy), \quad x \in B(z, r).$$

Hence, by (2.6) and (2.13), we have

$$\mathbf{P}_x(\widehat{T}_{A^z} < \widehat{\tau}_{B(z,r)}) \leq c_4 \left(\sup_{y \in K_r^z} G_{B(z,r)}(y, x) \right) \text{Cap}_{\widehat{Y}}^{B(z,r)}(A^z), \quad x \in B(z, r)$$

for some constant $c_4 > 0$. Now applying Proposition 2.7 and (2.14) to the right hand side above and using the scaling property of Brownian motion, we get the desired assertion. \square

Note that the result in Lemma 2.1 in [2] (with $T_{\partial B(z,r)}$ instead of $\tau_{B(z,r)}$) may not be valid for our processes. This is because our processes might have killing inside V and so $T_{\partial B(z,r)}$ may be different from $\tau_{B(z,r)}$.

LEMMA 2.9. *There exists $c > 0$ such that for every $r < r_1$ and $B(z, r) \subset \check{V}$,*

$$(2.18) \quad \mathbf{E}_z[\tau_{B(z,r)}] \vee \mathbf{E}_z[\widehat{\tau}_{B(z,r)}] < cr^2.$$

PROOF. By Proposition 2.7 and (2.6), the lemma is clear, in fact,

$$\mathbf{E}_z[\widehat{\tau}_{B(z,r)}] = \int_{B(z,r)} \frac{G_{B(z,r)}(y, z)H(y)}{H(z)} dy \leq c \int_{B(z,r)} G_{B(z,r)}^0(z, y) dy \leq c_1 r^2$$

for some constants $c, c_1 > 0$. \square

Using the above lemma and the Markov property, we can easily get the following result.

LEMMA 2.10. *Suppose $r < r_1$, $B(z, r) \subset \check{V}$ and $U \subset D$. Then*

$$\mathbf{P}_z(\tau_U < \tau_{B(z,r)}) > c_1, \quad (\mathbf{P}_z(\widehat{\tau}_U < \widehat{\tau}_{B(z,r)}) > c_1 \text{ respectively}), \quad \forall z$$

for some $c_1 > 0$ implies

$$\mathbf{E}_z[\tau_U] \leq c_2 r^2, \quad (\mathbf{E}_z[\widehat{\tau}_U] \leq c_2 r^2 \text{ respectively}), \quad \forall z$$

for some $c_2 > 0$.

PROOF. Using (2.18) and the Markov property, the lemma can be proved using an argument similar to the one in the proof of Lemma 3.3 in [2] (with τ_U and $\tau_{B(z,r)}$ instead of the hitting times there). We omit the proof. \square

3. Parabolic and Elliptic Harnack Inequalities. In this section we shall prove a small time parabolic Harnack inequality for Y and \hat{Y} . We will get a scale invariant version of the elliptic Harnack inequality as a corollary. These Harnack inequalities will be used later to prove the main results of this paper.

Recall that D is a bounded domain with $\bar{D} \subset V$, $\gamma_1 = \frac{1}{2} \text{dist}(\partial V, \bar{D})$ and $\check{V} = \{z \in V; \rho_V(z) > \gamma_1\}$. In [17], we proved a uniform Gaussian estimates for the density (with respect to the Lebesgue measure) of Y^D when D is a bounded smooth domain. We recall here part of the result from [17]: there exist positive constants t_0, t_1, c_1 and c_2 such that for every $R \leq \sqrt{t_0}, t \leq R^2 t_1$ and $(x, y) \in B(z, R) \times B(z, R)$,

$$(3.1) \quad r^{B(z,R)}(t, x, y) \geq c_1 t^{-\frac{d}{2}} \left(1 \wedge \frac{\rho_{B(z,R)}(y)}{\sqrt{t}}\right) \left(1 \wedge \frac{\rho_{B(z,R)}(x)}{\sqrt{t}}\right) e^{-\frac{c_2|x-y|^2}{2t}}$$

whenever $B(z, R) \subset V$ (see Theorem 4.4(2) in [17]). In the remainder of this paper, t_0 and t_1 will always stand for the constants above.

With the density estimates (3.1) in hand, one can follow the ideas in [13] (see also [15, 27]) to prove the parabolic Harnack inequality. For this reason, the proofs of this section will be a little sketchy.

LEMMA 3.1. *For each $0 < \delta, u < 1$, there exists $\varepsilon = \varepsilon(d, \delta, u, t_1) > 0$ such that*

$$(3.2) \quad r^{B(x_0,R)}(t, x, y) \wedge \hat{r}^{B(x_0,R)}(t, x, y) \geq \frac{\varepsilon}{|B(x_0, \delta R)|}$$

for all $x, y \in B(x_0, \delta R) \subset \check{V}$, $R \leq \sqrt{t_0}$ and $(1-u)R^2 t_1 \leq t \leq R^2 t_1$.

PROOF. Fix $0 < \delta, u < 1$ and $B(x_0, \delta R) \subset \check{V}$. Let $B_R := B(x_0, R)$ and assume that $R \leq \sqrt{t_0}$ and $t \leq R^2 t_1$. By (2.6) and (3.1), there exist c_1 and c_2 such that

$$(3.3) \quad \begin{aligned} \hat{r}^{B_R}(t, x, y) &= \frac{r^{B_R}(t, y, x)H(y)}{H(x)} \\ &\geq c_1 t^{-\frac{d}{2}} \left(1 \wedge \frac{\rho_{B_R}(y)}{\sqrt{t}}\right) \left(1 \wedge \frac{\rho_{B_R}(x)}{\sqrt{t}}\right) e^{-\frac{c_2|x-y|^2}{2t}}. \end{aligned}$$

If $|x - x_0| < \delta R$, $|y - x_0| < \delta R$ and $(1-u)R^2 t_1 \leq t \leq R^2 t_1$, then

$$\left(1 \wedge \frac{\rho_{B_R}(y)}{\sqrt{t}}\right) \left(1 \wedge \frac{\rho_{B_R}(x)}{\sqrt{t}}\right) \geq \frac{(1-\delta)^2}{t_1} \quad \text{and} \quad \frac{c_2|x-y|^2}{2t} \leq \frac{2c_2\delta^2}{(1-u)t_1}.$$

So the right-hand side of (3.3) is bounded below by

$$c_1(R^2 t_1)^{-\frac{d}{2}} \frac{(1-\delta)^2}{t_1} e^{-\frac{2c_2 \delta^2}{(1-u)t_1}} = c_3 c_1 t_1^{-\frac{d}{2}-1} \frac{(1-\delta)^2 \delta^d}{|B(0, \delta R)|} e^{-\frac{2c_2 \delta^2}{(1-u)t_1}} =: \frac{\varepsilon}{|B(0, \delta)|}$$

where c_3 depends only on d . \square

We define space-time processes $Z_s := (T_s, Y_s)$ and $\widehat{Z}_s := (T_s, \widehat{Y}_s)$, where $T_s = T_0 - s$. The law of the space-time processes Z_s (and \widehat{Z}_s) starting from (t, x) will be denoted as $\mathbf{P}_{t,x}$.

DEFINITION 3.2. *For any $(t, x) \in [0, \infty) \times V$, $u > 0$ and bounded subdomain U of V , we say that a non-negative continuous function g defined on $[t, t+u] \times U$ is parabolic for Y in $[t, t+u] \times U$ if for any $[s_1, s_2] \subset (t, t+u)$ and $B(y, \delta) \subset \overline{B(y, \delta)} \subset D$ we have*

$$(3.4) \quad g(s, z) = \mathbf{E}_{s,z} \left[g(Z_{\tau_{(s_1, s_2] \times B(y, \delta)}}); Z_{\tau_{(s_1, s_2] \times B(y, \delta)}} \in (0, \infty) \times V \right],$$

for every $(s, z) \in (s_1, s_2] \times B(y, \delta)$ where $\tau_{(s_1, s_2] \times B(y, \delta)} = \inf\{s > 0 : Z_s \notin (s_1, s_2] \times B(y, \delta)\}$. The definition of parabolic functions for \widehat{Y} is similar.

LEMMA 3.3. *Suppose that U is a subdomain of V . For each $T > 0$ and $y \in U$, $(t, x) \rightarrow r^U(t, x, y)$ and $(t, x) \rightarrow \widehat{r}^U(t, x, y)$ are parabolic in $(0, T] \times U$ for Y and \widehat{Y} respectively.*

PROOF. See the proof of Lemma 4.5 in [6]. \square

COROLLARY 3.4. *Suppose that U is a subdomain of V . For each $T > 0$ and $y \in U$ and any nonnegative bounded function f on U , the functions*

$$g(t, x) := \mathbf{E}_x [f(Y_t^U)] = \int_U r^U(t, x, y) f(y) dy$$

and

$$\widehat{g}(t, x) := \mathbf{E}_x [f(\widehat{Y}_t^U)] = \int_U \widehat{r}^U(t, x, y) f(y) dy$$

are parabolic in $(0, T] \times U$ for Y and \widehat{Y} respectively.

PROOF. The continuity of \widehat{g} follows from the continuity of \widehat{r}^U . (3.4) follows from Lemma 3.3 and Fubini's theorem. \square

For $s \geq 0$, $R > 0$ and $B(x, R) \subset V$ and we define the oscillation of a function g on $(s - t_1 R^2, s) \times B(x, R)$ by

$$\begin{aligned} & \text{Osc}(g; s, x, R) \\ &= \sup \{ |g(s_1, x_1) - g(s_2, x_2)| : s_1, s_2 \in (s - t_1 R^2, s), x_1, x_2 \in B(x, R) \}. \end{aligned}$$

LEMMA 3.5. *For any $0 < \delta < 1$, there exists $0 < \rho < 1$ such that for all $s \in (-\infty, \infty)$, $0 < R \leq \sqrt{t_0}$, $B(x_0, R) \subset \check{V}$ and function g which is parabolic for Y (\hat{Y} , respectively) in $(s - t_1 R^2, s) \times B(x_0, R)$ and continuous in $[s - t_1 R^2, s] \times \overline{B(x_0, R)}$*

$$\text{Osc}(g; s, x_0, \delta R) \leq \rho \text{Osc}(g; s, x_0, R).$$

PROOF. Fix $s \geq 0$, $0 < R \leq \sqrt{t_0}$ and $B(x_0, R) \subset \check{V}$, and consider a function g which is parabolic for \hat{Y} in $(s - t_1 R^2, s) \times B(x_0, R)$ and continuous in $[s - t_1 R^2, s] \times \overline{B(x_0, R)}$. Without loss of generality, we may assume that

$$\min_{(t,x) \in [s-t_1 R^2, s] \times B(x_0, R)} g(t, x) = 0 \quad \text{and} \quad \max_{(t,x) \in [s-t_1 R^2, s] \times B(x_0, R)} g(t, x) = 1.$$

Since \hat{Y} is a Hunt process, it is easy to see that \hat{Z}^Ω is a Hunt process for any bounded open subset Ω of $[0, \infty) \times V$. So g and $1 - g$ are excessive with respect to the process obtained by killing \hat{Z} upon exiting from $(s - t_1 R^2, s) \times B(x_0, R)$. First, we assume that δ satisfies

$$\int_{B(x_0, \delta R)} g(s - \frac{1}{2}(\delta^2 + 1)t_1 R^2, y) dy \geq \frac{|B(x_0, \delta R)|}{2}.$$

By Lemma 3.1, we have that for $(t, x) \in (s - \delta^2 t_1 R^2, s) \times B(x_0, \delta R)$,

$$\begin{aligned} & g(t, x) \\ & \geq \mathbf{E}_{t,x} \left[g(\hat{Z}_{t+\frac{1}{2}(\delta^2+1)t_1 R^2-s}) : \hat{Z}_{t+\frac{1}{2}(\delta^2+1)t_1 R^2-s} \in (t_1 R^2 - s, s) \times B(x_0, \delta R) \right] \\ & \geq \int_{B(x_0, \delta R)} \hat{q}^{B(x_0, R)}(t + \frac{1}{2}(\delta^2 + 1)t_1 R^2 - s, x, y) g(s - \frac{1}{2}(\delta^2 + 1)t_1 R^2, y) dy \\ & \geq \frac{\varepsilon}{|B(x_0, \delta R)|} \frac{|B(x_0, \delta R)|}{2} = \frac{\varepsilon}{2} \end{aligned}$$

Therefore $\text{Osc}(g; s, x_0, \delta R) \leq 1 - \varepsilon$.

If

$$\int_{B(x_0, \delta R)} g(s - \frac{1}{2}(\delta^2 + 1)t_1 R^2, y) dy \leq \frac{|B(x_0, \delta R)|}{2},$$

we consider $1 - g$ and use the same argument as above. \square

The above lemma implies the Hölder continuity of parabolic functions.

THEOREM 3.6. *For any $0 < \delta < 1$, there exist $c > 0$ and $\beta \in (0, 1)$ such that for all $s \in (-\infty, \infty)$, $0 < R \leq \sqrt{t_0}$, $B(x_0, R) \subset \check{V}$ and function g which is parabolic for Y (\hat{Y} , respectively) in $[s - t_1 R^2, s] \times B(x_0, R)$ and continuous in $[s - t_1 R^2, s] \times \overline{B(x_0, R)}$ we have*

$$|g(s_1, x_1) - g(s_2, x_2)| \leq c \|g\|_{L^\infty([s - t_1 R^2, s] \times \overline{B(x_0, R)})} \left(\frac{|s_1 - s_2|^2 + |x_1 - x_2|}{R} \right)^\beta$$

for any $(s_1, x_1), (s_2, x_2) \in [s - t_1 \delta^2 R^2, s] \times \overline{B(x_0, \delta R)}$.

PROOF. See Theorem 5.3 in [13]. □

Using Lemmas 3.1 and 3.5, the proof of the next theorem is almost identical to that of Theorem 5.4 in [13]. So we omit the proof.

THEOREM 3.7. *For any $0 < \alpha < \beta < 1$ and $0 < \delta < 1$, there exists $c > 0$ such that for all $s \in (-\infty, \infty)$, $0 < R \leq \sqrt{t_0}$, $B(x_0, R) \subset \check{V}$ and function g which is parabolic for Y (\hat{Y} , respectively) in $(s - t_1 R^2, s] \times B(x_0, R)$ and continuous in $(s - t_1 R^2, s] \times \overline{B(x_0, R)}$*

$$g(t, y) \leq c g(s, x_0), \quad (t, y) \in [s - \beta t_1 R^2, s - \alpha t_1 R^2] \times \overline{B(x_0, \delta R)}.$$

Now the parabolic Harnack inequality is an easy corollary of the theorem above.

THEOREM 3.8 (Parabolic Harnack inequality). *For any $0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < 1$ and $0 < \delta < 1$, there exist $c > 0$ and such that for all $0 < R \leq \sqrt{t_0}$, $B(x_0, R) \subset \check{V}$ and function g which is parabolic for Y (\hat{Y} , respectively) in $[0, t_1 R^2] \times B(x_0, R)$ and continuous in $[0, t_1 R^2] \times \overline{B(x_0, R)}$*

$$\sup_{(t, y) \in B_1} g(t, y) \leq c \inf_{(t, y) \in B_2} g(t, y),$$

where $B_i = \{(t, y) \in [\alpha_i t_1 R^2, \beta_i t_1 R^2] \times B(x_0, \delta R)\}$.

The scale invariant Harnack inequality is an easy corollary of the parabolic Harnack inequality.

THEOREM 3.9 (Scale invariant Harnack inequality). *Every harmonic function for Y (\hat{Y} , respectively) is Hölder continuous. There exists $c =$*

$c(D, V) > 0$ such that for every harmonic function f for Y (\widehat{Y} , respectively) in $B(z_0, R)$ with $B(z_0, R) \subset \check{V}$, we have

$$\sup_{y \in B(z_0, R/2)} f(y) \leq c \inf_{y \in B(z_0, R/2)} f(y).$$

PROOF. By Proposition 2.6, any harmonic function f for \widehat{Y} in $B(z_0, r)$ is parabolic in $(0, T] \times B(z_0, r)$ with respect to \widehat{Y} for any $T > 0$. Thus f is Hölder continuous by Theorem 3.6 and the Harnack inequality above is true for small R by Theorem 3.8. When R is large and $B(z_0, R) \subset \check{V}$, we use a Harnack chain argument and the fact that V is bounded. \square

4. Analysis on various rough domains. In this section, we recall the definitions of various rough domains from [1, 2] and prove the main lemma (Lemma 4.7). We will use the probabilistic methods used in [2]. For this reason, we follow the notations and the definitions from [2]. Unlike [2], we do not have the scaling property here and Lemma 2.8 works only for small balls. Moreover, our processes Y and \widehat{Y} may have killing inside V . All these make our argument more complicated than that of [2]. For the reader's convenience, we will spell out some of proofs, especially the parts where things are more complicated.

A bounded domain D is said to be a Hölder domain of order $\beta \in (0, 1]$ if the boundary of D is locally the graph of a function ϕ which is Hölder continuous of order β , i.e., $|\phi(x) - \phi(z)| \leq c|x - z|^\beta$. The concept of twisted Hölder domains, which is a natural generalization of the concept of Hölder domains, was introduced in [2]. Twisted Hölder domains have canals no longer and no thinner than Hölder domains, but do not have local representation of their boundaries as graphs of functions. For a rectifiable Jordan arc γ and $x, y \in \gamma$, we denote the length of the piece of γ between x and y by $l(\gamma(x, y))$. Recall the capacity defined in (2.12).

DEFINITION 4.1. *A bounded domain $D \subset \mathbf{R}^d$ is called a twisted Hölder domain of order $\alpha \in (0, 1]$, if there exist positive constants c_1, \dots, c_5 , a point $z_0 \in D$ and a continuous function $\delta : D \rightarrow (0, \infty)$ with the following properties.*

- (1) $\delta(x) \leq \rho_D(x)^\alpha$ for all $x \in D$;
- (2) for every $x \in D$, there exists a rectifiable Jordan arc γ connecting x and z_0 in D such that

$$\delta(y) \geq c_2(l(\gamma(x, y)) + \delta(x)), \quad \text{for all } y \in \gamma;$$

(3)

$$\frac{\text{Cap}^{B(x, 2c_3a)}(B(x, c_3a) \cap F(a)^c)}{\text{Cap}^{B(x, 2c_3a)}(B(x, c_3a))} \geq c_4 \quad \text{for all } x \in F(a), a \leq c_5$$

where $F(a) = \{y \in D : \delta(y) \leq a\}$.

One interesting fact is that the class of John domains (see page 422 of [2] for the definition) and the class of twisted Hölder domains of order 1 are identical (Proposition 3.2 of [2]). The boundary of a twisted Hölder domain can be highly nonrectifiable and, in general, no regularity of its boundary can be inferred. We refer [2] for some elementary results on twisted Hölder domains.

Under some regularity assumption on the boundary of D , Bañuelos considered in [1] another natural generalization of Hölder domains. Let $k_D(x, y)$ be the quasi-hyperbolic distance:

$$k_D(x, y) := \inf_{\gamma} \int_{\gamma} \frac{ds}{\rho_D(z)}$$

where the infimum is taken over all rectifiable curves joining x to y in D . The following definition is taken from [1].

DEFINITION 4.2. *A bounded domain $D \subset \mathbf{R}^d$ is called a uniformly Hölder domain of order $\alpha > 0$ if there exist positive constants c_1, \dots, c_5 and a point $z_1 \in D$ with the following properties.*

- (1) $k_D(x, z_1) \leq c_1 \rho_D(x)^{-\alpha} + c_2$ for all $x \in D$;
- (2) for every $Q \in \partial D$ and $r > 0$,

$$\text{Cap}^{B(Q, 2r)}(B(Q, r) \cap D^c) \geq c_3 r^{d-2}.$$

The class of uniformly Hölder domains is slightly more general than that of uniformly regular twisted L^p -domains defined in [2].

LEMMA 4.3. *(1) If D is a twisted Hölder domain of order $\alpha \in (0, 1]$, there exist $c_1 > 0$, $a_1 > 0$ and $b_1 > 0$ such that for every $a \leq a_1$,*

$$\sup_{y \in F(a)} \mathbf{P}_y(T_{F(a)^c \cap B(y, ab_1)} < \tau_{B(y, 2ab_1)}) > c_1.$$

and

$$\sup_{y \in F(a)} \mathbf{P}_y(\widehat{T}_{F(a)^c \cap B(y, ab_1)} < \widehat{\tau}_{B(y, 2ab_1)}) > c_1.$$

(2) If D is a uniformly Hölder domain of order $\alpha > 0$, there exist $c_2 > 0$ and $a_2 > 0$ such that for every $r \leq a_2$,

$$\sup_{y \in B(Q, \frac{2r}{3}) \cap D} \mathbf{P}_y(T_{B(Q,r) \cap D^c} < \tau_{B(Q,2r)}) > c_2.$$

and

$$\sup_{y \in B(Q, \frac{2r}{3}) \cap D} \mathbf{P}_y(\widehat{T}_{B(Q,r) \cap D^c} < \widehat{\tau}_{B(Q,2r)}) > c_2.$$

PROOF. Note that $\text{Cap}^{B(x,2r)}(B(x,r)) \geq cr^{d-2}$. Thus, to prove (i), we only need to use Lemma 2.8 and Definition 4.1 (3) with $K = \overline{B(0, 1/2)}$ and $A^z = \partial F(a) \cap \overline{B(z, ab_1)} \subset \overline{B(z, ab_1)}$.

To prove (ii), we use Lemma 2.8 and Definition 4.2 (2) with $K = \overline{B(0, 2/3)}$ and $A^z = \overline{B(z, 2r/3)} \cap \partial D \subset \overline{B(z, 2r/3)}$. \square

DEFINITION 4.4. We say that a bounded domain $D \subset \mathbf{R}^d$ can be locally represented as the region above the graph of a function if there exist a positive constant a_0 , a finite family of orthonormal coordinate systems CS_j 's, positive b_j 's and functions

$$f_j : \mathbf{R}^{d-1} \rightarrow (-\infty, 0], \quad j = 1, \dots, m_0$$

such that

$$D = \bigcup_{j=1}^{m_0} \{x = (x_1, \dots, x_{d-1}, x_d) =: (\tilde{x}, x_d) \text{ in } CS_j : |\tilde{x}| < b_j, f_j(\tilde{x}) < x_d \leq a_0\}.$$

LEMMA 4.5. Suppose that D is a bounded domain which can be locally represented as the region above the graph of a function. Assume that $a \leq r_1$ and that $y \in D$ is in $\{x = (\tilde{x}, x_d) \text{ in } CS_j : |\tilde{x}| < b_j, f_j(\tilde{x}) < x_d \leq a_0\}$ for some $j = 1, \dots, m_0$. If U and M are subsets of \mathbf{R}^d that can be written as

$$\begin{aligned} U &:= \{(\tilde{x}, x_d) \text{ in } CS_j : |\tilde{x} - \tilde{y}| < a, |x_d - y_d| < a\}, \\ M &:= \left\{ (\tilde{x}, x_d) \text{ in } CS_j : |\tilde{x} - \tilde{y}| < \frac{a}{2}, x_d = a + y_d \right\}, \end{aligned}$$

then there exists a constant $c_1 > 0$ independent of a , y and CS_j such that

$$\left(\inf_{|\tilde{x} - \tilde{y}| < \frac{a}{2}, x_d = y_d} \mathbf{P}_x(T_M = \tau_U) \right) \wedge \left(\inf_{|\tilde{x} - \tilde{y}| < \frac{a}{2}, x_d = y_d} \mathbf{P}_x(\widehat{T}_M = \widehat{\tau}_U) \right) > c_1.$$

PROOF. By our Harnack inequality (Theorem 3.9), it is enough to show that

$$\mathbf{P}_y(T_M \leq \tau_U) \wedge \mathbf{P}_y(\widehat{T}_M \leq \widehat{\tau}_U) > c_1$$

for some $c_1 > 0$ independent of a and CS_j . Fix the coordinate systems CS_j . Let $B_1 := B(y, a)$ and

$$\begin{aligned} B_2 &:= B((\tilde{y}, y_d + a/2), a/\sqrt{2}), \\ M_1 &:= \left\{ (\tilde{x}, x_d); |\tilde{x} - \tilde{y}| < \frac{a}{2}, x_d = \frac{a}{2} + y_d \right\}, \\ M_2 &:= \left\{ (\tilde{x}, x_d); |\tilde{x} - \tilde{y}| < \frac{a}{4}, x_d = a + y_d \right\}. \end{aligned}$$

Note that $B_2 \cap \{x_d = a + y_d\} = M$. Thus

$$\begin{aligned} \mathbf{P}_y(\widehat{T}_M = \widehat{\tau}_U) &\geq \mathbf{E}_y \left[\mathbf{P}_{\widehat{Y}_{\widehat{T}_{M_1}}}(\widehat{T}_{M_2} < \widehat{\tau}_{B_2}); \widehat{T}_{M_1} < \widehat{\tau}_{B_1} \right] \\ &\geq \mathbf{P}_y(\widehat{T}_{M_1} < \widehat{\tau}_{B_1}) \left(\inf_{z \in M_1} \mathbf{P}_z(\widehat{T}_{M_2} < \widehat{\tau}_{B_2}) \right). \end{aligned}$$

Now applying Lemma 2.8 to both factors on the right hand side the equation above, we arrive at our desired conclusion. \square

For a bounded domain which can be locally represented as the region above the graph of a function, we put

$$\Theta := \frac{1}{2} \left(1 + \frac{1}{4d-2} \right).$$

For any $k < 0$ and $y \in D$ such that

$$y \in \{x = (\tilde{x}, x_d) \text{ in } CS_j : |\tilde{x}| < b_j, f_j(\tilde{x}) < x_d < 0\}$$

for some $j = 1, \dots, m_0$, we let $l_0^{j,k}(y)$ be the smallest integer greater than $10|k|^\Theta(a_0/2 - y_d)/b_j$ and define

$$(4.1) \quad D_1^{j,k}(y) := \left\{ x \text{ in } CS_j : |\tilde{x} - \tilde{y}| < \frac{b_j}{4|k|^\Theta}, f_j(\tilde{x}) < x_d < a_0 \right\},$$

$$(4.2) \quad D_2^{j,k}(y) := \left\{ x \text{ in } CS_j : |\tilde{x} - \tilde{y}| < \frac{b_j}{4|k|^\Theta}, |x_d - y_d| < \frac{b_j}{4|k|^\Theta} \right\},$$

$$(4.3) \quad M^{j,k}(y) := \left\{ x \text{ in } CS_j : |\tilde{x} - \tilde{y}| < \frac{b_j}{20|k|^\Theta}, x_d = y_d + \frac{b_j}{10|k|^\Theta} l_0^{j,k}(y) \right\},$$

where a_0, b_j, CS_j and f_j are the quantities from Definition 4.4.

LEMMA 4.6. *Suppose that D is a bounded domain which can be locally represented as the region above the graph of a function. There exists $p_0 \in (0, 1)$ such that if $p \in [p_0, 1)$ and*

$$k \leq - \max_{1 \leq j \leq m_0} \left(\frac{b_j}{10r_1} \right)^{\frac{1}{\Theta}},$$

then for any $j = 1, \dots, m_0$,

$$\mathbf{P}_y(T_{\partial D} < \tau_{B(y, b_j |k|^{-\Theta})}) \leq 1 - p$$

$$\left(\mathbf{P}_y(\widehat{T}_{\partial D} < \widehat{\tau}_{B(y, b_j |k|^{-\Theta})}) \leq 1 - p \quad \text{respectively} \right)$$

for every $y \in \{(\tilde{x}, x_d) \text{ in } CS_j : |\tilde{x}| < b_j, f_j(\tilde{x}) < x_d < 0\}$ implies

$$\mathbf{P}_y(T_{M^{j,k}(y)} < \tau_{D_1^{j,k}(y)}) \geq \exp \left(-c_1 \frac{8(a_0 - y_d) |k|^\Theta}{b_j} \right)$$

$$\left(\mathbf{P}_y(\widehat{T}_{M^{j,k}(y)} < \widehat{\tau}_{D_1^{j,k}(y)}) \geq \exp \left(-c_1 \frac{8(a_0 - y_d) |k|^\Theta}{b_j} \right) \quad \text{respectively} \right)$$

for some $c_1 = c_1(p_0) > 0$ independent of j, f_j and y .

PROOF. Fix j and k satisfying the assumption of the lemma. We also fix an $y \in \{(\tilde{x}, x_d) \text{ in } CS_j : |\tilde{x}| < b_j, f_j(\tilde{x}) < x_d < 0\}$. Let $a := b_j 10^{-1} |k|^{-\Theta} \leq r_1$ and

$$D_l := \{x \text{ in } CS_j : |\tilde{x} - \tilde{y}| < \frac{5a}{2}, (y_d - a) \vee f_j(\tilde{x}) < x_d < y_d + al\}, \quad l \geq 1,$$

$$\tilde{D}_l := \{x \text{ in } CS_j : |\tilde{x} - \tilde{y}| < \frac{5a}{2}, y_d - a < x_d < y_d + al\}, \quad l \geq 1,$$

$$W_l := \{x \text{ in } CS_j : |\tilde{x} - \tilde{y}| < \frac{5a}{2}, y_d + a(l - 5) \leq x_d < y_d + al\}, \quad l \geq 4,$$

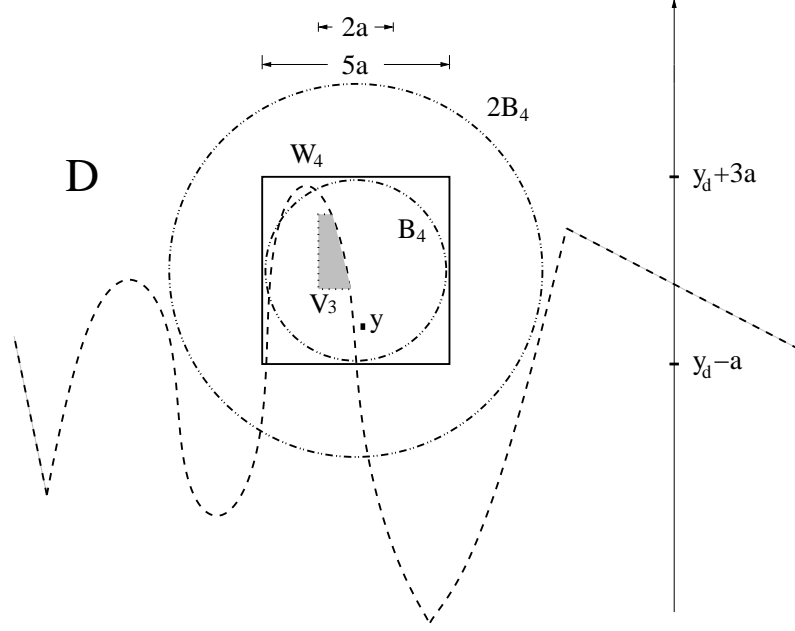
$$V_l := \{x \text{ in } CS_j : x \in D_l^c, |\tilde{x} - \tilde{y}| < a, y_d + a(l - 2) < x_d < y_d + al\}, \quad l \geq 3,$$

$$B_l := \{x \text{ in } CS_j : |x - (\tilde{y}, a(l - \frac{5}{2}))| < \frac{5a}{2}\}, \quad l \geq 4,$$

$$\text{and } 2B_l := \{x \text{ in } CS_j : |x - (\tilde{y}, a(l - \frac{5}{2}))| < 5a\}, \quad l \geq 4.$$

Note that $V_l \subset B_{l+1} \subset W_{l+1} \subset 2B_{l+1}$ (see Figure 1). Since $\mathbf{P}_y(\widehat{T}_{\partial D} < \widehat{\tau}_{B(y, b_j |k|^{-\Theta})}) \leq 1 - p$, we have

$$(4.4) \quad \mathbf{P}_y(\widehat{T}_{V_3} < \widehat{\tau}_{B_4}) \leq \mathbf{P}_y(\widehat{T}_{\partial D_4} < \widehat{\tau}_{\tilde{D}_4}) \leq (1 - p).$$


 FIG 1. $V_3 \subset B_4 \subset W_4 \subset 2B_4$

Thus by Lemma 2.8,

$$\text{Cap}^{B_4}(V_3) \leq c_1 \left(\inf_{w \in K} G_{B(0,1)}^0 \left(\left(0, -\frac{3}{5} \right), w \right) \right)^{-1} a^{d-2} (1-p) \leq c_2 a^{d-2} (1-p)$$

for some constants $c_1, c_2 > 0$ and where

$$(4.5) \quad K := \{ |\tilde{x}| < 1/10, -1/5 < x_d < 3/5 \}.$$

By the translation invariance of Cap and the definition of V_l ,

$$\text{Cap}^{B_{l+1}}(V_l) \leq c_2 a^{d-2} (1-p).$$

Since $W_{l+1} \subset 2B_{l+1}$, by Lemma 2.8, for $y_l := y + (\tilde{0}, (l-3)a)$,

$$\begin{aligned} \mathbf{P}_{y_l}(\widehat{T}_{V_l} < \widehat{\tau}_{W_{l+1}}) &\leq \mathbf{P}_{y_l}(\widehat{T}_{V_l} < \tau_{2B_{l+1}}) \\ &\leq c_3 a^{2-d} \left(\sup_{w \in \frac{1}{2}K} G_{B(0,1)}^0 \left(\left(0, -\frac{3}{10} \right), w \right) \right) \text{Cap}^{2B_{l+1}}(V_l) \\ &\leq c_4 a^{2-d} \text{Cap}^{2B_{l+1}}(V_l) \end{aligned}$$

where K is defined in (4.5). But, by the definition of Cap , $\text{Cap}^{2B_{l+1}}(V_l) \leq \text{Cap}^{B_{l+1}}(V_l)$. Therefore

$$\mathbf{P}_{y_l}(\widehat{T}_{V_l} < \widehat{\tau}_{W_{l+1}}) \leq c_4 a^{2-d} \text{Cap}^{B_{l+1}}(V_l) \leq c_5(1-p).$$

Applying the Harnack inequality (Theorem 3.9), we get

$$(4.6) \quad \mathbf{P}_x(\widehat{T}_{V_l} < \widehat{\tau}_{W_{l+1}}) \leq c_6(1-p), \quad |\tilde{x} - \tilde{y}| < \frac{a}{2}, \quad x_d = y_d + a(l-3).$$

Using our Lemma 4.5 and (4.6) instead of Lemma 2.3 and (2.5) of [2], the remaining part of the proof is similar to the proof of Lemma 2.4 on page 414 starting from the line 3 in [2] (after rescaling) with

$$\begin{aligned} \widehat{D}_l &:= \{x \text{ in } CS_j : |\tilde{x} - \tilde{y}| < a, y_d - a < x_d < y_d + al\}, \\ M_l &:= \{x \text{ in } CS_j : |\tilde{x} - \tilde{y}| < \frac{a}{2}, x_d = y_d + al\}. \end{aligned}$$

However, due to the possible killing inside the domain in our case, things are more delicate. We include the details of the remaining part of the proof for the reader's convenience.

Let θ be the usual shift operator for Markov processes and define

$$A_l := \bigcap_{m=1}^l \left\{ \widehat{\tau}_{D_m} = \widehat{T}_{M_m}, \widehat{T}_{\partial \widehat{D}_{m-1}} \circ \theta_{\widehat{T}_{M_m}} > \widehat{\tau}_{\widehat{D}_l} \right\}.$$

Note that by the strong Markov property applied at \widehat{T}_{M_1} ,

$$\begin{aligned} & \mathbf{P}_y \left(\bigcap_{m=1}^4 \left\{ \widehat{\tau}_{\widehat{D}_m} = \widehat{T}_{M_m}, \widehat{T}_{\partial \widehat{D}_{m-1}} \circ \theta_{\widehat{T}_{M_m}} > \widehat{\tau}_{\widehat{D}_4} \right\} \right) \\ &= \mathbf{E}_y \left[\mathbf{P}_{\widehat{Y}_{\widehat{T}_{M_1}}^D} \left(\bigcap_{m=2}^4 \left\{ \widehat{\tau}_{\widehat{D}_m} = \widehat{T}_{M_m}, \widehat{T}_{\partial \widehat{D}_{m-1}} \circ \theta_{\widehat{T}_{M_m}} > \widehat{\tau}_{\widehat{D}_4} \right\} \right. \right. \\ & \quad \left. \left. \cap \left\{ \widehat{T}_{\partial \widehat{D}_0} > \widehat{\tau}_{\widehat{D}_4} \right\} : \widehat{\tau}_{\widehat{D}_1} = \widehat{T}_{M_1} \right) \right]. \end{aligned}$$

Thus by Lemma 4.5 and the strong Markov property applied at \widehat{T}_{M_m} , $m =$

1, \dots , 4, we get

$$\begin{aligned}
(4.7) \quad & \mathbf{P}_y \left(\bigcap_{m=1}^4 \left\{ \hat{\tau}_{\hat{D}_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_4} \right\} \right) \\
& \geq c_7 \inf_{x \in M_1} \mathbf{P}_x \left(\bigcap_{m=2}^4 \left\{ \hat{\tau}_{\hat{D}_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_4} \right\} \right. \\
& \quad \left. \cap \left\{ \hat{T}_{\partial \hat{D}_0} > \hat{\tau}_{\hat{D}_4} \right\} \right) \\
& = c_7 \inf_{x \in M_1} \mathbf{E}_x \left[\mathbf{P}_{\hat{Y}_{\hat{T}_{M_2}}^D} \left(\bigcap_{m=3}^4 \left\{ \hat{\tau}_{\hat{D}_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_4} \right\} \right. \right. \\
& \quad \left. \left. \cap \left\{ \hat{T}_{\partial \hat{D}_0} > \hat{\tau}_{\hat{D}_4}, \hat{T}_{\partial \hat{D}_1} > \hat{\tau}_{\hat{D}_4} \right\} \right) : \hat{\tau}_{\hat{D}_2} = \hat{T}_{M_2} \right] \\
& \geq c_7^2 \dots \\
& \geq c_7^4 \inf_{x \in M_4} \mathbf{P}_x \left(\bigcap_{m=1}^4 \left\{ \hat{T}_{\partial \hat{D}_{m-1}} > \hat{\tau}_{\hat{D}_4} \right\} \right) \\
& = c_7^4 \inf_{x \in M_4} \mathbf{P}_x \left(\bigcap_{m=1}^4 \left\{ \hat{T}_{\partial \hat{D}_{m-1}} > 0 \right\} \right) = c_7^4.
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
& \left\{ \hat{\tau}_{\hat{D}_4} \leq \hat{T}_{\partial D_4}, \hat{\tau}_{\hat{D}_4} = \hat{T}_{M_4} \right\} = \left\{ \hat{\tau}_{\hat{D}_4} \leq \hat{T}_{\partial D_4} \leq \hat{T}_{M_4} = \hat{\tau}_{\hat{D}_4} \leq \hat{\tau}_{\hat{D}_4} \right\} \\
& = \left\{ \hat{\tau}_{\hat{D}_4} = \hat{T}_{\partial D_4} = \hat{T}_{M_4} \right\} \subset \left\{ \hat{\tau}_{D_4} = \hat{T}_{M_4} \right\},
\end{aligned}$$

we have

$$\begin{aligned}
& \mathbf{P}_y \left(\bigcap_{m=1}^4 \left\{ \hat{\tau}_{\hat{D}_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_4} \right\} \right) \\
&= \mathbf{P}_y \left(\bigcap_{m=1}^4 \left\{ \hat{\tau}_{\hat{D}_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_4} \right\} \cap \left\{ \hat{\tau}_{\hat{D}_4} \leq \hat{T}_{\partial D_4} \right\} \right) \\
&\quad + \mathbf{P}_y \left(\bigcap_{m=1}^4 \left\{ \hat{\tau}_{\hat{D}_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_4} \right\} \cap \left\{ \hat{\tau}_{\hat{D}_4} > \hat{T}_{\partial D_4} \right\} \right) \\
&\leq \mathbf{P}_y \left(\bigcap_{m=1}^3 \left\{ \hat{\tau}_{\hat{D}_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_4} \right\} \cap \left\{ \hat{\tau}_{\hat{D}_4} = \hat{T}_{M_4} \right\} \right) \\
&\quad + \mathbf{P}_y \left(\hat{\tau}_{\hat{D}_4} > \hat{T}_{\partial D_4} \right) \\
&\leq \mathbf{P}_y \left(\bigcap_{m=1}^3 \left\{ \hat{\tau}_{D_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_4} \right\} \cap \left\{ \hat{\tau}_{D_4} = \hat{T}_{M_4} \right\} \right) \\
&\quad + \mathbf{P}_y \left(\hat{\tau}_{\hat{D}_4} > \hat{T}_{\partial D_4} \right) \\
&= \mathbf{P}_y(A_4) + \mathbf{P}_y \left(\hat{\tau}_{\hat{D}_4} > \hat{T}_{\partial D_4} \right) \leq \mathbf{P}_y(A_4) + 1 - p.
\end{aligned}$$

In the last inequality above, we have used (4.4). Let $p > 1 - c_7^4/2$ and combine the inequality above with (4.7), we have

$$(4.8) \quad \mathbf{P}_y(A_2) \geq \mathbf{P}_y(A_3) \geq \mathbf{P}_y(A_4) \geq c_7^4/2.$$

We claim that there exist c_8 and p_0 , which will be chosen later, such that for every $p > p_0$

$$(4.9) \quad \mathbf{P}_y(A_{l+1}) \geq c_8 \mathbf{P}_y(A_l), \quad l \geq 2.$$

We will prove this claim by induction. By (4.8), we know that the claim is valid for $l = 2, 3$. First, we note that, by Lemma 4.5 and the strong Markov property applied at $\hat{T}_{M_{l+1}}$, we get

$$\begin{aligned}
& \mathbf{P}_y \left(A_{l+1} \cap \left\{ \hat{\tau}_{\hat{D}_{l+2}} = \hat{T}_{M_{l+2}}, \hat{T}_{\partial \hat{D}_l} \circ \theta_{\hat{T}_{M_{l+1}}} > \hat{\tau}_{\hat{D}_{l+2}} \right\} \right) \\
&= \mathbf{E}_y \left[\mathbf{P}_{\hat{Y}_{\hat{T}_{M_{l+1}}}} \left(\hat{\tau}_{\hat{D}_{l+2}} = \hat{T}_{M_{l+2}}, \hat{T}_{\partial \hat{D}_l} > \hat{\tau}_{\hat{D}_{l+2}} \right) : A_{l+1} \right] \\
&\geq \inf_{x \in M_{l+1}} \mathbf{P}_x \left(\hat{\tau}_{\hat{D}_{l+2}} = \hat{T}_{M_{l+2}}, \hat{T}_{\partial \hat{D}_l} > \hat{\tau}_{\hat{D}_{l+2}} \right) \mathbf{P}_y(A_{l+1}) \geq c_7 \mathbf{P}_y(A_{l+1}).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \mathbf{P}_y \left(A_{l+1} \cap \left\{ \hat{\tau}_{\hat{D}_{l+2}} = \hat{T}_{M_{l+2}}, \hat{T}_{\partial \hat{D}_l} \circ \theta_{\hat{T}_{M_{l+1}}} > \hat{\tau}_{\hat{D}_{l+2}} \right\} \right) \\
&= \mathbf{P}_y \left(A_{l+1} \cap \left\{ \hat{\tau}_{D_{l+2}} = \hat{T}_{M_{l+2}} = \hat{\tau}_{\hat{D}_{l+2}}, \hat{T}_{\partial \hat{D}_l} \circ \theta_{\hat{T}_{M_{l+1}}} > \hat{\tau}_{\hat{D}_{l+2}} \right\} \right) \\
&\quad + \mathbf{P}_y \left(A_{l+1} \cap \left\{ \hat{\tau}_{D_{l+2}} \neq \hat{T}_{M_{l+2}}, \hat{\tau}_{\hat{D}_{l+2}} = \hat{T}_{M_{l+2}}, \hat{T}_{\partial \hat{D}_l} \circ \theta_{\hat{T}_{M_{l+1}}} > \hat{\tau}_{\hat{D}_{l+2}} \right\} \right) \\
&= \mathbf{P}_y \left(\bigcap_{m=1}^{l+1} \left\{ \hat{\tau}_{D_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_{l+1}} \right\} \right. \\
&\quad \left. \cap \left\{ \hat{\tau}_{D_{l+2}} = \hat{T}_{M_{l+2}} = \hat{\tau}_{\hat{D}_{l+2}}, \hat{T}_{\partial \hat{D}_l} \circ \theta_{\hat{T}_{M_{l+1}}} > \hat{\tau}_{\hat{D}_{l+2}} \right\} \right) \\
&\quad + \mathbf{P}_y \left(\bigcap_{m=1}^{l-1} \left\{ \hat{\tau}_{D_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_{l+1}} \right\} \right. \\
&\quad \left. \cap \left\{ \hat{\tau}_{D_l} = \hat{T}_{M_l}, \hat{T}_{\partial \hat{D}_{l-1}} \circ \theta_{\hat{T}_{M_l}} > \hat{\tau}_{\hat{D}_{l+1}}, \hat{\tau}_{D_{l+1}} = \hat{T}_{M_{l+1}}, \right. \right. \\
&\quad \quad \left. \left. \hat{T}_{\partial \hat{D}_l} \circ \theta_{\hat{T}_{M_{l+1}}} > \hat{\tau}_{\hat{D}_{l+2}}, \hat{\tau}_{D_{l+2}} \neq \hat{T}_{M_{l+2}}, \hat{\tau}_{\hat{D}_{l+2}} = \hat{T}_{M_{l+2}} \right\} \right) \\
&\leq \mathbf{P}_y \left(\bigcap_{m=1}^{l+1} \left\{ \hat{\tau}_{D_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_{l+2}} \right\} \right. \\
&\quad \left. \cap \left\{ \hat{\tau}_{D_{l+2}} = \hat{T}_{M_{l+2}} = \hat{\tau}_{\hat{D}_{l+2}} \right\} \right) \\
&\quad + \mathbf{P}_y \left(\bigcap_{m=1}^{l-1} \left\{ \hat{\tau}_{D_m} = \hat{T}_{M_m}, \hat{T}_{\partial \hat{D}_{m-1}} \circ \theta_{\hat{T}_{M_m}} > \hat{\tau}_{\hat{D}_{l-1}} \right\} \right. \\
&\quad \left. \cap \left\{ \hat{\tau}_{D_l} = \hat{T}_{M_l}, \hat{T}_{\partial \hat{D}_{l-1}} \circ \theta_{\hat{T}_{M_l}} > \hat{\tau}_{\hat{D}_{l+1}}, \hat{\tau}_{D_{l+1}} = \hat{T}_{M_{l+1}}, \right. \right. \\
&\quad \quad \left. \left. \hat{T}_{\partial \hat{D}_l} \circ \theta_{\hat{T}_{M_{l+1}}} > \hat{\tau}_{\hat{D}_{l+2}}, \hat{\tau}_{D_{l+2}} < \hat{\tau}_{\hat{D}_{l+2}} = \hat{T}_{M_{l+2}} \right\} \right) \\
&\leq \mathbf{P}_y(A_{l+2}) + \mathbf{P}_y \left(A_{l-1} \cap \left\{ \hat{T}_{V_{l+2}} \circ \theta_{\hat{T}_{M_{l-1}}} < \hat{\tau}_{W_{l+3}} \circ \theta_{\hat{T}_{M_{l-1}}} \right\} \right),
\end{aligned}$$

which is less than equal to $\mathbf{P}_y(A_{l+2}) + c_6(1-p)\mathbf{P}_y(A_{l-1})$ by (4.6). Combining the two inequalities above, we get by induction

$$\begin{aligned}
\mathbf{P}_y(A_{l+2}) &\geq c_7 \mathbf{P}_y(A_{l+1}) - c_6(1-p)\mathbf{P}_y(A_{l-1}) \\
&\geq c_7 \mathbf{P}_y(A_{l+1}) - c_6(1-p)c_8^{-2} \mathbf{P}_y(A_{l+1}) \\
&= (c_7 - c_6(1-p)c_8^{-2}) \mathbf{P}_y(A_{l+1}).
\end{aligned}$$

Choose $c_8 < c_7^4/2$ small then choose $p_0 < 1$ large so that for every $p \in [p_0, 1)$,

$$c_7 - c_6(1-p)c_8^2 > c_8.$$

Thus the claim (4.9) is valid. Recall that $l_0 := l_0^{j,k}(y)$ is the smallest integer greater than $(a_0/2 - y_d)/a$. From (4.8) and (4.9), we conclude that

$$\begin{aligned} \mathbf{P}_y(\widehat{T}_{M^{j,k}(y)} < \widehat{\tau}_{D_1^{j,k}(y)}) &\geq \mathbf{P}_y(\widehat{\tau}_{D_{l_0}} = \widehat{T}_{M_{l_0}}) \geq \mathbf{P}_y(A_{l_0}) \\ &\geq c_8^{l_0-2} \mathbf{P}_y(A_2) \geq \frac{c_7^4}{2} c_8^{l_0-2} \geq \exp\left(-c_9 \frac{8(a_0 - y_d)|k|^\Theta}{b_j}\right) \end{aligned}$$

for some positive constant c_9 . \square

For any positive function h which is harmonic in D for either Y or \widehat{Y} , we let $S_k := \{x \in D : h(x) \leq 2^{k+1}\}$

LEMMA 4.7. *Suppose that D is one of the following types of bounded domains:*

- (a) a twisted Hölder domain of order $\alpha \in (1/3, 1]$ or
- (b) a uniformly Hölder domain of order $\alpha \in (0, 2)$ or
- (c) a domain which can be locally represented as the region above the graph of a function.

Then for any positive bounded function h which is harmonic in D for Y (\widehat{Y} , respectively), there exist $c > 0$ and $\beta > 0$ such that

$$(4.10) \quad \sup_{x \in D} \mathbf{E}_x[\tau_{S_k}] \leq c|k|^{-1-\beta} \quad \left(\sup_{x \in D} \mathbf{E}_x[\widehat{\tau}_{S_k}] \leq c|k|^{-1-\beta}, \quad \text{respectively} \right).$$

PROOF. Note that, by (2.4) and (2.6), we have

$$\widehat{G}_D(x, y) = \frac{G_D(y, x)H(y)}{H(x)} \leq c|x - y|^{-d+2},$$

which implies that

$$\sup_{x \in D} \mathbf{E}_x[\widehat{\tau}_{S_k}] \leq \sup_{x \in D} \mathbf{E}_x[\widehat{\tau}_D] \leq c_1 \sup_{x \in D} \int_D |x - y|^{-d+2} dy < \infty.$$

Thus, we only need to show (4.10) for negative k with $|k|$ large.

(i) Assume D is twisted Hölder domain of order $\alpha \in (1/3, 1)$. Recall that z_0 is the point from the Definition 4.1 (2). By Lemma 3.1 in [2], there exists $c_1 = c_1(D) > 0$ such that for every $x \in D$ there exists a sequence of

open balls contained in D , with centers $z^1 = x, z^2, \dots, z^k = z_0$ and radii $a_j \leq \text{dist}(z^j, \partial D)$, such that $|z^j - z^{j+1}| < (a_j \wedge a_{j+1})/2$ and $k \leq c_1 \delta(x)^{1-1/\alpha}$. Thus, by the Harnack inequality (Theorem 3.9), there exists $c_2 = c_2(z_0) > 0$ such that

$$(4.11) \quad h(x) \geq \exp(-c_2 \delta(x)^{1-1/\alpha}).$$

If $x \in S_k$, from (4.11) we have

$$2^{k+1} \geq h(x) \geq \exp(-c_2 \delta(x)^{1-1/\alpha}),$$

which implies that there exists $c_3 > 0$ such that

$$\delta(x) \leq c_3 |k|^{-\frac{\alpha}{1-\alpha}}.$$

Therefore $S_k \subset F(a)$ with $a \leq c_3 |k|^{-\frac{\alpha}{1-\alpha}}$. We consider negative k with $|k|$ large enough such that

$$c_3 |k|^{-\frac{\alpha}{1-\alpha}} \leq a_1 \quad \text{and} \quad 2c_3 b_1 |k|^{-\frac{\alpha}{1-\alpha}} \leq |k|^{-\frac{\alpha+1}{4(1-\alpha)}}$$

where a_1 and b_1 are the constant in Lemma 4.3. Note that the above is always possible because $\frac{1}{4}(\alpha+1) < \alpha$. For those k , we apply Lemma 4.3 and get

$$\begin{aligned} & \mathbf{P}_x \left(\widehat{\tau}_{S_k} < \widehat{\tau}_{B(x, |k|^{-\frac{\alpha+1}{4(1-\alpha)}})} \right) \\ & \geq \mathbf{P}_x \left(\widehat{\tau}_{S_k} < \widehat{\tau}_{B(x, 2c_3 b_1 |k|^{-\frac{\alpha}{1-\alpha}})} \right) \\ & \geq \mathbf{P}_x \left(\widehat{\tau}_{F(c_3 |k|^{-\frac{\alpha}{1-\alpha}})} < \widehat{\tau}_{B(x, 2c_3 b_1 |k|^{-\frac{\alpha}{1-\alpha}})} \right) \\ & \geq \mathbf{P}_x \left(\widehat{T}_{F(c_3 |k|^{-\frac{\alpha}{1-\alpha}})^c \cap B(x, c_3 b_1 |k|^{-\frac{\alpha}{1-\alpha}})} < \widehat{\tau}_{B(x, 2c_3 b_1 |k|^{-\frac{\alpha}{1-\alpha}})} \right) \geq c_4 \end{aligned}$$

for some $c_4 > 0$. Thus by Lemma 2.10, we have

$$\mathbf{E}_x[\widehat{\tau}_{S_k}] \leq c_5 |k|^{-\frac{\alpha+1}{2(1-\alpha)}} = c_5 |k|^{-1-\beta},$$

where $\beta = (3\alpha - 1)/(2 - 2\alpha) > 0$.

(ii) Assume that D is a John domain (i.e., a twisted Hölder domain of order $\alpha = 1$). It is well-known that $k_D(x, z_0) \leq -c_6 \ln \rho_D(x) + c_7$ for some positive constants c_6, c_7 (for example, see page 185 in [1]). It is easy to see that the shortest length of a Harnack chain connecting x and z_1 is comparable to $k_D(x, z_0)$. Thus, by our Harnack inequality (Theorem 3.9)

$$h(x) \geq \exp(-c_8 k_D(x, z_1)) \geq c_9 \rho_D(x)^{c_{10}} \geq c_{11} \delta(x)^{c_{10}},$$

for some positive constants c_8, c_9, c_{10} . Using the above instead (4.11), we can repeat the argument in (i) to arrive at the desired conclusion. We omit the details.

(iii) Now we assume that D is a uniformly Hölder domain of order $\alpha \in (0, 2)$. Recall that z_1 is the point from the Definition 4.2. Since the shortest length of a Harnack chain connecting x and z_1 is comparable to $k_D(x, z_1)$, by the Harnack inequality (Theorem 3.9) and the Definition 4.2 (1), there exists $c_{11} = c_{11}(z_1) > 0$ such that

$$(4.12) \quad h(x) \geq \exp(-ck_D(x, z_1)) \geq \exp(-c_{11}\rho_D(x)^{-\alpha}).$$

If $x \in S_k$, from (4.12) we have

$$2^{k+1} \geq h(x) \geq \exp(-c_{11}\rho_D(x)^{-\alpha}),$$

which implies that there exists $c_{12} > 0$ such that

$$\rho_D(x) \leq c_{12} |k|^{-\frac{1}{\alpha}}.$$

Therefore $S_k \subset D(a) := \{x \in D : \rho_D(x) < a\}$ with $a \leq c_{12}|k|^{-\frac{1}{\alpha}}$. For each $x \in S_k$, choose a point $Q_x \in \partial D$ such that

$$|Q_x - x| = \frac{3c_{12}}{2} |k|^{-\frac{1}{\alpha}}.$$

We consider negative k with $|k|$ large enough such that

$$c_{12}|k|^{-\frac{1}{\alpha}} \leq a_2 \quad \text{and} \quad \frac{7}{2}c_{12}|k|^{-\frac{1}{\alpha}} \leq |k|^{-\frac{\alpha+2}{4\alpha}}$$

where a_2 is the constant in Lemma 4.3 (2). Note that the above is always possible because $\frac{1}{4}(\alpha + 2) < 1$. Also we note that for those negative k 's

$$B(Q_x, 2c_{12}|k|^{-\frac{1}{\alpha}}) \subset B(x, |k|^{-\frac{\alpha+2}{4\alpha}}).$$

For those negative k 's, we apply Lemma 4.3 and get

$$\begin{aligned} & \mathbf{P}_x \left(\widehat{\tau}_{S_k} < \widehat{\tau}_{B(x, |k|^{-\frac{\alpha+2}{4\alpha}})} \right) \\ & \geq \mathbf{P}_x \left(\widehat{\tau}_{S_k} < \widehat{\tau}_{B(Q_x, 2c_{12}|k|^{-\frac{1}{\alpha}})} \right) \\ & \geq \mathbf{P}_x \left(\widehat{\tau}_{D(c_{12}|k|^{-\frac{1}{\alpha}})} < \widehat{\tau}_{B(Q_x, 2c_{12}|k|^{-\frac{1}{\alpha}})} \right) \\ & = \mathbf{P}_x \left(\widehat{\tau}_{D(c_{12}|k|^{-\frac{1}{\alpha}}) \cap B(Q_x, 2c_{12}|k|^{-\frac{1}{\alpha}})} < \widehat{\tau}_{B(Q_x, 2c_{12}|k|^{-\frac{1}{\alpha}})} \right) \\ & \geq \mathbf{P}_x \left(\widehat{T}_{D^c \cap B(Q_x, c_{12}|k|^{-\frac{1}{\alpha}})} < \widehat{\tau}_{B(Q_x, 2c_{12}|k|^{-\frac{1}{\alpha}})} \right) \geq c_4 \end{aligned}$$

for some constant $c_{13} > 0$. Thus by Lemma 2.10, we have

$$\mathbf{E}_x[\widehat{\tau}_{S_k}] \leq c_{14} |k|^{-\frac{\alpha+2}{2\alpha}} = c_{14} |k|^{-1-\beta},$$

for some constant $c_{14} > 0$, where $\beta = \frac{1}{2}(2 - \alpha)/\alpha > 0$.

(iv) Finally, we assume that D is a bounded domain which can be locally represented as the region above the graph of a function. Without loss of generality, we may assume that $\max_{1 \leq i \leq m_0} f_i < -\varepsilon$ for some positive $\varepsilon = \varepsilon(D)$ so that

$$\bigcup_{i=1}^{m_0} \{(\tilde{x}, x_d) \text{ in } CS_i : |\tilde{x}| < b_i, 0 \leq x_d \leq a_0\}$$

is a compact subset of D . Thus by the continuity of h , there exists $k_0 > 0$ such that $h(x) \geq 2^{-k_0+1}$ for $x \in K$. We let

$$k_1 := k_0 \vee \max_{1 \leq i \leq m_0} \left(\frac{b_i}{r_1} \right)^{\frac{1}{\Theta}} \quad \text{where } \Theta = \frac{1}{2} \left(1 + \frac{1}{4d-2} \right).$$

Fix j and f_j , and consider $y \in \{(\tilde{x}, x_d) \text{ in } CS_j : |\tilde{x}| < b_j, f_j(\tilde{x}) < x_d < 0\}$. Recall that p_0 is the constant in Lemma 4.6.

We claim that there exists $p_1 \in (p_0, 1)$ and $k_2 \geq k_1$ such that for every

$$y \in S_k \cap \{(\tilde{x}, x_d) \text{ in } CS_j : |\tilde{x}| < b_j, f_j(\tilde{x}) < x_d < 0\}$$

and $k < -k_2$, we have

$$\mathbf{P}_y \left(\widehat{T}_{\partial D} < \widehat{\tau}_{B(y, b_j |k|^{-\Theta})} \right) > 1 - p_1.$$

Recall that $D_1^{j,k}(y)$, $D_2^{j,k}(y)$ and $M^{j,k}(y)$ are defined in (4.1)-(4.3). Suppose that

$$\mathbf{P}_y \left(\widehat{T}_{\partial D} < \widehat{\tau}_{B(y, b_j |k|^{-\Theta})} \right) \leq 1 - p,$$

then

$$\mathbf{P}_y \left(\widehat{T}_{D^c \cap B(y, \frac{1}{2} b_j |k|^{-\Theta})} < \widehat{\tau}_{B(y, b_j |k|^{-\Theta})} \right) \leq \mathbf{P}_y \left(\widehat{T}_{\partial D} < \widehat{\tau}_{B(y, b_j |k|^{-\Theta})} \right) \leq 1 - p.$$

Since $b_j |k|^{-\Theta} \leq r_1$, by Lemma 2.8 with $K := \overline{B(0, 1/2)}$, we have

$$\begin{aligned} & c_{15}^{-1} (b_j |k|^{-\Theta})^{2-d} \left(\inf_{w \in K} G_{B(0,1)}^0(0, w) \right) \text{Cap}^{B(y, b_j |k|^{-\Theta})}(D^c \cap \overline{B(y, b_j |k|^{-\Theta}/2)}) \\ & \leq \mathbf{P}_y \left(\widehat{T}_{D^c \cap B(y, \frac{1}{2} b_j |k|^{-\Theta})} < \widehat{\tau}_{B(y, b_j |k|^{-\Theta})} \right) \leq 1 - p. \end{aligned}$$

Thus

$$(4.13) \quad \text{Cap}^{B(y, b_j |k|^{-\Theta})} \left(D^c \cap \overline{B(y, b_j |k|^{-\Theta}/2)} \right) \leq c_{16} (1-p) b_j^{d-2} |k|^{-(d-2)\Theta}.$$

Using the facts that $D^c \cap D_2^{j,k}(y) \subset D^c \cap B(y, \frac{1}{2} b_j |k|^{-\Theta})$ and

$$|A \cap \overline{B(z, r/2)}|^{\frac{d-2}{d}} \leq c_{17} \text{Cap}^{B(z, r)}(A \cap \overline{B(z, r/2)}), \quad z \in \mathbf{R}^d,$$

we have from (4.13) that

$$\begin{aligned} |D^c \cap D_2^{j,k}(y)| &\leq |D^c \cap B(y, \frac{1}{2} b_j |k|^{-\Theta})| \\ &\leq c_{18} \left(\text{Cap}^{B(y, b_j |k|^{-\Theta})} \left(D^c \cap \overline{B(y, b_j |k|^{-\Theta}/2)} \right) \right)^{\frac{d}{d-2}} \\ &\leq c_{19} (1-p)^{\frac{d}{d-2}} b_j^d |k|^{-d\Theta}. \end{aligned}$$

Choose $p_1 \in (p_0, 1)$ and let $c_{20} := c_{19} (1-p_1)^{\frac{d}{d-2}} \max_{1 \leq i \leq m_0} b_i^d$ be such that

$$|D^c \cap D_2^{j,k}(y)| \leq c_{20} |k|^{-d\Theta} = \frac{1}{2} |D_2^{j,k}(y)|,$$

then

$$|D \cap D_2^{j,k}(y)| > c_{20} |k|^{-d\Theta} / 2.$$

Note that, since D is bounded, D is an L^d -domain (a domain which can be locally represented as the region above the graph of an L^d function). Now we can follow the proof of Lemma 2.6 (with $p = d$ and $\Theta = r$ there) on the second half of page 417 in [2] (after rescaling) to get

$$(4.14) \quad (a_0 - y_d) / |k|^{-\Theta} \leq c_{21} |k|^{\Theta(d-1)/d} |k|^\Theta = c_{21} |k|^{1-1/(4d)}.$$

Since $p_1 \in (p_0, 1)$, by Lemma 4.6,

$$(4.15) \quad \mathbf{P}_y(\widehat{T}_{M^{j,k}(y)} < \widehat{\tau}_{D_1^{j,k}(y)}) \geq \exp\left(-c_{22} \frac{8(a_0 - y_d) |k|^\Theta}{b_j}\right).$$

Using our (4.14)-(4.15) instead of (2.10)-(2.11) in [2], we can follow the argument in the proof of Lemma 2.6 after (2.11) in [2] (after rescaling) to conclude that $y \notin S_k$ if $-k$ is sufficiently large. Thus we have proved the claim by a contradiction. Moreover,

$$\mathbf{P}_y\left(\widehat{\tau}_{S_k} < \widehat{\tau}_{B(y, b_j |k|^{-\Theta})}\right) \geq \mathbf{P}_y\left(\widehat{\tau}_D < \widehat{\tau}_{B(y, b_j |k|^{-\Theta})}\right) > 1 - p_1, \quad y \in S_k.$$

Thus by Lemma 2.10, we have

$$\mathbf{E}_y[\widehat{\tau}_{S_k}] \leq c_{23} \left(\max_{1 \leq i \leq m_0} b_i \right)^{-\Theta} |k|^{-2\Theta} = c_{24} |k|^{-1-\beta},$$

where $\beta = 1/(4d - 2) > 0$. □

5. Parabolic boundary Harnack principle and Intrinsic Ultracontractivity. Throughout this section, we will assume that D is one of the following types of bounded domains:

- (a) a twisted Hölder domain of order $\alpha \in (1/3, 1]$ or
- (b) a uniformly Hölder domain of order $\alpha \in (0, 2)$ or
- (c) a bounded domain which can be locally represented as the region above the graph of a function.

Recall that t_1 is the constant from (3.1) and $\widehat{\tau}_B = \inf\{t > 0 : \widehat{Y}_t \notin B\}$. For any $\delta > 0$, we put $D_\delta := \{x \in D : \rho_D(x) < \delta\}$.

LEMMA 5.1. *There exist constants $c, R_1 > 0$ and a point x_1 in D such that $B_1 := B(x_1, \frac{1}{2}R_1) \subset D \setminus D_{\frac{1}{4}R_1}$ and for every $R \leq R_1$, $r^D(t, x, y) \wedge \widehat{r}^D(t, x, y) \geq cR^{-d}$ for all $x, y \in B(x_1, \frac{1}{2}R)$ and $\frac{1}{3}t_1R^2 \leq t \leq t_1R^2$.*

PROOF. Choose $R_1 = R_1(D) \leq \sqrt{t_0}$ and $x_1 \in D$ such that $B(x_1, R_1) \subset D$. Then we apply Lemma 3.1 with $\delta = \frac{1}{3}$ and use the monotonicity of the density to get the desired assertion. \square

We fix x_1, R_1 and B_1 in the lemma above for the remainder of this section. Let $h_1(x) := G_D(x, x_1)$ and $h_2(x) := \widehat{G}_D(x, x_1)$. h_1 and h_2 are regular harmonic for Y and \widehat{Y} in $D \setminus B_1$ respectively. Moreover, by (2.4) and (2.6), h_1 and h_2 are bounded by 2^{k_0+1} for some $k_0 = k_0(R_1)$ on $D \setminus B_1$. Let (\mathbf{P}_x^h, Y_t^D) and $(\mathbf{P}_x^h, \widehat{Y}_t^D)$ be the h -transform of (\mathbf{P}_x, Y_t^D) and $(\mathbf{P}_x, \widehat{Y}_t^D)$ respectively.

LEMMA 5.2. *For every $s > 0$, there exists a positive constant $\delta_0 = \delta_0(s) \leq \frac{1}{4}R_1$ such that*

$$\left(\inf_{x \in D} \mathbf{P}_x^{h_1}(T_{D \setminus D_\delta} < \frac{s}{4}) \right) \wedge \left(\inf_{x \in D} \mathbf{P}_x^{h_2}(\widehat{T}_{D \setminus D_\delta} < \frac{s}{4}) \right) \geq \frac{1}{2}.$$

PROOF. For $k \leq k_0$, let

$$V_k^\delta := \{x \in D_\delta : h_2(x) \leq 2^{k+1}\}, \quad U_k := \{x \in D \setminus B_1 : h_2(x) \leq 2^{k+1}\}.$$

Clearly, $V_k^\delta \subset U_k$ for $\delta \leq \frac{1}{4}R_1$. For each k , by (2.4) and (2.6), we have

$$\sup_{x \in D} \mathbf{E}_x[\widehat{\tau}_{V_k^\delta}] \leq c \sup_{x \in D} \int_{V_k^\delta} \frac{dy}{|x-y|^{d-2}}$$

for some $c > 0$. So $\sup_{x \in D} \mathbf{E}_x[\widehat{\tau}_{V_k^\delta}]$ goes to zero as $\delta \rightarrow 0$ by the uniform integrability of $|x-y|^{-d+2}$ over D . Note that $D \setminus B_1$ is also one of the types

of domains we assumed at the beginning of this section. So by Lemma 4.7,

$$\sum_{k=-\infty}^{k_0} \sup_{x \in D} \mathbf{E}_x[\widehat{\tau}_{U_k}] < \infty.$$

Thus by the dominated convergence theorem, we have

$$(5.1) \quad \lim_{\delta \downarrow 0} \sum_{k=-\infty}^{k_0} \sup_{x \in D} \mathbf{E}_x[\widehat{\tau}_{V_k^\delta}] = 0.$$

On the other hand, since 1 is excessive for \widehat{Y}^D , it is easy to see that $(1/h_2(\widehat{Y}^D), \mathcal{F}_t)$ is a super-martingale with respect to $\mathbf{P}_x^{h_2}$ where \mathcal{F}_t is the natural filtration of $\{\widehat{Y}^D\}$ (For example, see page 83 in [14]). Thus with the same proof, one can see that the first inequality in equation (8) on page 179 of [8] is true. Thus there exists c_1 independent of h_2 and δ such that

$$(5.2) \quad \sup_{x \in D} \mathbf{E}_x^{h_2}[\widehat{\tau}_{D_\delta}] \leq c_1 \sum_{k=-\infty}^{k_0} \sup_{x \in D} \mathbf{E}_x[\widehat{\tau}_{V_k^\delta}].$$

Combining (5.1)-(5.2), we have that for each $s > 0$, there exists $\delta > 0$ such that $\sup_{x \in D} \mathbf{E}_x^{h_2}[\widehat{\tau}_{D_\delta}] < s/8$. We can now apply Chebyshev's inequality to get

$$\mathbf{P}_x^{h_2}(\widehat{\tau}_{D_\delta} < \frac{s}{4}) \geq \frac{1}{2}.$$

On the other hand, using (2.2), (2.4) and (2.6), it is elementary to show that the strictly positive function $\widehat{G}_D(x, y)$ is ∞ if and only if $x = y \in D$, and for every $x \in D$, $\widehat{G}_D(x, \cdot)$ and $\widehat{G}_D(\cdot, x)$ are extended continuous in D (see the proof in Theorem 2.6 in [16]). Thus the condition (H) in [22] holds. Also the strict positivity of $\widehat{G}_D(x, y)$ and Proposition 2.6 imply that the set W on page 5 in [22] and the set Z defined in [9] (equation (12) on page 179) are empty. Thus by Theorem 2 in [22], for every $x \neq x_1$, the lifetime $\widehat{\zeta}^{h_2}$ of \widehat{Y}^D is finite $\mathbf{P}_x^{h_2}$ -a.s. and

$$(5.3) \quad \lim_{t \uparrow \widehat{\zeta}^{h_2}} \widehat{Y}_t^D = x_1 \quad \mathbf{P}_x^{h_2}\text{-a.s.}$$

Thus for $x \in D_\delta$, the conditioned process \widehat{Y}^D with respect to $\mathbf{P}_x^{h_2}$ can not be killed before hitting $D \setminus D_\delta$ due to the continuity of \widehat{Y}^D . Therefore we have

$$\mathbf{P}_x^{h_2}(\widehat{T}_{D \setminus D_\delta} < \frac{s}{4}) = \mathbf{P}_x^{h_2}(\widehat{\tau}_{D_\delta} < \frac{s}{4}) \geq \frac{1}{2}.$$

□

For a parabolic function $g(t, x)$ in $\Omega = (T_1, T_2] \times D$ for Y (\widehat{Y} , respectively), let $(\mathbf{P}_{t,x}^g, Z_s^\Omega)$ ($(\mathbf{P}_{t,x}^g, \widehat{Z}_s^\Omega)$, respectively) be the killed space-time process $(\mathbf{P}_{t,x}, Z_s^\Omega)$ ($(\mathbf{P}_{t,x}, \widehat{Z}_s^\Omega)$, respectively) conditioned by g . For each $u > 0$, we let

$$W_k = W_k(u) := \left\{ (s, y) \in [u/2, u] \times D : 2^k \leq g(s, x) \leq 2^{k+1} \right\}$$

and

$$W^n = W^n(u) := \bigcup_{k=-\infty}^n W_k.$$

LEMMA 5.3. *For every $M > 0$ and $u > 0$, there exists $k_1 = k_1(M, u, h_1, h_2, B_1) < -3$ such that for every positive parabolic function $g(t, x)$ in $(u/2, u] \times D$ for Y (and \widehat{Y} , respectively),*

$$g(s, x) \geq Mh_1(x) \quad (g(s, x) \geq Mh_2(x), \text{ respectively}), \quad (s, x) \in [u/2, u] \times (D \setminus B_1)$$

implies

$$\mathbf{E}_{u,x}^g [\tau_{W^{k_1}}] \leq \frac{u}{8} \quad \left(\mathbf{E}_{u,x}^g [\widehat{\tau}_{W^{k_1}}] \leq \frac{u}{8}, \text{ respectively} \right), \quad x \in D$$

where $\tau_{W^{k_1}} = \inf\{t > 0 : Z_t \notin W^{k_1}\}$ and $\widehat{\tau}_{W^{k_1}} = \inf\{t > 0 : \widehat{Z}_t \notin W^{k_1}\}$.

PROOF. Let m_1 be the smallest integer greater than $\log_2 M$ and $U_k := \{x \in D \setminus B_1 : h_2(x) \leq 2^{k+1}\}$ so that $W_k \subset U_{k+m_1} \times [u/2, u]$ for small k . By Lemma 4.7, we get for small n

$$(5.4) \quad \sum_{k=-\infty}^n \sup_{(s,y) \in W_k} \mathbf{E}_{s,y} [\widehat{\tau}_{W_k}] \leq \sum_{k=-\infty}^n \sup_{(s,y) \in U_{k+m_1}} \mathbf{E}_{s,y} [\widehat{\tau}_{U_{k+m_1}}] < \infty.$$

Similar to the argument in the proof of the previous lemma, using the estimates in [8], there exists c_1 independent of g , n and u such that

$$(5.5) \quad \sup_{y \in D} \mathbf{E}_{u,y}^g [\widehat{\tau}_{W^n}] \leq c_1 \sum_{k=-\infty}^n \sup_{(s,y) \in W_k} \mathbf{E}_{s,y} [\widehat{\tau}_{W_k}].$$

Combining (5.4)-(5.5), we have that for small n ,

$$\sup_{x \in D} \mathbf{E}_{u,x}^g [\widehat{\tau}_{W^n}] < \infty.$$

Now choose $k_1 = k_1(u) < 0$ small so that

$$\sup_{x \in D} \mathbf{E}_{u,x}^g [\widehat{\tau}_{W^{k_1}}] < \frac{u}{8}.$$

□

The idea of the proof of the next lemma comes from the proof of Lemma 5.1 in [2]. We spell out the details for the reader's convenience.

LEMMA 5.4. *For every $u \in (0, \frac{1}{2}t_1R_1^2)$ there exists $c > 0$ such that for all $x \in D$,*

$$\mathbf{P}_x(Y_u \in B_1, \tau_D > u) \geq c \mathbf{P}_x(\tau_D > u)$$

and

$$\mathbf{P}_x(\widehat{Y}_u \in B_1, \widehat{\tau}_D > u) \geq c \mathbf{P}_x(\widehat{\tau}_D > u).$$

PROOF. In this proof, for $A \subset [0, \infty) \times V$, \widehat{T}_A will denote the first hitting time of A for \widehat{Z}_s .

We fix $u \leq \frac{1}{2}t_1R_1^2$ and let $\delta_0 = \delta_0(u) \leq \frac{1}{4}R_1$ be the constant from Lemma 5.2 and let $D_2 := D_{\delta_0}$. Note that $B_1 \subset D \setminus D_2$. Let $f_\varepsilon(x) = \varepsilon$ on $D \setminus B_1$ and 1 on B_1 . Define a parabolic function g_ε on $(0, \infty) \times D$ by

$$g_\varepsilon(t, x) := \int_D \widehat{r}^D(t, x, y) f_\varepsilon(y) dy = \mathbf{E}_x[f_\varepsilon(\widehat{Y}_t^D) : \widehat{Y}_t^D \in D], \quad 0 < \varepsilon < 1.$$

Clearly,

$$(5.6) \quad \varepsilon \mathbf{P}_x(\widehat{\tau}_D > t) \leq g_\varepsilon(t, x) \leq \mathbf{P}_x(\widehat{\tau}_D > t).$$

We claim that there exists $c_1 > 0$ independent of ε such that

$$(5.7) \quad g_\varepsilon(t, x) \geq c_1 h_2(x), \quad (x, t) \in (D \setminus B_1) \times [u/2, u].$$

First, we note that, since $2u \leq t_1R_1^2$, by Theorem 3.8 and a chain argument, we get

$$(5.8) \quad \inf_{(t,x) \in [u/4, u] \times (D \setminus D_2)} g_\varepsilon(t, x) \geq c_1 g_\varepsilon(u/8, x_1) \geq c_1 \int_{B_1} \widehat{r}^D(u/8, x_1, y) dy = c_2$$

for some $c_2 > 0$. Let $h(t, x) := h_2(x)$ for $(t, x) \in [u/4, u] \times (D \setminus B_1)$. Since $h(t, x) \leq 2^{k_0+1}$, by (5.8), we have

$$g_\varepsilon(t, x) \geq c_2 2^{-k_0-1} h(t, x), \quad (t, x) \in [u/4, u] \times (D \setminus (D_2 \cup B_1)).$$

Let $\Omega := (0, \infty) \times D$. For $(s, x) \in [u/2, u] \times D_2$,

$$\begin{aligned} g_\varepsilon(s, x) &\geq \mathbf{E}_{s,x} \left[g_\varepsilon(\widehat{Z}_{\widehat{T}_{(0,\infty) \times (D \setminus D_2)}^\Omega}^\Omega) : \widehat{T}_{(0,\infty) \times (D \setminus D_2)} \leq u/4 \right] \\ &\geq c_2 2^{-k_0-1} h(s, x) \mathbf{P}_{s,x}^h \left(\widehat{T}_{(0,\infty) \times (D \setminus D_2)} \leq u/4 \right) \\ &= c_2 2^{-k_0-1} h_2(x) \mathbf{P}_{s,x}^h \left(\widehat{T}_{(0,\infty) \times (D \setminus D_2)} \leq u/4 \right) \\ &= c_2 2^{-k_0-1} h_2(x) \mathbf{P}_x^{h_2} \left(\widehat{T}_{D \setminus D_2} \leq u/4 \right), \end{aligned}$$

which is greater than equal to $c_2 2^{-k_0-2} h_2(x)$ by Lemma 5.2. The claim is proved.

Now we apply Lemma 5.3 to $g_\varepsilon(s, x)$ and get

$$(5.9) \quad \mathbf{E}_{u,x}^{g_\varepsilon} [\widehat{\tau}_{W^{k_1}}(\varepsilon)] \leq \frac{u}{8}, \quad x \in D.$$

Let $\varepsilon_1 := 2^{k_1-1} < \frac{1}{4}$, $g(s, x) := g_{\varepsilon_1}(s, x)$ and

$$E := W^{k_1} = \{(s, x) \in [u/2, u] \times D : g(s, x) \leq 4\varepsilon_1\}.$$

By Chebyshev's inequality, from (5.9) we get

$$(5.10) \quad \mathbf{P}_{u,x}^g \left(\widehat{\tau}_E \leq \frac{u}{4} \right) \geq \frac{1}{2}, \quad x \in D.$$

Let S_1 be the first hitting time of $\partial(D \times [0, \infty))$ of \widehat{Z} . The conditioned process $(\mathbf{P}_{t,x}^g, \widehat{Z}^\Omega)$ can not be killed before time t . In fact,

$$\begin{aligned} \mathbf{P}_{t,x}^g(\widehat{Z}_{S_1-}^\Omega \in \{0\} \times D) &= \mathbf{E}_{t,x} \left[\frac{g(\widehat{Z}_{S_1-}^\Omega)}{g(t, x)} : Z_{S_1-}^\Omega \in \{0\} \times D \right] \\ &= \mathbf{E}_x \left[\frac{g(0, \widehat{Y}_t^D)}{g(t, x)} : \widehat{Y}_t^D \in D \right] \\ &= \frac{1}{g(t, x)} \mathbf{E}_x \left[f_{\varepsilon_1}(\widehat{Y}_t^D) : \widehat{Y}_t^D \in D \right] = 1. \end{aligned}$$

Thus we get

$$(5.11) \quad \mathbf{P}_{u,x}^{g_\varepsilon} \left(\widehat{T}_{\partial_1 E} \leq \frac{u}{4} \right) = \mathbf{P}_{u,x}^{g_\varepsilon} \left(\widehat{\tau}_E \leq \frac{u}{4} \right) \geq \frac{1}{2}, \quad x \in D,$$

where $\partial_1 E := \partial E \cap ((0, \infty) \times D)$.

Note that by (5.6)

$$(5.12) \quad \begin{aligned} \mathbf{P}_x(\widehat{Y}_u \in B_1, \widehat{\tau}_D > u) / \mathbf{P}_x(\widehat{\tau}_D > u) &\geq \varepsilon_1 \mathbf{P}_x(\widehat{Y}_u \in B_1, \widehat{\tau}_D > u) / g(u, x) \\ &\geq \varepsilon_1 \mathbf{P}_{u,x}^g(\widehat{Z}_{S_1-}^\Omega \in \{0\} \times B_1). \end{aligned}$$

Thus it is enough to bound $\mathbf{P}_{u,x}^g(\widehat{Z}_{S_1-}^\Omega \in \{0\} \times B_1)$. By the strong Markov property and (5.11),

$$(5.13) \quad \begin{aligned} &\mathbf{P}_{u,x}^g \left(\widehat{Z}_{S_1-}^\Omega \in \{0\} \times B_1 \right) \\ &\geq \mathbf{P}_{u,x}^g \left(\widehat{Z}_{S_1-}^\Omega \in \{0\} \times B_1, \widehat{T}_{\partial_1 E} \leq \frac{u}{4} \right) \\ &= \mathbf{E}_{u,x}^g \left[\mathbf{P}_{\widehat{Z}_{\widehat{T}_{\partial_1 E}}^\Omega}^g \left(\widehat{Z}_{S_1-}^\Omega \in \{0\} \times B_1 \right) : \widehat{T}_{\partial_1 E} \leq \frac{u}{4} \right] \\ &\geq \frac{1}{2} \inf_{(s,x) \in \partial_1 E} \mathbf{P}_{s,x}^g \left(\widehat{Z}_{S_1-}^\Omega \in \{0\} \times B_1 \right). \end{aligned}$$

Since $g = 4\varepsilon_1$ on $\partial_1 E$ by the continuity of g , for $(s, x) \in \partial_1 E$,

$$\begin{aligned} 4\varepsilon_1 &= \int_D \widehat{r}^D(s, x, y) f_{\varepsilon_1}(y) dy \\ &= \mathbf{P}_x(\widehat{Y}_s^D \in B_1) + \varepsilon_1 \mathbf{P}_x(\widehat{Y}_s^D \in D \setminus B_1) \\ &= \mathbf{P}_{s,x}(\widehat{Z}_{S_1} \in \{0\} \times B_1) + \varepsilon_1 \mathbf{P}_{s,x}(\widehat{Z}_{S_1} \in \{0\} \times (D \setminus B_1)) \\ &\leq \mathbf{P}_{s,x}(\widehat{Z}_{S_1} \in \{0\} \times B_1) + \varepsilon_1. \end{aligned}$$

Thus

$$\mathbf{P}_{s,x}(\widehat{Z}_{S_1} \in \{0\} \times B_1) \geq 3\varepsilon_1.$$

Since $\mathbf{P}_{s,x}^g(\widehat{Z}_{S_1-}^\Omega \in \{0\} \times D) = 1$, applying the above inequality we get

$$\begin{aligned} &\mathbf{P}_{s,x}^g(\widehat{Z}_{S_1-}^\Omega \in \{0\} \times B_1) \\ &= \frac{1}{4\varepsilon_1} \mathbf{E}_{s,x} \left[g(\widehat{Z}_{S_1-}^\Omega); \widehat{Z}_{S_1-}^\Omega \in \{0\} \times B_1 \right] \\ (5.14) \quad &= \frac{1}{4\varepsilon_1} \mathbf{P}_{s,x}(\widehat{Z}_{S_1} \in \{0\} \times B_1) \geq \frac{3}{4} > 0, \quad (s, x) \in \partial_1 E. \end{aligned}$$

Combining (5.11)-(5.14), we have finished the proof. \square

Let $p(t, x, y) := r^D(t, x, y)/H(y)$. Recall that $H(y) = \int_V G(x, y) dx$ and $\xi(dy) = H(y) dy$. For any $t > 0$, define

$$P_t^D f(x) := \int_D r^D(t, x, y) f(y) dy = \int_D p(t, x, y) f(y) \xi(dy)$$

and

$$\widehat{P}_t^D f(x) := \int_D \widehat{r}^D(t, x, y) f(y) dy = \int_D p(t, y, x) f(y) \xi(dy).$$

By definition, we have

$$\int_D f(x) P_t^D g(x) \xi(dx) = \int_D g(x) \widehat{P}_t^D f(x) \xi(dx).$$

It is easy to check that $\{P_t\}$ and $\{\widehat{P}_t\}$ are both strongly continuous contraction semigroups in $L^2(D, \xi(dx))$. We will use \mathcal{L} and $\widehat{\mathcal{L}}$ to denote the $L^2(D, \xi(dx))$ -infinitesimal generators of $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ respectively.

LEMMA 5.5. (1)

$$\frac{p(t, x, y)}{p(t, x, z)} \geq c_1 \frac{p(t, w, y)}{p(t, w, z)}, \quad \forall w, x, y, z \in D$$

implies that for every $s > t$ and $w, x, y, z \in D$

$$\frac{p(s, y, x)}{p(s, z, x)} \geq c_1 \frac{p(t, y, w)}{p(t, z, w)} \quad \text{and} \quad \frac{p(s, x, y)}{p(s, x, z)} \leq c_1^{-1} \frac{p(t, w, y)}{p(t, w, z)}.$$

(2)

$$\frac{p(t, y, x)}{p(t, z, x)} \geq c_2 \frac{p(t, y, v)}{p(t, z, v)}, \quad \forall v, x, y, z \in D$$

implies that for every $s > t$ and $v, x, y, z \in D$

$$\frac{p(s, x, y)}{p(s, x, z)} \geq c_2 \frac{p(t, v, y)}{p(t, v, z)} \quad \text{and} \quad \frac{p(s, y, x)}{p(s, z, x)} \leq c_2^{-1} \frac{p(t, y, v)}{p(t, z, v)}.$$

PROOF. We give the proof of (2) only. The proof of (1) is similar. Since

$$p(t, w, y) \geq c_2 \frac{p(t, w, z)}{p(t, v, z)} p(t, v, y), \quad \forall w, x, y, z \in D,$$

we get

$$\begin{aligned} p(s, x, y) &= \int_D p(s-t, x, w) p(t, w, y) \xi(dw) \\ &\geq c_2 \frac{p(t, v, y)}{p(t, v, z)} \int_D p(s-t, x, w) p(t, w, z) \xi(dw) \\ &= \frac{p(t, v, y)}{p(t, v, z)} p(s, x, z). \end{aligned}$$

On the other hand, since

$$p(t, y, w) \leq c_2^{-1} \frac{p(t, y, v)}{p(t, z, v)} p(t, z, w), \quad \forall w, x, y, z \in D,$$

we get

$$\begin{aligned} p(s, y, x) &= \int_D p(t, y, w) p(s-t, w, x) \xi(dw) \\ &\leq c_2^{-1} \frac{p(t, y, v)}{p(t, z, v)} \int_D p(t, z, w) p(s-t, w, x) \xi(dw) \\ &= \frac{p(t, y, v)}{p(t, z, v)} p(s, z, x). \end{aligned}$$

□

THEOREM 5.6. For each $u \in (0, \frac{1}{2}t_1R_1^2)$ there exists $c = c(D, u) > 0$ such that

$$(5.15) \quad \frac{p(t, x, y)}{p(t, x, z)} \geq c \frac{p(s, v, y)}{p(s, v, z)}, \quad \frac{p(t, y, x)}{p(t, z, x)} \geq c \frac{p(s, y, v)}{p(s, z, v)}$$

for every $s, t \geq u$ and $v, x, y, z \in D$.

PROOF. Let $\tau_1 := \inf\{t > 0 : Y_t \notin D\}$, $\tau_2 := \inf\{t > 0 : \widehat{Y}_t \notin D\}$, $\varphi_1(x) := \mathbf{P}_x(\tau_1 > u/3)$ and $\varphi_2(y) := \mathbf{P}_y(\tau_2 > u/3)$. By (2.1) with $T = \frac{1}{2}t_1R_1^2$, there exists $c_1 > 0$ such that

$$\begin{aligned} p(u, x, y) &= \int_D p\left(\frac{u}{3}, x, z\right) \int_D p\left(\frac{u}{3}, z, w\right) p\left(\frac{u}{3}, w, y\right) \xi(dw) \xi(dz) \\ &\leq c_1 u^{-\frac{d}{2}} \int_D p\left(\frac{u}{3}, x, z\right) \xi(dz) \int_D p\left(\frac{u}{3}, w, y\right) \xi(dw) \\ &= c_1 u^{-\frac{d}{2}} \varphi_1(x) \varphi_2(y). \end{aligned}$$

For the lower bound, we use Lemmas 5.1 and 5.4, and get

$$\begin{aligned} p(u, x, y) &\geq \int_{B_1} p\left(\frac{u}{3}, x, z\right) \int_{B_1} p\left(\frac{u}{3}, z, w\right) p\left(\frac{u}{3}, w, y\right) \xi(dw) \xi(dz) \\ &\geq c_2 u^{-\frac{d}{2}} \int_{B_1} p\left(\frac{u}{3}, x, z\right) \xi(dz) \int_{B_1} p\left(\frac{u}{3}, w, y\right) \xi(dw) \\ &= c_2 u^{-\frac{d}{2}} \mathbf{P}_x(Y_{\frac{u}{3}} \in B_1, \tau_1 > u) \mathbf{P}_y(\widehat{Y}_{\frac{u}{3}} \in B_1, \tau_2 > u) \\ &\geq c_3 u^{-\frac{d}{2}} \varphi_1(x) \varphi_2(y). \end{aligned}$$

for some positive constants c_2 and c_3 . Thus both inequalities in (5.15) are true for $s = t = u \leq \frac{1}{2}t_1R_1^2$. Now we apply Lemma 5.5 (1)-(2) and we get for $s > u$ and $v, x, y, z \in D$

$$(5.16) \quad \frac{p(s, y, x)}{p(s, z, x)} \geq c_4 \frac{p(u, y, v)}{p(u, z, v)}, \quad \frac{p(s, x, y)}{p(s, x, z)} \leq c_4^{-1} \frac{p(u, v, y)}{p(u, v, z)}$$

and

$$(5.17) \quad \frac{p(s, x, y)}{p(s, x, z)} \geq c_4 \frac{p(u, v, y)}{p(u, v, z)}, \quad \frac{p(s, y, x)}{p(s, z, x)} \leq c_4^{-1} \frac{p(u, y, v)}{p(u, z, v)}.$$

Thus both inequalities in (5.15) are true for $s > t = u$. Moreover, Combining (5.16)-(5.17), both inequalities in (5.15) are true for $t = s > u$ too. Now applying Lemma 5.5 (1)-(2) again, we get our conclusion. \square

By (2.6), we have proved the parabolic boundary Harnack principle for Y^D .

COROLLARY 5.7. *For each positive $u \in (0, \frac{1}{2}t_1R_1^2)$ there exists $c = c(D, u) > 0$ such that*

$$\frac{r^D(t, x, y)}{r^D(t, x, z)} \geq c \frac{r^D(s, w, y)}{r^D(s, w, z)}, \quad \frac{r^D(t, y, x)}{r^D(t, z, x)} \geq c \frac{r^D(s, y, w)}{r^D(s, z, w)}$$

for every $s, t \geq u$ and $w, x, y, z \in D$.

Since for each $t > 0$, $p(t, x, y)$ is bounded in $D \times D$, it follows from Jentzsch's Theorem (Theorem V.6.6 on page 337 of [26]) that the common value $\lambda_0 := \sup \operatorname{Re}(\sigma(\mathcal{L})) = \sup \operatorname{Re}(\sigma(\widehat{\mathcal{L}}))$ is an eigenvalue of multiplicity 1 for both \mathcal{L} and $\widehat{\mathcal{L}}$, and that an eigenfunction ϕ_0 of \mathcal{L} associated with λ_0 can be chosen to be strictly positive with $\|\phi_0\|_{L^2(D, \xi(dx))} = 1$ and an eigenfunction ψ_0 of $\widehat{\mathcal{L}}$ associated with λ_0 can be chosen to be strictly positive with $\|\psi_0\|_{L^2(D, \xi(dx))} = 1$.

DEFINITION 5.8. *The semigroups $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ are said to be intrinsic ultracontractive if, for any $t > 0$, there exists a constant $c_t > 0$ such that*

$$p(t, x, y) \leq c_t \phi_0(x) \psi_0(y), \quad \forall (x, y) \in D \times D.$$

Now the next theorem, which is the main result of this paper, can be proved easily from Lemma 5.4 and the continuity of ϕ_0 and ψ_0 . But we give the proof of that Theorem 5.6 implies the intrinsic ultracontractivity.

THEOREM 5.9. *The semigroups $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ are intrinsic ultracontractive. Moreover, for any $t > 0$, there exists a constant $c_t > 0$ such that*

$$(5.18) \quad c_t^{-1} \phi_0(x) \psi_0(y) \leq p(t, x, y) \leq c_t \phi_0(x) \psi_0(y), \quad \forall (x, y) \in D \times D.$$

PROOF. Integrating both sides of (5.15) with respect to y over D for $t = s = u \leq \frac{1}{2}t_1R_1^2$, we get

$$(5.19) \quad \frac{p(t, x, z)}{\int_D p(t, x, y) \xi(dy)} \leq c_t \frac{p(t, w, z)}{\int_D p(t, w, y) \xi(dy)}$$

and

$$(5.20) \quad \frac{p(t, z, x)}{\int_D p(t, y, x) \xi(dy)} \leq c_t \frac{p(t, z, w)}{\int_D p(t, y, w) \xi(dy)}$$

for all $w, x, z \in D$. We fix $x_0 \in D$. The above (5.20) implies that for any positive function f and $z \in D$,

$$\begin{aligned} P_t^D f(z) &= \int_D p(t, z, x) f(x) \xi(dx) \\ &\leq c_t \left(\int_D p(t, y, x_0) \xi(dy) \right)^{-1} \int_D \int_D p(t, y, x) \xi(dy) p(t, z, x_0) f(x) \xi(dx) \\ &= c_t \frac{p(t, z, x_0)}{\int_D p(t, y, x_0) \xi(dy)} \int_D \int_D p(t, y, x) \xi(dy) f(x) \xi(dx) \\ &= c_t \frac{p(t, z, x_0)}{\int_D p(t, y, x_0) \xi(dy)} \int_D P_t^D f(y) \xi(dy). \end{aligned}$$

Similarly (5.20) also implies the lower bound

$$P_t^D f(z) \geq c_t^{-1} \frac{p(t, z, x_0)}{\int_D p(t, y, x_0) \xi(dy)} \int_D P_t^D f(y) \xi(dy), \quad z \in D.$$

Using (5.19), we also get the corresponding result for \widehat{P}_t^D . Thus we have for all $z, w \in D$,

$$(5.21) \quad c_t^{-1} \frac{p(t, z, x_0)}{\int_D p(t, y, x_0) \xi(dy)} \leq \frac{P_t^D f(z)}{\int_D P_t^D f(y) \xi(dy)} \leq c_t \frac{p(t, z, x_0)}{\int_D p(t, y, x_0) \xi(dy)}$$

and

$$(5.22) \quad c_t^{-1} \frac{p(t, x_0, w)}{\int_D p(t, x_0, y) \xi(dy)} \leq \frac{\widehat{P}_t^D f(w)}{\int_D \widehat{P}_t^D f(y) \xi(dy)} \leq c_t \frac{p(t, x_0, w)}{\int_D p(t, x_0, y) \xi(dy)}.$$

Applying (5.21) to ϕ_0 and a sequence of functions approaching the point mass at w appropriately, we get that for any $z, w \in D$,

$$c_t^{-2} \phi_0(z) \leq \frac{p(t, z, w)}{\int_D p(t, y, w) \xi(dy)} \leq c_t^2 \phi_0(z),$$

which implies that

$$(5.23) \quad c_t^{-4} \frac{\phi_0(z)}{\phi_0(x_0)} \leq \frac{p(t, z, w)}{p(t, x_0, w)} \leq c_t^4 \frac{\phi_0(z)}{\phi_0(x_0)}, \quad z, w \in D.$$

Similarly, applying (5.22) to ψ_0 and a sequence of functions approaching point mass at z , we get that for any $z, w \in D$,

$$(5.24) \quad c_t^{-4} \frac{\psi_0(w)}{\psi_0(x_0)} \leq \frac{p(t, z, w)}{p(t, z, x_0)} \leq c_t^4 \frac{\psi_0(w)}{\psi_0(x_0)}.$$

Thus, combining (5.23)-(5.24), we conclude that for any $t \leq \frac{1}{2}t_1R_1^2$ and any $z, w \in D$,

$$p(t, z, w) = p(t, x_0, x_0) \frac{p(t, x_0, w)}{p(t, x_0, x_0)} \frac{p(t, z, w)}{p(t, x_0, w)} \leq c_t^8 p(t, x_0, x_0) \frac{\phi_0(z)\psi_0(w)}{\phi_0(x_0)\psi_0(x_0)}.$$

Let $T := \frac{1}{2}t_1R_1^2$. Since

$$\begin{aligned} p(s, x, y) &= \int_D p(T, x, z)p(s-T, z, y)\xi(dz) \\ &\leq c_T^8 c_{s-T}^8 \phi_0(x)\psi_0(y) \int_D \phi_0(z)\psi_0(z)\xi(dz) \\ &\leq c_T^8 c_{s-T}^8 \phi_0(x)\psi_0(y), \quad s \in (T, 2T], \end{aligned}$$

we can easily get the intrinsic ultracontractivity by induction. The fact that intrinsic ultracontractivity implies the lower bound is proved in [18] (Proposition 2.5 in [18]). \square

Let

(5.25)

$$\phi(x) := \phi_0(x) / \int_D \phi_0(y)^2 dy, \quad \psi(x) := \psi_0(x)H(x) / \int_D \psi_0(y)^2 H(y)^2 dy.$$

Note that $0 < \int_D \psi_0(y)^2 H(y)^2 dy < \infty$ because of (2.6). Since

$$e^{\lambda_0 t} \phi_0(x) = \int_D p(t, x, y)\phi_0(y)\xi(dy) = \int_D r^D(t, x, y)\phi_0(y)dy$$

and

$$e^{\lambda_0 t} \psi_0(x)H(x) = H(x) \int_D p(t, y, x)\psi_0(y)\xi(dy) = \int_D r^D(t, y, x)\psi_0(y)H(y)dy,$$

we have

$$(5.26) \quad e^{\lambda_0 t} \phi(x) = \int_D r^D(t, x, y)\phi(y)dy, \quad e^{\lambda_0 t} \psi(x) = \int_D r^D(t, y, x)\psi(y)dy.$$

We say that the common value $e^{\lambda_0 t}$ is an eigenvalue for $r^D(t, x, y)$ and the pair (ϕ, ψ) are the corresponding eigenfunctions if (5.26) is true and ϕ are ψ strictly positive with $\|\phi\|_{L^2(D, dx)} = 1$ and $\|\psi\|_{L^2(D, dx)} = 1$. So the intrinsic ultracontractivity of $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ can be rephrased as follows.

COROLLARY 5.10. *For any $t > 0$, there exists a constant $c_t > 0$ such that*

$$(5.27) \quad c_t^{-1} \phi(x)\psi(y) \leq r^D(t, x, y) \leq c_t \phi(x)\psi(y), \quad \forall (x, y) \in D \times D.$$

PROOF. It is clear from (5.18) and (5.25). \square

Applying Theorem 2.7 in [18], we have the following.

THEOREM 5.11. *There exist positive constants c and a such that for every $(t, x, y) \in (1, \infty) \times D \times D$,*

$$(5.28) \quad \left| \left(e^{-\lambda_0 t} \int_D \phi_0(z) \psi_0(z) \xi(dz) \right) \frac{r^D(t, x, y)}{\phi_0(x) \psi_0(y) H(y)} - 1 \right| \leq c e^{-at}$$

We are going to use SH^+ to denote families of nonnegative superharmonic functions of Y in D . For any $h \in SH^+$, we use \mathbf{P}_x^h to denote the law of the h -conditioned diffusion process Y^D and use \mathbf{E}_x^h to denote the expectation with respect to \mathbf{P}_x^h . Let ζ^h be the lifetime of the h -conditioned diffusion process Y^D .

The bound for the lifetime of the conditioned Y^D is proved using Theorem 5.11 in [18].

THEOREM 5.12. *(Theorem 2.8 in [18])*

(1)

$$\sup_{x \in D, h \in SH^+} \mathbf{E}_x^h[\zeta^h] < \infty.$$

(2) *For any $h \in SH^+$, we have*

$$\lim_{t \uparrow \infty} e^{-\lambda_0 t} \mathbf{P}_x^h(\zeta^h > t) = \frac{\phi_0(x)}{h(x)} \int_D \psi_0(y) h(y) \xi(dy) / \int_D \phi_0(y) \psi_0(y) \xi(dy).$$

In particular,

$$\lim_{t \uparrow \infty} \frac{1}{t} \log \mathbf{P}_x^h(\zeta^h > t) = \lambda_0.$$

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