

Intrinsic Ultracontractivity for Non-symmetric Lévy Processes

Panki Kim*

Department of Mathematics
Seoul National University
Seoul 151-742, Republic of Korea
Email: pkim@math.snu.ac.kr

and

Renming Song[†]

Department of Mathematics
University of Illinois
Urbana, IL 61801, USA
Email: rsong@math.uiuc.edu

September 28, 2008

Abstract

Recently in [17, 18], we extended the concept of intrinsic ultracontractivity to non-symmetric semigroups and proved that for a large class of non-symmetric diffusions Z with measure-valued drift and potential, the semigroup of Z^D (the process obtained by killing Z upon exiting D) in a bounded domain is intrinsic ultracontractive under very mild assumptions.

In this paper, we study the intrinsic ultracontractivity for non-symmetric discontinuous Lévy processes. We prove that, for a large class of non-symmetric discontinuous Lévy processes X such that the Lebesgue measure is absolutely continuous with respect to the Lévy measure of X , the semigroup of X^D in any bounded open set D is intrinsic ultracontractive. In particular, for the non-symmetric stable process X discussed in [25], the semigroup of X^D is intrinsic ultracontractive for any bounded set D . Using the intrinsic ultracontractivity, we show that the parabolic boundary Harnack principle is true for those processes. Moreover, we get that the supremum of the expected

*Research partially supported by the Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MEST) (No. R01-2008-000-20010-0).

[†]The research of this author is supported in part by a joint US-Croatia grant INT 0302167.

conditional lifetimes in a bounded open set is finite. We also have results of the same nature when the Lévy measure is compactly supported.

AMS 2000 Mathematics Subject Classification: Primary: 47D07, 60J25; Secondary: 60J45

Keywords and phrases: stable processes, non-symmetric stable process, Lévy process, non-symmetric Lévy process, semigroups, non-symmetric semigroups, parabolic boundary Harnack principle, intrinsic ultracontractivity

1 Introduction

Suppose that H is a semi-bounded self-adjoint operator on $L^2(D)$ with D being an open set in \mathbf{R}^d and that $\{e^{Ht}\}$ is an irreducible positivity-preserving semigroup with integral kernel $a(t, x, y)$. We assume that the top of the spectrum λ_1 of H is an eigenvalue. In this case, λ_1 has multiplicity one and the corresponding eigenfunction ϕ_1 , normalized by $\|\phi_1\|_{L^2(D)} = 1$, is positive almost everywhere on D . $\{e^{Ht}\}$ is said to be intrinsic ultracontractive if for every $t > 0$, there exists $c_t \in (0, \infty)$ such that $a(t, x, y) \leq c_t \phi_1(x) \phi_1(y)$.

The notion of the intrinsic ultracontractivity above was introduced in [11]. It is a very important concept in both analysis and probability, and has been studied extensively. When H is the Dirichlet Laplacian in a domain D (equivalently, the corresponding process is a killed Brownian motion), the semigroup $\{e^{Ht}\}$ is intrinsic ultracontractive for a large class of non-smooth domains (see, for instance [1, 3]). For symmetric α -stable processes with $\alpha \in (0, 2)$, the intrinsic ultracontractivity has been discussed in [6, 7, 20]. After obtaining the main results of this paper, we found out from [13] that the intrinsic ultracontractivity for some large classes of symmetric Lévy processes was studied in [12].

Very recently in [17], we extended the concept of intrinsic ultracontractivity to non-symmetric semigroups and, by using an analytic method, we proved there that the semigroup of a killed diffusion process in a bounded Lipschitz domain is intrinsic ultracontractive if the coefficients of the generator of the diffusion process are smooth. In [18], by using a probabilistic method we proved that for a non-symmetric diffusion with measure-valued drift and potential belonging to appropriate Kato classes, the semigroup of the killed process in a bounded domain is intrinsic ultracontractive when the bounded domain is one of the following types: twisted Hölder domains of order $\alpha \in (1/3, 1]$, uniformly Hölder domains of order $\alpha \in (0, 2)$ and domains which can be locally represented as the region above the graph of a function (see [18] for details).

In this paper, we continue our discussion of intrinsic ultracontractivity for non-symmetric semigroups. We study the intrinsic ultracontractivity for non-symmetric discontinuous Lévy

processes under one of the following two non-overlapping assumptions on the Lévy measure: the first case is that the Lebesgue measure is absolutely continuous with respect to the Lévy measure and the Radon-Nikodym derivative is locally integrable away from 0 and the second case is that the Lévy measure is compactly supported. In the first case, we show that for any bounded open set, the semigroup of the killed process is intrinsic ultracontractive if the transition density of the killed process is strictly positive, bounded and continuous. In particular, the semigroup of the killed strictly α -stable process in any bounded open set is intrinsic ultracontractive. In the second case we put some mild assumptions on both the open set and the Lévy measure: We assume that the open set is bounded κ -fat (a disconnected analogue of John domain, for the definition see Definition 3.1) and that the Radon-Nikodym derivative of the absolutely continuous part of Lévy measure is bounded below by a positive constant near the origin. We show that in this case, the intrinsic ultracontractivity is true if the transition density of the killed process is strictly positive, bounded and continuous. We do not assume that our non-symmetric Lévy process is a purely discontinuous process. It may contain diffusion and drift parts.

The content of this paper is organized as follows. In Section 2, we recall some preliminary facts about non-symmetric Lévy processes. Section 3 contains the proof of the intrinsic ultracontractivity. We also show in Section 3 that the intrinsic ultracontractivity implies the parabolic boundary Harnack principle and that the supremum of the expected conditional lifetimes is finite. In the last section we collect some concrete examples of non-symmetric Lévy processes satisfying the assumptions of this paper.

In this paper we use the convention $f(\partial) = 0$. In this paper we will also use the following convention: the values of the constants c_1, c_2, \dots might change from one appearance to another. The labeling of the constants c_1, c_2, \dots starts anew in the statement of each result.

In this paper, we use “:=” to denote a definition, which is read as “is defined to be”.

2 Non-symmetric Lévy Processes

Let $X = (X_t, \mathbf{P}_x)$ be a Lévy process in \mathbf{R}^d with the generating triplet (A, ν, γ) . i.e., for every $z \in \mathbf{R}^d$,

$$\mathbf{E}_0 [e^{iz \cdot X_1}] = \exp \left(-\frac{1}{2} z \cdot A z + i\gamma \cdot z + \int_{\mathbf{R}^d} (e^{iz \cdot x} - 1 - iz \cdot x 1_{\{|x| \leq 1\}}(x)) \nu(dx) \right)$$

where A is a symmetric nonnegative definite $d \times d$ matrix, $\gamma \in \mathbf{R}^d$, and ν is a measure on \mathbf{R}^d satisfying

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbf{R}^d} (|x|^2 \wedge 1) \nu(dx) < \infty. \quad (2.1)$$

γ is called the drift of X and ν is called the Lévy measure of X .

$-X$ is also a Lévy process and it is the dual of X . For this reason we sometimes use \widehat{X} to denote this process. From the above definition, it is clear that \widehat{X} is a Lévy process in \mathbf{R}^d with the generating triplet $(A, \nu(-dx), -\gamma)$.

Let

$$P_t f(x) := \mathbf{E}_x[f(X_t)] \quad \text{and} \quad \widehat{P}_t f(x) := \mathbf{E}_x[f(\widehat{X}_t)].$$

Then for any non-negative Borel functions f and g ,

$$\int_{\mathbf{R}^d} P_t f(x) g(x) dx = \int_{\mathbf{R}^d} f(x) \widehat{P}_t g(x) dx.$$

Throughout this paper, we assume the following.

(A1) The Lévy measure ν satisfies either (a) or (b) below:

(a) The Lebesgue measure in \mathbf{R}^d is absolutely continuous with respect to ν . i.e., there exists a non-negative Borel function $L(x)$ such that for any Borel set B ,

$$|B| = \int_B L(x) \nu(dx). \quad (2.2)$$

Moreover, we assume that L is locally integrable on $\mathbf{R}^d \setminus \{0\}$ with respect to the Lebesgue measure.

(b) Let $M(x)$ be the Radon-Nikodym derivative of the absolutely continuous part of ν . We assume that there exists $R_0 > 0$ such that

$$\inf_{x \in B(0, R_0)} M(x) > 0. \quad (2.3)$$

In [18], we have already discussed the case when $\nu = 0$. The second assumption in (a) is the same as assuming that L is locally L^2 -integrable on $\mathbf{R}^d \setminus \{0\}$ with respect to ν . (b) covers the case where the Lévy measures have compact supports.

Let $C_0(\mathbf{R}^d)$ be the class of bounded continuous functions f on \mathbf{R}^d with $\lim_{|x| \rightarrow \infty} f(x) = 0$. We say a Markov process Y in \mathbf{R}^d has the Feller property if for every $g \in C_0(\mathbf{R}^d)$, $\mathbf{E}_x[g(Y_t)]$ is in $C_0(\mathbf{R}^d)$ and $\lim_{t \downarrow 0} \mathbf{E}_x[g(Y_t)] = g(x)$. Any Lévy process in \mathbf{R}^d has the Feller property (for example, see [4, 22]).

For any open set U , we use τ_U to denote the first exit time of U for X , i.e., $\tau_U := \inf\{t > 0 : X_t \notin U\}$. Given an open set $U \subset \mathbf{R}^d$, we define $X_t^U(\omega) = X_t(\omega)$ if $t < \tau_U(\omega)$ and $X_t^U(\omega) = \partial$ if $t \geq \tau_U(\omega)$, where ∂ is a cemetery state. The process X^U is called a killed process in U . We use $\widehat{\tau}_U$ to denote the first exit time of U for \widehat{X} . i.e., $\widehat{\tau}_U := \inf\{t > 0 : \widehat{X}_t \notin U\}$. We similarly define \widehat{X}^U .

For any $t > 0$, define

$$P_t^D f(x) := \mathbf{E}_x[f(X_t^D)] \quad \text{and} \quad \widehat{P}_t^D f(x) := \mathbf{E}_x[f(\widehat{X}_t^D)].$$

The next equality is known as Hunt's switching identity (for example, see Theorem II.5 in [4]).

$$\int_D f(x)P_t^D g(x)dx = \int_D g(x)\widehat{P}_t^D f(x)dx.$$

For the remainder of this section, D is a fixed bounded open set in \mathbf{R}^d . The next assumption is needed to define intrinsic ultracontractivity for non-symmetric semigroups (see [17]).

(A2) The transition density function $p^D(t, x, y)$ for X_t^D exists. Moreover each $t > 0$, $p^D(t, \cdot, \cdot)$ is continuous in $D \times D$.

We further assume that $p^D(t, \cdot, \cdot)$ is bounded.

(A3) $\{P_t^D\}$ is ultracontractive. i.e., for $t > 0$, there exists positive constant c_t such that

$$p^D(t, x, y) \leq c_t < \infty, \quad (x, y) \in D \times D.$$

Remark 2.1. We do not know any necessary and sufficient conditions for (A2)-(A3) in terms of the Lévy measure. In fact, no necessary and sufficient condition in terms of the Lévy measure for the existence of transition density for Lévy process is known (see [22] for some sufficient conditions).

In the remainder of this section, we discuss some elementary consequences of (A2)-(A3).

From Hunt's switching identity and the continuity of $p^D(t, x, y)$, we see that $\widehat{p}^D(t, x, y) := p^D(t, y, x)$ is the transition density for \widehat{X} . So for every $t > 0$ and Borel set $A \subset D$,

$$\mathbf{P}_x(X_t^D \in A) = \int_A p^D(t, x, y)dy \quad \text{and} \quad \mathbf{P}_x(\widehat{X}_t^D \in A) = \int_A p^D(t, y, x)dy. \quad (2.4)$$

If $U \subset D$, then for every $t > 0$, $x \in U$ and nonnegative Borel function f ,

$$P_t^U f(x) \leq \int_U p^D(t, x, y)f(y)dy \leq c_t \int_U f(y)dy.$$

Thus $\mathbf{P}_x(X_t^U \in dy)$ is absolutely continuous with respect to the Lebesgue measure and for every $t > 0$, $x \in U$ the density $p^U(t, x, \cdot)$ exists. Similarly, if $U \subset D$, $\mathbf{P}_x(\widehat{X}_t^U \in dy)$ is absolutely continuous with respect to the Lebesgue measure and for every $t > 0$, $x \in D$ the density $\widehat{p}^U(t, x, \cdot)$ exists. Moreover, from Hunt's switching identity, we see that for every $t > 0$,

$$p^U(t, x, y) = \widehat{p}^U(t, y, x), \quad \text{a.e. } (x, y) \in D \times D.$$

Note that in general we do not know whether $p^U(t, x, y)$ and $\widehat{p}^U(t, y, x)$ are continuous and strictly positive.

From Lemma 48.3 in [22], it is easy to see that for any bounded open subset U , there exists $t_1 > 0$ such that $\sup_{x \in \mathbf{R}^d} \mathbf{P}_x(X_{t_1} \in U) < 1$. Thus

$$\theta := \sup_{x \in \mathbf{R}^d} \mathbf{P}_x(\tau_U > t_1) \leq \sup_{x \in \mathbf{R}^d} \mathbf{P}_x(X_{t_1} \in U) < 1.$$

By the Markov property and an induction argument,

$$\sup_{x \in \mathbf{R}^d} \mathbf{P}_x(\tau_U > nt_1) \leq \theta^n.$$

Thus

$$\sup_{x \in U} \mathbf{E}_x[\tau_U] \leq \frac{t_1}{1 - \theta} < \infty \quad (2.5)$$

(see [8] for the details).

For any bounded open subset $U \subset D$, we will use $G_U(x, y)$ to denote the Green function of X^U , i.e.,

$$G_U(x, y) := \int_0^\infty p^U(t, x, y) dt, \quad (x, y) \in U \times U.$$

By (2.5),

$$\mathbf{E}_x[\tau_U] = \int_U G_U(x, y) dy < \infty, \quad x \in U$$

and $G_U(x, \cdot)$ is well-defined a.e. U .

Also (2.5) implies that for every open set $U \subset D$ and $A \subset U^c$ with $\text{dist}(A, U) > 0$, we have

$$\mathbf{P}_x(X_{\tau_U} \in A) = \int_U G_U(x, y) \nu(y - A) dy. \quad (2.6)$$

(for example, see [14]).

Similarly the Green function $\widehat{G}_U(x, y)$ of \widehat{X}^U is defined as

$$\widehat{G}_U(x, y) := \int_0^\infty \widehat{p}^U(t, y, x) dt, \quad (x, y) \in U \times U,$$

which is well-defined a.e. U . For every $A \subset U^c$ with $\text{dist}(A, U) > 0$, we have

$$\mathbf{P}_x(\widehat{X}_{\tau_U} \in A) = \int_U \widehat{G}_U(x, y) \nu(A - y) dy. \quad (2.7)$$

Clearly,

$$G_U(x, y) = \widehat{G}_U(y, x), \quad \text{a.e. } (x, y) \in U \times U.$$

3 Intrinsic Ultracontractivity for Non-symmetric Lévy Processes

In this section, we first recall the definition of the intrinsic ultracontractivity for non-symmetric semigroups from [17] and then prove that the intrinsic ultracontractivity is true if the killed non-symmetric Lévy process X^D satisfies (A1)-(A3) in the previous section and (A4)-(A5) below. We will use some ideas from [20].

Many results in this section are stated for both X^D and its dual \widehat{X}^D . Since the proofs for the two processes are similar, we only present the proofs for X^D .

The following definition is taken from [24].

Definition 3.1. *Let $\kappa \in (0, 1/2]$. We say that an open set D in \mathbf{R}^d is κ -fat if there exists $R > 0$ such that for each $Q \in \partial D$ and $r \in (0, R]$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$ for some $A_r(Q) \in D$. The pair (R, κ) is called the characteristics of the κ -fat open set D .*

Note that every Lipschitz domain and every non-tangentially accessible domain (see [15] for the definition of non-tangentially accessible domains) are κ -fat. Moreover, every John domain is κ -fat (see Lemma 6.3 in [21]). The boundary of a κ -fat open set can be highly nonrectifiable and, in general, no regularity of its boundary can be inferred. Bounded κ -fat open sets may be disconnected.

Depending on whether (A1)(a) or (A1)(b) is valid, our assumptions on the open set D are different. In both cases, we will need to define some subsets B_0 , C_1 and B_2 of D . The following assumptions on D will always be in force in the remainder of this section.

(A4)(a) If ν satisfies (A1)(a), we assume that D is an arbitrary bounded open set. Choose a point x_0 in D and $r_0 \in (0, \infty)$ such that $B(x_0, 2r_0) \subset \overline{B(x_0, 2r_0)} \subset D$. We put $B_0 := B(x_0, r_0/2)$, $C_1 := \overline{B(x_0, r_0)}$ and $B_2 := B(x_0, 2r_0)$.

(A4)(b) If ν satisfies (A1)(b), then we assume that D is a bounded κ -fat open set with the characteristics (R, κ) . Without loss of generality, we assume $R \leq \frac{1}{2}R_0$ where R_0 is the constant in (A1)(b). Let $\rho(x)$ be the distance of a point x to the boundary of D , i.e., $\rho(x) = \text{dist}(x, \partial D)$. Define

$$\begin{aligned} B_0 &:= \{x \in D : \rho(x) > R\kappa/2\}, \\ C_1 &:= \{x \in D : \rho(x) \geq R\kappa/4\}, \\ B_2 &:= \{x \in D : \rho(x) > R\kappa/8\}. \end{aligned}$$

The distinction between (A4)(a) and (A4)(b) will be made only in the proof of Lemma 3.2 below.

Define

$$\eta_U := \inf\{t \geq 0 : X_t \notin U\} \quad \text{and} \quad \widehat{\eta}_U := \inf\{t \geq 0 : \widehat{X}_t \notin U\}.$$

Note that $\eta_U \leq \tau_U$ and $\widehat{\eta}_U \leq \widehat{\tau}_U$. Moreover, $\eta_U(\omega) = \tau_U(\omega)$ and $\widehat{\eta}_U(\omega) = \widehat{\tau}_U(\omega)$ if $X_0(\omega) \in U$ and $\widehat{X}_0(\omega) \in U$ respectively.

Lemma 3.2. *If (A1)-(A4) are true, then there exists a constant $c > 0$ such that for every $x \in \mathbf{R}^d \setminus C_1$,*

$$\mathbf{P}_x \left(X_{\eta_{D \setminus C_1}} \in C_1 \right) \geq c \mathbf{E}_x [\eta_{D \setminus C_1}] \quad \text{and} \quad \mathbf{P}_x \left(\widehat{X}_{\widehat{\eta}_{D \setminus C_1}} \in C_1 \right) \geq c \mathbf{E}_x [\widehat{\eta}_{D \setminus C_1}].$$

Proof. If $x \in \mathbf{R}^d \setminus D$, $\mathbf{P}_x(\eta_{D \setminus C_1} = 0) = 1$. Thus $\mathbf{E}_x[\eta_{D \setminus C_1}] = 0$ and the assertions of the lemma are trivial in this case. Now we assume $x \in D \setminus C_1$.

- (1) First we deal with the case that ν satisfies (A1)(a). If $w \in B_0$ and $y \in D \setminus C_1$, then $|w - y| \geq |y - x_0| - |w - x_0| > r_0/2$ and $|w - y| < 2\text{diam}(D)$. So the set

$$A := \bigcup_{y \in D \setminus C_1} (y - B_0)$$

is a relatively compact subset of $\mathbf{R}^d \setminus \{0\}$. By (A1), for every $y \in D \setminus C_1$ we have

$$|B_0| = |y - B_0| \leq \int_A 1_{y-B_0}(z) L(z) \nu(dz) \leq (\nu(y - B_0))^{1/2} \|1_A L\|_{L^2(\nu)}.$$

We know from our assumption (A1)(a) that

$$\|1_A L\|_{L^2(\nu)}^2 = \int_A L^2(z) \nu(dz) = \int_A L(z) dz < \infty.$$

Therefore from (2.6), we have

$$\begin{aligned} \mathbf{P}_x \left(X_{\eta_{D \setminus C_1}} \in C_1 \right) &\geq \mathbf{P}_x \left(X_{\tau_{D \setminus C_1}} \in B_0 \right) \\ &= \int_{D \setminus C_1} G_{D \setminus C_1}(x, y) \int_{y-B_0} \nu(dz) dy \geq \int_{D \setminus C_1} G_{D \setminus C_1}(x, y) \frac{|B_0|^2}{\|1_A L\|_{L^2(\nu)}^2} dy \\ &= \frac{|B_0|^2}{\|1_A L\|_{L^2(\nu)}^2} \mathbf{E}_x [\tau_{D \setminus C_1}] = \frac{|B_0|^2}{\|1_A L\|_{L^2(\nu)}^2} \mathbf{E}_x [\eta_{D \setminus C_1}]. \end{aligned}$$

- (2) Now we deal with the case that ν satisfies (A1)(b). For each $y \in D \setminus C_1$, choose a point $Q_y \in \partial D$ such that $\rho(y) = |y - Q_y| < \kappa R/4$. Since D is κ -fat, there exists a point $A_y \in D$ such that $B(A_y, \kappa R) \subset D \cap B(Q_y, R)$. It is easy to see that

$$B(A_y, \frac{1}{2}\kappa R) \subset B_0 \cap B(Q_y, R) \subset B_0 \cap B(y, R_0). \quad (3.1)$$

In fact, if $|w - A_y| < \frac{1}{2}\kappa R$, then $\rho(w) \geq \rho(A_y) - |w - A_y| > \kappa R - \frac{1}{2}\kappa R = \frac{1}{2}\kappa R$. If $|w - Q_y| < R$, then $|y - w| \leq |y - Q_y| + |w - Q_y| < R + \kappa R/4 < 2R \leq R_0$. Thus by (3.1) and (A1)(2), we have for every $y \in D \setminus C_1$

$$\begin{aligned} \nu(y - B_0) &\geq \int_{B_0} M(y - z) dz \geq \int_{B(A_y, \frac{1}{2}\kappa R)} M(y - z) dz \\ &\geq \left(\inf_{w \in B(0, R_0)} M(w) \right) |B(0, \frac{1}{2}\kappa R)| =: c_1(R, R_0, \kappa, d) > 0. \end{aligned}$$

Now by (2.6), we get

$$\begin{aligned} \mathbf{P}_x \left(X_{\eta_{D \setminus C_1}} \in C_1 \right) &\geq \mathbf{P}_x \left(X_{\tau_{D \setminus C_1}} \in B_0 \right) \\ &= \int_{D \setminus C_1} G_{D \setminus C_1}(x, y) \nu(y - B_0) dy \geq c_1 \mathbf{E}_x [\tau_{D \setminus C_1}] = c_1 \mathbf{E}_x [\eta_{D \setminus C_1}]. \end{aligned}$$

□

Let θ be the usual shift operator for Markov processes, and we define stopping times S_n and T_n recursively by

$$S_1 := 0, \quad T_n := S_n + \eta_{D \setminus C_1} \circ \theta_{S_n} \quad \text{and} \quad S_{n+1} := T_n + \eta_{B_2} \circ \theta_{T_n}, \quad n \geq 1.$$

Similarly we define \widehat{T}_n and \widehat{S}_n for \widehat{X} .

Lemma 3.3. *If (A1)-(A4) are true, then there exists a constant $c > 0$ such that for every $x \in D$*

$$\mathbf{P}_x(X_{T_n} \in C_1) \geq c \mathbf{E}_x[T_n - S_n] \quad \text{and} \quad \mathbf{P}_x(\widehat{X}_{\widehat{T}_n} \in C_1) \geq c \mathbf{E}_x[\widehat{T}_n - \widehat{S}_n].$$

Proof. Since $T_n = S_n + \eta_{D \setminus C_1} \circ \theta_{S_n}$, by the strong Markov property,

$$\mathbf{P}_x(X_{T_n} \in C_1) = \mathbf{P}_x \left(X_{S_n + \eta_{D \setminus C_1} \circ \theta_{S_n}} \in C_1 \right) = \mathbf{E}_x \left[\mathbf{P}_{X_{S_n}} (\eta_{D \setminus C_1} \in C_1) \right].$$

Applying Lemma 3.2 to the equation above, we get

$$\mathbf{P}_x(X_{T_n} \in C_1) \geq c \mathbf{E}_x \left[\mathbf{E}_{X_{S_n}} [\eta_{D \setminus C_1}] \right] = c \mathbf{E}_x [\eta_{D \setminus C_1} \circ \theta_{S_n}] = c \mathbf{E}_x [T_n - S_n].$$

□

Lemma 3.4. *For every $x \in D$,*

$$\mathbf{P}_x \left(\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} T_n = \tau_D \right) = \mathbf{P}_x \left(\lim_{n \rightarrow \infty} \widehat{S}_n = \lim_{n \rightarrow \infty} \widehat{T}_n = \widehat{\tau}_D \right) = 1.$$

Proof. Recall that $\mathbf{P}_x(\tau_D < \infty) = 1$, Clearly $S_n \leq \tau_D$, Let $S := \lim_{n \rightarrow \infty} S_n \leq \tau_D$. we define a subprocess Z of X^D by letting $Z_t(\omega) = X_t(\omega)$ if $t < S(\omega)$ and $Z_t(\omega) = \partial$ if $t \geq S(\omega)$. By Corollary III.3.16 in [5], Z is a Hunt process. Thus by the quasi-left continuity,

$$\begin{aligned} \mathbf{P}_x(S < \tau_D) &= \mathbf{P}_x(S < \tau_D, \tau_D < \infty) = \mathbf{P}_x(Z_{S-} \in D, \tau_D < \infty) \\ &= \mathbf{P}_x(\lim_{n \rightarrow \infty} Z_{S_n} \in D, S_n < S_{n+1}, \tau_D < \infty) \leq \mathbf{P}_x(Z_S \in D, S < \infty) = 0. \end{aligned}$$

□

By the separation property for Feller processes, there exists t_0 such that

$$\inf_{y \in C_1} \mathbf{P}_y(\tau_{B_2} > t) \geq \frac{1}{2} \quad \text{and} \quad \inf_{y \in C_1} \mathbf{P}_y(\widehat{\tau}_{B_2} > t) \geq \frac{1}{2}$$

for any $t \leq t_0$ (see Exercise 2 on page 73 of [9]).

Thus we have

$$\inf_{y \in C_1} \mathbf{E}_y[\tau_{B_2}] \geq t_0 \inf_{y \in C_1} \mathbf{P}_y(\tau_{B_2} > t_0) \geq \frac{t_0}{2} \quad \text{and} \quad \inf_{y \in C_1} \mathbf{E}_y[\widehat{\tau}_{B_2}] \geq t_0 \inf_{y \in C_1} \mathbf{P}_y(\widehat{\tau}_{B_2} > t_0) \geq \frac{t_0}{2}. \quad (3.2)$$

Lemma 3.5. *If (A1)-(A4) are true, then there exists $c > 0$ such that*

$$\int_{B_2} G_D(x, y) dy \geq c \int_{D \setminus B_2} G_D(x, y) dy, \quad x \in D$$

and

$$\int_{B_2} \widehat{G}_D(x, y) dy \geq c \int_{D \setminus B_2} \widehat{G}_D(x, y) dy, \quad x \in D.$$

Proof. Note that, since $X_t^D \in B_2$ for $T_n < t < S_{n+1}$, we have by Lemma 3.4,

$$\begin{aligned} \int_{B_2} G_D(x, y) dy &= \mathbf{E}_x \left[\int_0^{\tau_D} \mathbf{1}_{B_2}(X_t^D) dt \right] \\ &= \mathbf{E}_x \left[\sum_{n=1}^{\infty} \left(\int_{S_n}^{T_n} \mathbf{1}_{B_2}(X_t^D) dt + \int_{T_n}^{S_{n+1}} \mathbf{1}_{B_2}(X_t^D) dt \right) \right] \\ &\geq \mathbf{E}_x \left[\sum_{n=1}^{\infty} \left(\int_{T_n}^{S_{n+1}} \mathbf{1}_{B_2}(X_t^D) dt \right) \right] \\ &= \mathbf{E}_x \left[\sum_{n=1}^{\infty} (S_{n+1} - T_n) \right]. \end{aligned}$$

By the strong Markov property and (3.2),

$$\mathbf{E}_x [S_{n+1} - T_n] = \mathbf{E}_x \left[\mathbf{E}_{X_{T_n}^D} [\tau_{B_2}]; T_n < \tau_U \right] \geq \frac{t_0}{2} \mathbf{P}_x (X_{T_n} \in C_1)$$

which is larger than $c_1 \mathbf{E}_x [T_n - S_n]$ for some constant $c_1 > 0$ by Lemma 3.3.

Therefore by Lemma 3.4 and Fubini's theorem, for $x \in D$,

$$\begin{aligned} \int_{B_2} G_D(x, y) dy &\geq c_1 \sum_{n=1}^{\infty} \mathbf{E}_x [T_n - S_n] = c_1 \sum_{n=1}^{\infty} \mathbf{E}_x \left[\int_{S_n}^{T_n} 1_{\mathbf{R}^d}(X_t) dt \right] \\ &\geq c_1 \mathbf{E}_x \left[\sum_{n=1}^{\infty} \int_{S_n}^{T_n} 1_{D \setminus B_2}(X_t) dt \right] = c_1 \mathbf{E}_x \left[\sum_{n=1}^{\infty} \int_{S_n}^{T_n} 1_{D \setminus B_2}(X_t) dt + \sum_{n=1}^{\infty} \int_{T_n}^{S_{n+1}} 1_{D \setminus B_2}(X_t) dt \right] \\ &= c_1 \mathbf{E}_x \left[\int_0^{\tau_D} 1_{D \setminus B_2}(X_t) dt \right] = c_1 \int_{D \setminus B_2} G_D(x, y) dy. \end{aligned}$$

The other inequality in the lemma can be proved in exactly the same way. \square

The next proposition is elementary and should be well-known. But we could not find any reference for this. We include a proof here for completeness.

Proposition 3.6. *For any open set D with finite Lebesgue measure, $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ are both strongly continuous contraction semigroups in $L^2(D, dx)$.*

Proof. The contraction property follows easily from the duality and Hölder's inequality. So we only prove the strong continuity.

Recall that for any open subset U of \mathbf{R}^d and any $x \in U$, we have

$$\lim_{t \rightarrow 0} \mathbf{P}_x(\tau_U \leq t) = \mathbf{P}_x(\tau_U = 0) = 0. \quad (3.3)$$

We first consider f in $C_c(D)$, the class of continuous functions on D with compact supports. Fix $x \in D$. Given $\varepsilon > 0$, choose $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon/2$ for $y \in B(x, \delta) \subset D$. Then for $x \in D$,

$$\begin{aligned} &|P_t^D f(x) - f(x)| \\ &\leq \int_D p^D(t, x, y) |f(y) - f(x)| dy + |f(x)| \mathbf{P}_x(\tau_D \leq t) \\ &\leq \left(\int_{D \cap \{|x-y| < \delta\}} + \int_{D \cap \{|x-y| \geq \delta\}} \right) p^D(t, x, y) |f(y) - f(x)| dy + \|f\|_{\infty} \mathbf{P}_x(\tau_D \leq t) \\ &\leq \frac{\varepsilon}{2} + 2\|f\|_{\infty} \mathbf{P}_x(|X_t^D - x| \geq \delta) + \|f\|_{\infty} \mathbf{P}_x(\tau_D \leq t) \\ &\leq \frac{\varepsilon}{2} + 2\|f\|_{\infty} \mathbf{P}_x(\tau_{B(x, \delta)} \leq t) + \|f\|_{\infty} \mathbf{P}_x(\tau_D \leq t). \end{aligned}$$

Applying (3.3) to both $\mathbf{P}_x(\tau_{B(x,\delta)} \leq t)$ and $\mathbf{P}_x(\tau_D \leq t)$, we get that $P_t^D f$ converges pointwise to f . Since $\|P_t^D f\|_\infty \leq \|f\|_\infty$ and D has finite Lebesgue measure, by the bounded convergence theorem, $P_t^D f$ also converges to f in $L^2(D)$.

Now we assume $f \in L^2(D)$. Given $\varepsilon > 0$, choose $g \in C_c(D)$ with $\|f - g\|_{L^2(D)} < \varepsilon/4$. By the contraction property of P_t^D ,

$$\begin{aligned} \|P_t^D f - f\|_{L^2(D)} &\leq \|P_t^D(f - g)\|_{L^2(D)} + \|P_t^D g - g\|_{L^2(D)} + \|f - g\|_{L^2(D)} \\ &\leq 2\|f - g\|_{L^2(D)} + \|P_t^D g - g\|_{L^2(D)} \leq \frac{\varepsilon}{2} + \|P_t^D g - g\|_{L^2(D)}. \end{aligned}$$

Thus $P_t^D f$ converges to f in $L^2(D)$. \square

Our last assumption below will be used to define the intrinsic ultracontractivity for non-symmetric semigroups (see [17]).

(A5) The transition density function $p^D(t, x, y)$ for X_t^D is strictly positive in $D \times D$.

Remark 3.7. Even if the Lévy process has a smooth and strictly positive transition density, it is non-trivial to show (A5) (see [2, 10] for the case of killed Brownian motions in a domain, [6] for the case of killed symmetric stable processes in a domain and [25] for the case of killed non-symmetric stable processes in a domain). If the Lévy measure ν satisfies (A1)(b), the distance between connected components of D shouldn't be too far away, otherwise $p^D(t, x, y)$ will be zero there. In Section 4, we will show that for a large class of non-symmetric Lévy processes, (A5) is true.

In the remainder of this section we always assume that (A1)-(A5) are in force.

We use A_D and \widehat{A}_D to denote the L^2 generators of $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ respectively. Since for each $t > 0$, $p^D(t, x, y)$ is bounded in $D \times D$ by (A3), $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ are compact operators in $L^2(D, dx)$. Moreover $p^D(t, x, y)$ is strictly positive in $D \times D$ by (A5). Thus it follows from Jentzsch's Theorem (Theorem V.6.6 on page 337 of [23]) and the strong continuity of $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ that the common value $\lambda_0 := \sup \operatorname{Re}(\sigma(A_D)) = \sup \operatorname{Re}(\sigma(\widehat{A}_D)) < 0$ is an eigenvalue of multiplicity 1 for both A_D and \widehat{A}_D , and that an eigenfunction ϕ_0 of A_D associated with λ_0 can be chosen to be strictly positive a.e. with $\|\phi_0\|_{L^2(D)} = 1$ and an eigenfunction ψ_0 of \widehat{A}_D associated with λ_0 can be chosen to be strictly positive a.e. with $\|\psi_0\|_{L^2(D)} = 1$. Thus for a.e. $(x, y) \in D \times D$,

$$e^{\lambda_0 t} \phi_0(x) = \int_D p^D(t, x, z) \phi_0(z) dz, \quad -\frac{1}{\lambda_0} \phi_0(x) = \int_D G_D(x, z) \phi_0(z) dz, \quad (3.4)$$

$$e^{\lambda_0 t} \psi_0(y) = \int_D \widehat{p}^D(t, y, z) \psi_0(z) dz, \quad -\frac{1}{\lambda_0} \psi_0(y) = \int_D \widehat{G}_D(y, z) \psi_0(z) dz. \quad (3.5)$$

Proposition 3.8. $\phi_0(x)$ and $\psi_0(x)$ are strictly positive and continuous in D . Thus (3.4) and (3.5) are true for every $(x, y) \in D \times D$.

Proof. By (3.4),

$$\phi_0(x) = e^{-\lambda_0} \int_D p^D(1, x, z) \phi_0(z) dz.$$

Since $p^D(1, x, z)$ is bounded continuous and D is a bounded open set, the right hand side of the above equation is continuous by using the dominated convergence theorem and the fact $\|\phi_0\|_{L^2(D)} = 1$. Similarly, $e^{-\lambda_0} \int_D \widehat{p}^D(1, y, z) \psi_0(z) dz$ is continuous. Thus there exist continuous versions of ϕ_0 and ψ_0 , and (3.4)-(3.5) are true for every $(x, y) \in D \times D$. Now the strict positivity of ϕ_0 and ψ_0 follow from the strict positivity of $p^D(1, \cdot, \cdot)$ and (3.4)-(3.5). \square

Definition 3.9. The semigroups $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ are said to be intrinsic ultracontractive if, for any $t > 0$, there exists a constant $c_t > 0$ such that

$$p^D(t, x, y) \leq c_t \phi_0(x) \psi_0(y), \quad \forall (x, y) \in D \times D.$$

For results on intrinsic ultracontractivity for general non-symmetric semigroups, we refer our readers to Section 2 of [17].

We will show that the semigroup of any killed non-symmetric Lévy process X^D satisfying (A1)-(A5) is intrinsic ultracontractive.

Lemma 3.10. There exists a constant $c > 0$ such that

$$\mathbf{E}_x[\tau_D] \leq c \phi_0(x) \quad \text{and} \quad \mathbf{E}_y[\widehat{\tau}_D] \leq c \psi_0(y) \quad \forall (x, y) \in D \times D. \quad (3.6)$$

Proof. By Lemma 3.5, there exists a constant $c_1 > 0$ such that

$$\mathbf{E}_x[\tau_D] = \int_{B_2} G_D(x, z) dz + \int_{D \setminus B_2} G_D(x, z) dz \leq c_1 \int_{B_2} G_D(x, z) dz.$$

Thus by Proposition 3.8, we have

$$\begin{aligned} \int_{B_2} G_D(x, z) dz &\leq c_2 \int_{B_2} G_D(x, z) \phi_0(z) dz \\ &\leq c_2 \int_D G_D(x, z) \phi_0(z) dz = -\frac{c_2}{\lambda_0} \phi_0(x) \end{aligned}$$

for some positive constant c_2 . In the last equality above, we have used (3.4).

Using Lemma 3.5, Proposition 3.8 and (3.5), the second inequality in (3.6) can be proved similarly. \square

Theorem 3.11. *The semigroups $\{P_t^D\}$ and $\{\widehat{P}_t^D\}$ are intrinsic ultracontractive. Moreover, for any $t > 0$, there exists a constant $c_t > 0$ such that*

$$c_t^{-1} \phi_0(x)\psi_0(y) \leq p^D(t, x, y) \leq c_t \phi_0(x)\psi_0(y), \quad \forall (x, y) \in D \times D. \quad (3.7)$$

Proof. By (A3) and the semigroup property, there exists $c_1(t) > 0$ such that

$$\begin{aligned} p^D(t, x, y) &= \int_D p^D\left(\frac{t}{3}, x, z\right) \int_D p^D\left(\frac{t}{3}, z, w\right) p^D\left(\frac{t}{3}, w, y\right) dw dz \\ &\leq c_1(t) \int_D p^D\left(\frac{t}{3}, x, z\right) dz \int_D p^D\left(\frac{t}{3}, w, y\right) dw \\ &= c_1(t) \mathbf{P}_x(\tau_D > t/3) \mathbf{P}_y(\widehat{\tau}_D > t/3). \end{aligned}$$

By applying Chebyshev's inequality we get

$$p^D(t, x, y) \leq \frac{c_1(t)}{9t^2} \mathbf{E}_x[\tau_D] \mathbf{E}_y[\widehat{\tau}_D].$$

Thus the intrinsic ultracontractivity is proved by Lemma 3.10.

The fact that intrinsic ultracontractivity implies the lower bound is proved in [17] (Proposition 2.4 in [17]). \square

The following lower bound of $G_D(x, y)$ is an easy corollary of Lemma 3.10 and Theorem 3.11.

Corollary 3.12. *There exist constants $c_i > 0, i = 1, 2$ such that*

$$c_1 \mathbf{E}_x[\tau_D] \mathbf{E}_y[\widehat{\tau}_D] \leq c_2 \phi_0(x)\psi_0(y) \leq G_D(x, y), \quad (x, y) \in D \times D. \quad (3.8)$$

Moreover, there exists constant $c_3 > 0$ such that

$$c_3^{-1} \mathbf{E}_x[\tau_D] \leq \phi_0(x) \leq c_3 \mathbf{E}_x[\tau_D] \quad \text{and} \quad c_3^{-1} \mathbf{E}_x[\widehat{\tau}_D] \leq \psi_0(x) \leq c_3 \mathbf{E}_x[\widehat{\tau}_D], \quad \forall x \in D.$$

Applying Theorem 2.4 in [17], we have the following.

Theorem 3.13. *There exist positive constants c and ν such that*

$$\left| \left(e^{-\lambda_0 t} \int_D \phi_0(z)\psi_0(z) dz \right) \frac{p^D(t, x, y)}{\phi_0(x)\psi_0(y)} - 1 \right| \leq c e^{-\nu t}, \quad (t, x, y) \in (1, \infty) \times D \times D. \quad (3.9)$$

We recall the following simple lemma from [18].

Lemma 3.14. (Lemma 5.5 in [18])

(1)

$$\frac{p^D(t, x, y)}{p^D(t, x, z)} \geq c_1 \frac{p^D(t, v, y)}{p^D(t, v, z)}, \quad \forall v, x, y, z \in D$$

implies that for every $s > t$,

$$\frac{p^D(s, y, x)}{p^D(s, z, x)} \geq c_1 \frac{p^D(t, y, v)}{p^D(t, z, v)} \quad \text{and} \quad \frac{p^D(s, x, y)}{p^D(s, x, z)} \leq c_1^{-1} \frac{p^D(t, v, y)}{p^D(t, v, z)}, \quad \forall v, x, y, z \in D.$$

(2)

$$\frac{p^D(t, y, x)}{p^D(t, z, x)} \geq c_2 \frac{p^D(t, y, v)}{p^D(t, z, v)}, \quad \forall v, x, y, z \in D$$

implies that for every $s > t$,

$$\frac{p^D(s, x, y)}{p^D(s, x, z)} \geq c_2 \frac{p^D(t, v, y)}{p^D(t, v, z)} \quad \text{and} \quad \frac{p^D(s, y, x)}{p^D(s, z, x)} \leq c_2^{-1} \frac{p^D(t, y, v)}{p^D(t, z, v)}, \quad \forall v, x, y, z \in D.$$

The parabolic boundary Harnack principle is an easy corollary of Theorem 3.11.

Corollary 3.15. For each positive u there exists $c = c(D, u) > 0$ such that

$$\frac{p^D(t, x, y)}{p^D(t, x, z)} \geq c \frac{p^D(s, v, y)}{p^D(s, v, z)}, \quad \frac{p^D(t, y, x)}{p^D(t, z, x)} \geq c \frac{p^D(s, y, v)}{p^D(s, z, v)} \quad (3.10)$$

for every $s, t \geq u$ and $v, x, y, z \in D$.

Proof. By Theorem 3.11, both inequalities in (3.10) are true for $s = t = u$. Now we apply Lemma 3.14 (1)-(2) and we get for $s > u$,

$$\frac{p^D(s, y, x)}{p^D(s, z, x)} \geq c \frac{p^D(u, y, v)}{p^D(u, z, v)}, \quad \frac{p^D(s, x, y)}{p^D(s, x, z)} \leq c^{-1} \frac{p^D(u, v, y)}{p^D(u, v, z)}, \quad \forall v, x, y, z \in D \quad (3.11)$$

and

$$\frac{p^D(s, x, y)}{p^D(s, x, z)} \geq c \frac{p^D(u, v, y)}{p^D(u, v, z)}, \quad \frac{p^D(s, y, x)}{p^D(s, z, x)} \leq c^{-1} \frac{p^D(u, y, v)}{p^D(u, z, v)}, \quad \forall v, x, y, z \in D. \quad (3.12)$$

Thus both inequalities in (3.10) are true for $s > t = u$. Moreover, Combining (3.11)-(3.12), both inequalities in (3.10) are true for $t = s > u$ too. Now applying Lemma 3.14 (1)-(2) again, we get our conclusion. \square

A Borel function h defined on D is said to be superharmonic with respect to X^D if

$$h(x) \geq \mathbf{E}_x [h(X_{\tau_B}^D)], \quad x \in B,$$

for every bounded open set B with $\bar{B} \subset D$. We use SH^+ to denote families of nonnegative superharmonic functions of X^D . For any $h \in SH^+$, we use \mathbf{P}_x^h to denote the law of the h -conditioned process X^D and use \mathbf{E}_x^h to denote the expectation with respect to \mathbf{P}_x^h . i.e.,

$$\mathbf{E}_x^h [g(X_t^D)] = \mathbf{E}_x \left[\frac{h(X_t^D)}{h(x)} g(X_t^D) \right].$$

Let ζ^h be the lifetime of the h -conditioned process X^D .

The bound for the lifetime of the conditioned X^D can be proved using Theorem 3.13. It is proved in [17] for second order elliptic operators with smooth coefficients. Since the proof is similar, we omit the proof here.

Theorem 3.16. (*Theorem 3.8 in [17]*)

(1)

$$\sup_{x \in D, h \in SH^+} \mathbf{E}_x^h [\zeta^h] < \infty.$$

(2) For any $h \in SH^+$, we have

$$\lim_{t \uparrow \infty} e^{-\lambda_0 t} \mathbf{P}_x^h(\zeta^h > t) = \frac{\phi_0(x)}{h(x)} \int_D \psi_0(y) h(y) dy / \int_D \phi_0(y) \psi_0(y) dy.$$

In particular,

$$\lim_{t \uparrow \infty} \frac{1}{t} \log \mathbf{P}_x^h(\zeta^h > t) = \lambda_0.$$

4 Examples

In this section we collect some examples of Lévy processes X and open sets D so that X^D satisfies the assumptions (A1)-(A5).

Example 4.1. We first recall the definition of non-symmetric strictly α -stable processes. Let $\alpha \in (0, 2)$ and $d \geq 2$. The process X is said to be strictly α -stable if $(X_{at}, \mathbf{P}_0)_{t \geq 0}$ is equal to $(a^{1/\alpha} X_t, \mathbf{P}_0)_{t \geq 0}$ in distribution. Since $\alpha \in (0, 2)$, $A = 0$ and there is a finite measure η on the unit sphere $S = \{x \in \mathbf{R}^d : |x| = 1\}$. such that

$$\nu(U) = \int_S \int_0^\infty 1_U(rz) r^{-(1+\alpha)} dr \eta(dz)$$

for every Borel set U in \mathbf{R}^d . The measure η is called the spherical part of the Lévy measure ν . A strictly α -stable process X can be described using its characteristic function as follows:

(i) for $\alpha \in (0, 1)$, a Lévy process X in \mathbf{R}^d is strictly α -stable if and only if

$$\mathbf{E}_0 [e^{iz \cdot X_1}] = \exp \left(\int_S \eta(d\xi) \int_0^\infty (e^{irz \cdot \xi} - 1) r^{-(1+\alpha)} dr \right);$$

(ii) for $\alpha = 1$, a Lévy process X in \mathbf{R}^d is strictly α -stable if and only if

$$\mathbf{E}_0 [e^{iz \cdot X_1}] = \exp \left(\int_S \eta(d\xi) \int_0^\infty (e^{irz \cdot \xi} - 1 - irz \cdot \xi 1_{(0,1]}(r)) r^{-2} dr + iz \cdot \gamma \right)$$

for some $\gamma \in \mathbf{R}^d$ and $\int_S \xi \eta(d\xi) = 0$;

(iii) for $\alpha \in (1, 2)$, a Lévy process X in \mathbf{R}^d is strictly α -stable if and only if

$$\mathbf{E}_0 [e^{iz \cdot X_1}] = \exp \left(\int_S \eta(d\xi) \int_0^\infty (e^{irz \cdot \xi} - 1 - irz \cdot \xi) r^{-(1+\alpha)} dr \right).$$

Suppose that $X = (X_t, \mathbf{P}_x)$ is a strictly α -stable process with the spherical part η of its Lévy measure satisfying the following assumption: there exist $\varphi : S \rightarrow (0, \infty)$ and $\kappa > 0$ such that

$$\varphi = \frac{d\eta}{d\sigma} \quad \text{and} \quad \kappa \leq \varphi(z) \leq \kappa^{-1}, \quad \forall z \in S, \quad (4.1)$$

where σ is the surface measure on S . Thus the Lévy measure ν has a density $f(x) = \varphi(x/|x|)|x|^{-(d+\alpha)}$ with respect to the d -dimensional Lebesgue measure, and

$$\kappa |x|^{-(d+\alpha)} \leq f(x) \leq \kappa^{-1} |x|^{-(d+\alpha)}, \quad x \in \mathbf{R}^d. \quad (4.2)$$

Thus it is easy to see that (A1)(a) is true with $L(x) = |x|^{d+\alpha} \varphi(x/|x|)^{-1}$.

The process X has a jointly continuous and strictly positive transition density function $p(t, x, y) = p(t, x - y)$ and there exists $c > 0$ such that

$$p(t, x, y) \leq c t^{-\frac{d}{\alpha}}, \quad \forall (t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d \quad (4.3)$$

(see (2.6) in [25]). Moreover, for any $\gamma > 0$, there exists $c > 0$ such that

$$p(t, x, y) \leq ct, \quad |x - y| \geq \gamma, \quad t > 0. \quad (4.4)$$

(see (2.5) in [25]). Using the facts above, one can follow routine arguments (see, for instance, the proof of Theorem 2.4 in [10]) to show that, for every open subset D , the killed process X^D has a transition density $p^D(t, x, y)$ such that, for any $t > 0$, $p^D(t, x, y)$ is jointly continuous on $D \times D$. It follows from Theorem 3.2 in [25] that $p^D(t, x, y)$ is strictly positive on $(0, \infty) \times D \times D$ when D is connected. Now we are going to show that $p^D(t, x, y)$ is strictly positive on $(0, \infty) \times D \times D$ when D is not connected. It is enough to show that for any two connected

components D_1 and D_2 of D , $p^D(t, x, y)$ strictly positive on $(0, \infty) \times D_1 \times D_2$. Fix $(t, x, y) \in (0, \infty) \times D_1 \times D_2$ and choose open balls $B(x, 2\varepsilon) \subset D_1$ and $B(y, 2\varepsilon) \subset D_2$. By the semigroup property and the strict positivity and the continuity of $p^{B(y, 2\varepsilon)}$, we have

$$\begin{aligned}
p^D(t, x, y) &= \int_D p^D(t/2, x, z)p^D(t/2, z, y)dy \\
&\geq \int_{B(y, \varepsilon)} p^D(t/2, x, z)p^{B(y, 2\varepsilon)}(t/2, z, y)dy \\
&\geq c_1(t, \varepsilon) \int_{B(y, \varepsilon)} p^D(t/2, x, z)dy \\
&= c_1(t, \varepsilon)\mathbf{P}_x(X_{t/2}^D \in B(y, \varepsilon)).
\end{aligned}$$

By the strong Markov property, we have

$$\begin{aligned}
&\mathbf{P}_x(X_{t/2}^D \in B(y, \varepsilon)) \\
&\geq \mathbf{P}_x(X^D \text{ hits } B(y, \varepsilon/2) \text{ by time } t/2 \text{ and stays in } B(y, \varepsilon) \text{ until } t/2) \\
&\geq \left(\inf_{z \in B(y, \varepsilon/2)} \mathbf{P}_z(\tau_{B(y, \varepsilon)} > t/2) \right) \mathbf{P}_x(X^D \text{ hits } B(y, \varepsilon/2) \text{ by time } t/2.) \\
&\geq \mathbf{P}_0(\tau_{B(0, \varepsilon/2)} > t/2) \mathbf{P}_x(X_{(t/2) \wedge \tau_{B(x, \varepsilon)}}^D \in B(y, \varepsilon/2)) \\
&\geq c_2(t, \varepsilon)\mathbf{P}_x(X_{(t/2) \wedge \tau_{B(x, \varepsilon)}}^D \in B(y, \varepsilon/2)).
\end{aligned}$$

Now, by using the Lévy system and (4.2),

$$\begin{aligned}
&\mathbf{P}_x(X_{(t/2) \wedge \tau_{B(x, \varepsilon)}}^D \in B(y, \varepsilon/2)) \\
&\geq \mathbf{E}_x \left[\int_0^{(t/2) \wedge \tau_{B(x, \varepsilon)}} \int_{B(y, \varepsilon/2)} \frac{c_3}{|X_s - u|^{d+\alpha}} dud s \right] \\
&\geq c_4 \mathbf{E}_x [(t/2) \wedge \tau_{B(x, \varepsilon)}] \int_{B(y, \varepsilon/2)} \frac{1}{|x - y|^{d+\alpha}} du \\
&\geq c_5 t \mathbf{P}_x(\tau_{B(x, \varepsilon)} \geq t/2) |B(y, \varepsilon/2)| |x - y|^{-d-\alpha} \\
&= c_5 t \mathbf{P}_0(\tau_{B(0, \varepsilon)} \geq t/2) |B(0, \varepsilon/2)| |x - y|^{-d-\alpha} > 0
\end{aligned}$$

for some positive constants c_3, c_4 and c_5 . Therefore we get that $p^D(t, x, y)$ is strictly positive everywhere on $(0, \infty) \times D_1 \times D_2$. Thus in this case (A2), (A3) and (A5) are valid for any bounded open subset D as well.

Example 4.2. Assume that X is a non-symmetric strictly α -stable processes from the previous example and we will use the notations from the previous example. A Lévy process Y in \mathbf{R}^d is called truncated (non-symmetric) strictly α -stable process if

(i) when $\alpha \in (0, 1)$,

$$\mathbf{E}_0 [e^{iz \cdot Y_1}] = \exp \left(\int_S \eta(d\xi) \int_0^1 (e^{irz \cdot \xi} - 1) r^{-(1+\alpha)} dr \right);$$

(ii) when $\alpha = 1$,

$$\mathbf{E}_0 [e^{iz \cdot Y_1}] = \exp \left(\int_S \eta(d\xi) \int_0^1 (e^{irz \cdot \xi} - 1 - irz \cdot \xi) r^{-2} dr + iz \cdot \gamma \right)$$

for some $\gamma \in \mathbf{R}^d$ and $\int_S \xi \eta(d\xi) = 0$;

(iii) when $\alpha \in (1, 2)$,

$$\mathbf{E}_0 [e^{iz \cdot Y_1}] = \exp \left(\int_S \eta(d\xi) \int_0^1 (e^{irz \cdot \xi} - 1 - irz \cdot \xi) r^{-(1+\alpha)} dr \right).$$

We also assume that η satisfies (4.1). Then the Lévy density $g(x)$ for Y is

$$g(x) := \varphi(x/|x|) |x|^{-(d+\alpha)} 1_{\{|x| < 1\}} \quad (4.5)$$

and (A1)(b) is satisfied. In the case when Y is rotationally invariant, it has been studied recently by the authors [16]. (4.1) implies that the characteristic function of Y_t is integrable. Thus the process Y has a bounded and continuous density $q(t, x, y)$ (cf. [22]). Let

$$h(x) := f(x) - g(x) = \varphi(x/|x|) |x|^{-(d+\alpha)} 1_{\{|x| \geq 1\}}. \quad (4.6)$$

Note that $\lambda := \int_{\mathbf{R}^d} h(x) dx < \infty$. Thus we can write $X_t = Y_t + Z_t$ where Z_t is a compound Poisson process with the Lévy density $h(x)$, independent of Y_t . Let

$$T := \inf\{t \geq 0 : Z_t \neq 0\}.$$

T is an exponential random variable with intensity λ . Moreover, $Y_t = X_t$ for $t < T$ and $\{t < \tau_D^Y, t < T\} = \{t < \tau_D^X, t < T\}$ where $\tau_D^X := \inf\{t > 0 : X_t \notin D\}$ and $\tau_D^Y := \inf\{t > 0 : Y_t \notin D\}$. Thus, since Y and T are independent, for every open subsets U and D with $U \subset D$ we have

$$\begin{aligned} \mathbf{P}(Y_t^D \in U | Y_0 = x) \mathbf{P}(T > t) &= \mathbf{P}(Y_t \in U, t < \tau_D^Y, t < T | Y_0 = x) \\ &= \mathbf{P}(X_t \in U, t < \tau_D^X, t < T | X_0 = x) \end{aligned} \quad (4.7)$$

$$\leq \mathbf{P}(X_t^D \in U, t < \tau_D^X | X_0 = x). \quad (4.8)$$

One can find a similar argument for symmetric Lévy processes in the proof of Lemma 2.5 in [13].

From (4.8) with $D = \mathbf{R}^d$, we have

$$q(t, x, y) \leq e^{\lambda t} p(t, x, y), \quad \forall (t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d. \quad (4.9)$$

Combining (4.3), (4.4) and (4.9), we get

$$q(t, x, y) \leq c e^{\lambda t} t^{-\frac{d}{\alpha}}, \quad \forall (t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d \quad (4.10)$$

and

$$q(t, x, y) \leq c t e^{\lambda t}, \quad |x - y| \geq \gamma, t > 0. \quad (4.11)$$

Using the facts above, one can follow routine arguments (see, for instance, the proof of Theorem 2.4 in [10]) to show that, for every open subset D , the killed process Y^D has a transition density $q^D(t, x, y)$ such that, for any $t > 0$, $q^D(t, x, y)$ is jointly continuous on $D \times D$.

Due to our assumption on the Lévy measure of Y , $q^D(t, x, y)$ may not be strictly positive without further assumption on the open set D . Now we are going to show that, when D is a bounded roughly connected open set, $q^D(t, x, y)$ is strictly positive on $(0, \infty) \times D \times D$. Thus in this case, (A2), (A3) and (A5) are satisfied.

Definition 4.3. *We say that an open set D in \mathbf{R}^d is roughly connected if for every $x, y \in D$, there exist distinct connected components $U_1 \cdots, U_m$ of D such that $x \in U_1$, $y \in U_m$ and $\text{dist}(U_k, U_{k+1}) < 1$ for $1 \leq k \leq m - 1$.*

Proposition 4.4. *For every bounded roughly connected open set D , the transition density function $q^D(t, x, y)$ for Y in D is strictly positive in $(t, x, y) \in (0, \infty) \times D \times D$.*

Proof. We prove the proposition in several steps.

- (1) We first assume that $\text{diam}(D) < 1$. Fix $t > 0$. We recall from (4.7) that for every non-empty open set $U \subset D$

$$\mathbf{P}_x(Y_{t/2}^D \in U) = e^{\lambda t/2} \mathbf{P}(X_{t/2} \in U, t/2 < \tau_D^X, t/2 < T \mid X_0 = x).$$

Note that by (4.6), we know Z_t makes jumps with sizes great than or equal to 1 only. Thus, since $\text{diam}(D) < 1$, $\{t/2 < \tau_D^X, t/2 < T\} = \{t/2 < \tau_D^X\}$, which implies that

$$\int_U q^D(t/2, x, y) dy = \mathbf{P}_x(Y_{t/2}^D \in U) = e^{\lambda t/2} \mathbf{P}_x(X_{t/2}^D \in U) > 0.$$

Thus for each $x \in D$, $q^D(t/2, x, y) > 0$ for a.e. $y \in D$. Similarly,

$$\int_U q^D(t/2, x, y) dx = \mathbf{P}_y(\widehat{Y}_{t/2}^D \in U) = e^{\lambda t/2} \mathbf{P}_y(\widehat{X}_{t/2}^D \in U) > 0.$$

Thus, for each $y \in D$, $q^D(t/2, x, y) > 0$ for a.e. $x \in D$. Therefore the semigroup property implies that

$$q^D(t, x, y) = \int_D q^D(t/2, x, z) q^D(t/2, z, y) dz$$

is strictly positive for $(x, y) \in D \times D$ in this case.

- (2) Now we assume that D is connected. If $x, y \in D \cap B(x_0, r)$ where $x_0 \in D$ and $r < 1/2$, then by (1)

$$q^D(t, x, y) \geq q^{D \cap B(x_0, 1/2)}(t, x, y) > 0. \quad (4.12)$$

Thus by the semigroup property and (4.12), for $y \in B(x, 1/2)$,

$$\begin{aligned} q^D(t, x, y) &= \int_D q^D(t/2, x, z) q^D(t/2, z, y) dz \\ &\geq \int_{D \cap B(x, 1/2)} q^D(t/2, x, z) q^D(t/2, z, y) dz > 0. \end{aligned} \quad (4.13)$$

Using this and a simple chain argument one can easily show that $q^D(t, x, y)$ is strictly positive on $(0, \infty) \times D \times D$ in this case.

- (3) Finally we deal with the general case that D is a roughly connected open set. Fix $x, y \in D$. There exist distinct connected components U_1, \dots, U_m of D and $\varepsilon > 0$ such that $x \in U_1$, $y \in U_m$ and $\text{dist}(U_k, U_{k+1}) < 1 - 4\varepsilon$ for $1 \leq k \leq m - 1$. Choose points $x_k^1, x_k^2 \in U_k$ and $\delta_k^1, \delta_k^2 < \varepsilon$ where $1 \leq k \leq m$ such that $x = x_1^1$, $y = x_m^2$, $|x_k^2 - x_{k+1}^1| < 1 - 2\varepsilon$ and

$$V_{k,k+1} := B(x_k^2, \delta_k^2) \cup B(x_{k+1}^1, \delta_{k+1}^1) \subset U_k \cup U_{k+1},$$

for $1 \leq k \leq m - 1$. Let $t_m := t/(2m - 1)$. Now by the semigroup property,

$$\begin{aligned} &q^D(t, x, y) \\ &= \int_D \cdots \int_D q^D(t_m, x_1^1, y_1^2) q^D(t_m, y_1^2, y_2^1) \cdots q^D(t_m, y_{k-1}^1, y_k^2) \\ &\quad \times q^D(t_m, y_k^2, y_{k+1}^1) \cdots q^D(t_m, y_{m-1}^1, y_m^2) q^D(t_m, y_m^2, x_m^1) dy_1^2 dy_2^1 \cdots dy_{m-1}^1 dy_m^2 \\ &\geq \int_{U_1} q^D(t_m, x_1^1, y_1^2) \int_{V_{1,2}} q^D(t_m, y_1^2, y_2^1) \cdots \int_{U_k} q^D(t_m, y_k^1, y_k^2) \\ &\quad \times \int_{V_{k,k+1}} q^D(t_m, y_k^2, y_{k+1}^1) \cdots \int_{V_{m-1,m}} q^D(t_m, y_{m-1}^1, y_m^2) \\ &\quad \times \int_{U_m} q^D(t_m, y_m^2, x_m^1) dy_1^2 dy_2^1 \cdots dy_{m-1}^1 dy_m^2, \end{aligned}$$

which is strictly positive in $D \times D$ by (1)-(2).

□

The result above will be used in [19] to study the Martin boundary of truncated symmetric stable processes.

Example 4.5. Suppose that X is a strictly α -stable process in \mathbf{R}^d satisfying all the assumptions in Example 4.1, that B is a Brownian motion in \mathbf{R}^d and that X and B are independent. Then the process Z defined by $Z_t = B_t + X_t$ is also a Lévy process and it obviously satisfies (A1)(a). The transition density $q(t, x, y)$ of Z is given by the convolution of the transition densities of B and X . Using this, the explicit formula for the transition density of B , and (4.3) and (4.4) for the transition density of X , we can easily show that there exists $c > 0$ such that

$$q(t, x, y) \leq ct^{-\frac{d}{\alpha}}, \quad \forall (t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d. \quad (4.14)$$

Moreover, for any $\gamma > 0$, there exists $c > 0$ such that

$$q(t, x, y) \leq ct, \quad |x - y| \geq \gamma, t > 0. \quad (4.15)$$

Using the facts above, one can follow routine arguments (see, for instance, the proof of Theorem 2.4 in [10]) to show that, for every open subset D , the killed process Z^D has a transition density $q^D(t, x, y)$ such that, for any $t > 0$, $q^D(t, x, y)$ is jointly continuous on $D \times D$. It follows from the lemma below that $q^D(t, x, y)$ is strictly positive on $(0, \infty) \times D \times D$. Thus for any bounded open subset D , Z^D satisfied (A2), (A3) and (A5).

Lemma 4.6. *Suppose that Z is the process in Example 4.5 and that $q(t, x, y)$ is the transition density of Z . Then for any bounded open set D in \mathbf{R}^d , the transition density $q^D(t, x, y)$ of Z^D is strictly positive on $(0, \infty) \times D \times D$.*

Proof. For any bounded domain V and bounded open set U , let $p_1^V(t, x, y)$ and $p_2^U(t, x, y)$ be the density of the killed Brownian motion B^V and the killed strictly α -stable process X^U respectively. Note that the above densities are strictly positive.

Without loss of generality we assume $B_0 = 0$. For $x \in D$, let δ_x be a positive constant with $B(x, 2\delta_x) \subset D$. We will show that for every $B(x_0, \varepsilon) \subset D$, $\mathbf{P}_x(Z_t^D \in B(x_0, \varepsilon)) > 0$.

Choose $\delta = \delta(x_0, x, \varepsilon) < \delta_x$ such that $B(x_0, \varepsilon) \subset B(x_0, \varepsilon + \delta) \subset D$ and let $U := B(x_0, \varepsilon) \cup B(x, \delta_x)$. Then

$$\begin{aligned} \mathbf{P}_x(Z_t^D \in B(x_0, \varepsilon)) &= \mathbf{P}_x(B_t + X_t \in B(x_0, \varepsilon), \tau_D^Z > t) \\ &\geq \mathbf{P}_x(B_t + X_t \in B(x_0, \varepsilon), \tau_{B(0, \delta)}^B > t, \tau_U^X > t) \\ &= \int_{B(0, \delta)} \int_{B(x_0, \varepsilon)} p_2^{U+y}(t, x+y, z) p_1^{B(0, \delta)}(t, 0, y) dz dy, \end{aligned}$$

which is strictly positive. This implies that, for $t < \infty$ and $x \in D$, $q^D(t, x, \cdot)$ is strictly positive almost everywhere on D . By working with the dual process we get that for $t < \infty$ and $y \in D$, $q^D(t, \cdot, y)$ is strictly positive almost everywhere on D . Combining these with the semigroup property we get that $q^D(t, x, y)$ is strictly positive everywhere on $(0, \infty) \times D \times D$. \square

Example 4.7. Suppose that Y is a truncated strictly α -stable process in \mathbf{R}^d satisfying all the assumptions in Example 4.2, that B is a Brownian motion in \mathbf{R}^d and that Y and B are independent. Then the process Z defined by $Z_t = B_t + Y_t$ is also a Lévy process and it obviously satisfies (A1)(b). The transition density $k(t, x, y)$ of Z is given by the convolution of the transition densities of B and Y . Using this, the explicit formula for the transition density of B , and (4.10) and (4.11) for the transition density of Y , we can easily show that there exist $c > 0$ and $\lambda > 0$ such that

$$k(t, x, y) \leq c e^{\lambda t} t^{-\frac{d}{\alpha}}, \quad \forall (t, x, y) \in (0, \infty) \times \mathbf{R}^d \times \mathbf{R}^d. \quad (4.16)$$

Moreover, for any $\gamma > 0$, there exist $c > 0$ and $\lambda > 0$ such that

$$k(t, x, y) \leq c t e^{\lambda t}, \quad |x - y| \geq \gamma, t > 0. \quad (4.17)$$

Using the facts above, one can follow routine arguments (see, for instance, the proof of Theorem 2.4 in [10]) to show that, for every open subset D , the killed process Z^D has a transition density $k^D(t, x, y)$ such that, for any $t > 0$, $k^D(t, x, y)$ is jointly continuous on $D \times D$. It follows from the lemma below that for every bounded roughly connected open set D , $k^D(t, x, y)$ is strictly positive on $(0, \infty) \times D \times D$. Thus for any bounded roughly connected open subset D , Z^D satisfied (A2), (A3) and (A5).

Lemma 4.8. *Suppose that Z is the process in Example 4.7 and that $k(t, x, y)$ is the transition density of Z . Then for any bounded roughly connected open set D in \mathbf{R}^d , the transition density $k^D(t, x, y)$ of Z^D is strictly positive on $(0, \infty) \times D \times D$.*

Proof. For any bounded domain V and bounded open set U , let $p_1^V(t, x, y)$ and $p_2^U(t, x, y)$ be the density of the killed Brownian motion B^V and the killed truncated α -stable process Y^U respectively. Note that if $\text{diam}(U) < 1$, the above densities are strictly positive (Proposition 4.4). Thus through the same argument in the proof of Lemma 4.6, we have that for every open subset D with $\text{diam}(D) < 1$, $k^D(t, x, y)$ is strictly positive. Now following the step (2)-(3) in the proof of Proposition 4.4, we conclude that for any bounded roughly connected open set D , $k^D(t, x, y)$ is strictly positive on $(0, \infty) \times D \times D$. \square

We list here more examples of Lévy processes X and open sets D so that X^D satisfies the assumptions (A1)-(A5) without giving proofs. One can prove them easily using arguments similar to those in the previous examples and induction.

If $X^{(j)}, j = 1, \dots, n$, are independent strictly α_j -stable processes satisfying the assumptions of Example 4.1. Then the process X defined by $X_t = X_t^{(1)} + \dots + X_t^{(n)}$ is a Lévy process satisfying (A1)(a). For any bounded open subset of \mathbf{R}^d , the killed process X^D satisfies (A2), (A3) and (A5). Similarly, if $X^{(j)}, j = 1, \dots, n$, are independent truncated strictly α_j -stable processes all satisfying the assumptions of Example 4.1. Then the process X defined by $X_t = X_t^{(1)} + \dots + X_t^{(n)}$ is a Lévy process satisfying (A1)(b). For any bounded roughly connected open subset of \mathbf{R}^d , the killed process X^D satisfies (A2), (A3) and (A5). Of course, one can combine the examples above with Brownian motion to get more examples.

Acknowledgment: This paper was finalized while the first named author was visiting KIAS in Seoul, the hospitality of which is acknowledged.

References

- [1] R. Bañuelos, Intrinsic ultracontractivity and eigenfunction estimates for Schrödinger operators. *J. Funct. Anal.* **100** (1991), 181-206.
- [2] R. F. Bass, *Probabilistic Techniques in Analysis*. Springer-Verlag, 1995.
- [3] R. F. Bass and K. Burdzy, Lifetimes of conditioned diffusions. *Probab. Theory Related Fields* **91** (1992), 405-443.
- [4] J. Bertoin, *Lévy processes*, Cambridge University Press, Cambridge, 1996.
- [5] R. M. Blumenthal and R. K. Gettoor, *Markov Processes and Potential Theory*. Academic Press, 1968.
- [6] Z.-Q. Chen and R. Song, Intrinsic ultracontractivity and conditional gauge for symmetric stable processes. *J. Funct. Anal.* **150** (1997), 204-239.
- [7] Z.-Q. Chen and R. Song, Intrinsic ultracontractivity, conditional lifetimes and conditional gauge for symmetric stable processes on rough domains. *Illinois J. Math.* **44(1)** (2000), 138-160.
- [8] K. L. Chung, Greenian bounds for Markov processes, *Potential Anal.* **1(1)** (1992), 83-92.
- [9] K. L. Chung and J. B. Walsh, *Markov processes, Brownian motion, and time symmetry*. Springer, New York, 2005
- [10] K. L. Chung and Z. X. Zhao, *From Brownian motion to Schrödinger's equation*, Springer-Verlag, Berlin, 1995.
- [11] E. B. Davies and B. Simon. Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians. *J. Funct. Anal.* **59** (1984), 335-395.

- [12] T. Grzywny, Intrinsic ultracontractivity for Lévy processes, Preprint, 2006.
- [13] T. Grzywny and M. Ryznar, Estimates of Green function for some perturbations of fractional Laplacian, *Ill. J. Math.* (to appear), 2007.
- [14] N. Ikeda and S. Watanabe, On some relations between the harmonic measure and the Lévy measure for a certain class of Markov processes, *J. Math. Kyoto Univ.* **2** (1962), 79–95.
- [15] D. S. Jerison and C. E. Kenig, Boundary behavior of harmonic functions in non-tangentially accessible domains. *Adv. Math.*, **46** (1982), 80–147.
- [16] P. Kim and R. Song, Potential theory of truncated stable processes. *Math. Z.*, **256** (2007), 139–173.
- [17] P. Kim and R. Song, Intrinsic ultracontractivity of non-symmetric diffusion semigroups in bounded domains. *Tohoku Math. J.*, (to appear), 2008.
- [18] P. Kim and R. Song, Intrinsic ultracontractivity of non-symmetric diffusions with measure-valued drifts and potentials. *Ann. Probab.*, **36** (2008), 1904–1945
- [19] P. Kim and R. Song, Boundary behavior of harmonic functions for truncated stable processes. *J. Theoret. Probab.*, **21** (2008), 287–321.
- [20] T. Kulczycki, Intrinsic ultracontractivity for symmetric stable processes, *Bull. Polish Acad. Sci. Math.* **46(3)** (1998), 325–334.
- [21] O. Martio and M. Vuorinen, Whitney cubes, p -capacity, and Minkowski content. *Exposition. Math.*, **5(1)** (1987), 17–40.
- [22] K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge, 1999.
- [23] H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, New York, 1974.
- [24] R. Song and J. Wu, Boundary Harnack principle for symmetric stable processes, *J. Funct. Anal.* **168(2)** (1999), 403–427.
- [25] Zoran, Vondraček, Basic potential theory of certain nonsymmetric strictly α -stable processes. *Glas. Mat. Ser. III* **37(57)** (2002), 211–233.