

On the monotonicity of a function related to the local time of a symmetric Lévy process

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Abstract

Let ψ be the characteristic exponent of a symmetric Lévy process X . The function

$$h(x) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{\psi(\lambda)} d\lambda$$

appears in various studies on the local time of X . We study monotonicity properties of the function h . In case when X is a subordinate Brownian motion, we show that $x \mapsto h(\sqrt{x})$ is a Bernstein function.

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1 Introduction

Let X be a symmetric Lévy process in \mathbb{R} with the characteristic exponent ψ , i.e.,

$$\mathbb{E}e^{i\lambda X_t} = e^{-t\psi(\lambda)}.$$

Throughout this paper we assume that the point 0 is regular for itself, and that the characteristic exponent ψ satisfies

$$\int_0^\infty \frac{1}{1 + \psi(\lambda)} < \infty. \quad (1)$$

These two conditions guarantee that the process X admits a local time $L(0, t)$ at zero. Let $T_x = \inf\{s > 0 : X_s = x\}$ be the hitting time to $x \in \mathbb{R}$, and let

$$h(x) := \mathbb{E}(L(0, T_x)).$$

Then by Lemma 11 in Chapter 5 of [2]

$$h(x) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{\psi(\lambda)} d\lambda. \quad (2)$$

This function appears often in studies of the local time of Lévy processes. For instance, a monotone rearrangement of this function was used in [1] to formulate necessary and sufficient conditions for the joint continuity of the local time. In his study on the most visited sites of X [5], M. B. Marcus assumed that the function h is strictly increasing on $[0, \infty)$. This assumption on h is not always easy to check. In Section 5 of [5], Marcus showed that the so called stable mixtures satisfy the assumption.

The purpose of this note is to better understand the monotonicity of the function h , and to provide more examples of strictly increasing h . We also show that for subordinate Brownian motions, $x \mapsto h(\sqrt{x})$ is, in fact, a Bernstein function. Under a reasonable additional assumption, it is even a complete Bernstein function.

We start with a simple sufficient condition for h to be increasing. To this end we first rewrite the function h in a different way. It follows from Theorem 16 and Theorem 19 in Chapter 2 of [2] that under the assumptions stated in the first paragraph, the q -potential measure U^q of X has a density u^q which is bounded and continuous. From the proof of Lemma 11 in Chapter 5 of [2] we see that h defined by (2) may be written as

$$h(x) = 2 \lim_{q \downarrow 0} (u^q(0) - u^q(x)), \quad x \in \mathbb{R}. \quad (3)$$

Thus if we know that for any $q > 0$, the function u^q is decreasing in $[0, \infty)$, then the equation above immediately gives us that h is increasing in $[0, \infty)$. Using this fact and Theorem 54.2 of [9] we immediately get the following

Proposition 1.1 *If the Lévy measure ν of the process X is given by*

$$\nu(dx) = n(x)dx,$$

for some even function n which is decreasing on $(0, \infty)$, then h is increasing on $[0, \infty)$.

Proof. It follows from Theorem 54.2 of [9] that when the Lévy measure ν of the process X is given by

$$\nu(dx) = n(x)dx$$

for some even function n which is decreasing on $(0, \infty)$, the distribution of X_t is unimodal with mode 0 for every $t > 0$. This implies that, for any $q > 0$, u^q is a decreasing function in $[0, \infty)$. Therefore h is increasing on $[0, \infty)$. \square

2 Subordinate Brownian motion

In this section we first make a comment that a subordinate Brownian motion satisfies condition of Proposition 1.1, and then prove that a much stronger result than Proposition 1.1 holds in this case. Let us begin by recalling relevant definitions.

Let $T = (T_t : t \geq 0)$ be a subordinator with Laplace exponent f , that is,

$$\mathbb{E}e^{-\lambda T_t} = e^{-tf(\lambda)},$$

and let $B = (B_t : t \geq 0)$ be a Brownian motion with generator $\frac{d^2}{dx^2}$. If B and T are independent, then the process $X_t := B(T_t)$ is called a subordinate Brownian motion with subordinator T . It is well known (see, for instance, page 197 of [9]) that the characteristic exponent of this subordinate Brownian motion satisfies $\psi(\lambda) = f(\lambda^2)$, that is,

$$\mathbb{E}e^{i\lambda X_t} = e^{-tf(\lambda^2)}.$$

We still assume that (1) holds. This implies that $\lim_{\lambda \rightarrow \infty} f(\lambda) = \infty$, which means that T is not a compound Poisson process. It is well known that the Lévy measure of subordinate Brownian motion has the density n given by

$$n(x) = \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} \mu(dt),$$

where μ is the Lévy measure of the subordinator. Clearly, n is even and decreasing on $(0, \infty)$, hence the assumption of Proposition 1.1 is satisfied.

The Laplace exponent f is a Bernstein function, that is, $f \in C^\infty(0, \infty)$, and satisfies $(-1)^n D^n f \leq 0$ for every $n \in \mathbb{N}$. Any Bernstein function f can be written in the form

$$f(\lambda) = a + b\lambda + \int_0^\infty (1 - e^{-\lambda t})\mu(dt),$$

where a, b are nonnegative numbers and μ is a σ -finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (t \wedge 1) \mu(dt) < \infty.$$

The measure μ is called the Lévy measure of the Bernstein function f . For more results on Bernstein functions, the reader is referred to [4]. It follows from the representation of Bernstein functions above that a non-constant Bernstein function is strictly increasing. We will also need the notion of a complete Bernstein function: A function $f : (0, \infty) \rightarrow [0, \infty)$ is called a complete Bernstein function if there exists a Bernstein function g such that

$$f(\lambda) = \lambda^2 \mathcal{L}g(\lambda), \quad \lambda > 0,$$

where \mathcal{L} stands for the Laplace transform. A complete Bernstein function is a Bernstein function. The family of all complete Bernstein functions is a convex cone containing positive constants and it is closed under compositions. For more on complete Bernstein functions see [4].

Let V be the potential measure of T , that is,

$$V(A) = \mathbb{E} \int_0^\infty 1_{\{T_t \in A\}} dt.$$

Then it is well known (see, e.g., [2], p. 74) that

$$\frac{1}{f(\lambda)} = \int_0^\infty e^{-\lambda t} dV(t), \quad \lambda > 0. \quad (4)$$

Proposition 2.1 *The function $\phi : [0, \infty) \rightarrow [0, \infty)$, defined by*

$$\phi(x) := h(\sqrt{x}) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda\sqrt{x})}{f(\lambda^2)} d\lambda, \quad (5)$$

is a Bernstein function.

Proof. Using (4) we get

$$\begin{aligned} h(x) &= \frac{2}{\pi} \int_0^\infty (1 - \cos(\lambda x)) \int_0^\infty e^{-\lambda^2 t} dV(t) d\lambda \\ &= \frac{2}{\pi} \int_0^\infty dV(t) \int_0^\infty (1 - \cos(\lambda x)) e^{-\lambda^2 t} d\lambda \\ &= \frac{1}{\pi} \int_0^\infty dV(t) \int_0^\infty q^{-1/2} (1 - \cos(x\sqrt{q})) e^{-qt} dq \\ &= \frac{1}{\pi} \int_0^\infty dV(t) \left(\int_0^\infty q^{-1/2} e^{-qt} dq - \int_0^\infty q^{-1/2} \cos(x\sqrt{q}) e^{-qt} dq \right) \end{aligned} \quad (6)$$

Using formula (67) on page 158 of [3] we see that

$$\int_0^\infty q^{-1/2} \cos(x\sqrt{q}) e^{-qt} dq = \sqrt{\frac{\pi}{t}} e^{-\frac{x^2}{4t}}, \quad (7)$$

thus we have

$$h(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} (1 - e^{-\frac{x^2}{4t}}) dV(t).$$

Consequently we have

$$\phi(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{t}} (1 - e^{-\frac{x^2}{4t}}) dV(t).$$

Let \tilde{V} be the image measure of V with respect to the mapping $t \mapsto 1/4t$. Then

$$\begin{aligned} \phi(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \sqrt{4t} (1 - e^{-xt}) d\tilde{V}(t) \\ &= \frac{2}{\sqrt{\pi}} \int_0^\infty (1 - e^{-xt}) t^{1/2} d\tilde{V}(t) \end{aligned} \quad (8)$$

It is straightforward to check that the measure $t^{1/2} d\tilde{V}(t)$ is a Lévy measure, thus proving that ϕ is a Bernstein function. \square

Remark 2.2 Let $p(t, x) = (1/\sqrt{4\pi t}) \exp\{-x^2/4t\}$ be the transition density of Brownian motion B . By use of (6) and (7), we may rewrite the formula for the function h in the following explicit form:

$$h(x) = 2 \int_0^\infty (p(t, 0) - p(t, x)) dV(t). \quad (9)$$

This formula should be compared with the formula for the compensated potential density of Brownian motion.

In the same spirit as above we can show that, for any $q > 0$, the q -potential density $u^q(x)$ of the subordinate process X is strictly decreasing on $[0, \infty)$. Indeed, let $q > 0$, and let V^q denote the potential measure of the subordinator T killed at an independent exponential time with parameter q . Then

$$u^q(x) = \int_0^\infty p(t, x) dV^q(t).$$

This formula proves clearly that $x \rightarrow u^q(x)$ is strictly decreasing on $[0, \infty)$.

Proposition 2.1 can be strengthened as follows.

Proposition 2.3 *Suppose that T is a subordinator with Laplace exponent f such that*

$$\frac{x}{f(x)} = x^2 \mathcal{L}g(x)$$

for some Bernstein function g . If g is given by

$$g(x) = \int_0^\infty (1 - e^{-tx}) \rho(t) dt \tag{10}$$

for some Lévy density ρ such that $t\rho(t)$ is decreasing on $(0, \infty)$, then the function ϕ defined in (5) is a complete Bernstein function.

Proof. Since T corresponds to a complete Bernstein function and is not a compound Poisson process, we know from [8] that

$$dV(t) = v(t) dt$$

where v is a locally integrable decreasing function on $(0, \infty)$. The formula (8) tells us that under the present assumption we have

$$\phi(x) = \frac{1}{2\sqrt{\pi}} \int_0^\infty (1 - e^{-qx}) \frac{1}{q^{3/2}} v\left(\frac{1}{4q}\right) dq.$$

To show that ϕ is a complete Bernstein function, it suffices to show that the function

$$\frac{1}{2\sqrt{\pi}} \frac{1}{q^{3/2}} v\left(\frac{1}{4q}\right)$$

is the Laplace transform of some positive function. Assumption (10) implies that

$$g'(x) = \int_0^\infty e^{-tx} t \rho(t) dt.$$

From the proof of Theorem 2.3 in [8] we know that $v(x) = g'(x)$ and so v is the Laplace transform of the function $t\rho(t)$. Therefore by formula (30) of [3] we know that the function

$$\pi^{1/2} \lambda^{-3/2} v(\lambda^{-1})$$

is the Laplace transform of the function

$$\int_0^\infty \sin(2s^{1/2}t^{1/2}) s^{1/2} \rho(s) ds.$$

The function above can be rewritten as

$$2 \int_0^\infty \sin(2t^{1/2}r) r^2 \rho(r^2) dr.$$

Now splitting the integral above into a sum over intervals $(\frac{n\pi}{\sqrt{t}}, \frac{(n+1)\pi}{\sqrt{t}})$, $n = 0, 1, \dots$, and using the assumption that $r\rho(r)$ is decreasing, it follows easily that the preceding function is positive. Using this and properties of the Laplace transform it follows that the function

$$\frac{1}{2\sqrt{\pi}} \frac{1}{q^{3/2}} v\left(\frac{1}{4q}\right)$$

is the Laplace transform of some positive function. □

3 Examples

We first recall an example from [5] and show that it fits into the setting of Section 2. Take the Laplace exponent

$$f(\lambda) = \int_{1/2}^1 \lambda^s d\xi(s), \quad \lambda > 0$$

with ξ a finite measure on $(\frac{1}{2}, 1]$. Then

$$\psi(\lambda) = f(\lambda^2),$$

is a function of the type given in (5.1) of [5]. The function h is strictly increasing on $[0, \infty)$, and $x \mapsto h(\sqrt{x})$ is a Bernstein function.

The Lévy process with characteristic exponent ψ given by the above example is called a stable mixtures and it has been studied by Marcus and Rosen in [6] and [7]. We give now several examples of different type.

Example 3.1 Let

$$f(\lambda) = (\lambda^\alpha + 1)^\beta - 1$$

for $0 < \alpha \leq 1$ and $0 < \beta < 1$. Being a composition of complete Bernstein function, f itself is a Bernstein function. When $\alpha = 1$, the corresponding subordinator is a relativistic β -stable subordinator. In order for (1) to be satisfied, we assume that $\alpha\beta > 1/2$. By Proposition 2.1, the function

$$h(x) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos(\lambda x)}{f(\lambda^2)} d\lambda$$

is strictly increasing. Moreover, the characteristic exponent $\psi(\lambda) = f(\lambda^2)$ is regularly varying at 0 with index 2α . In [5], Marcus assumed another condition, namely that ψ is regularly varying at zero with index $\alpha \in (1, 2]$. Hence, by assuming $1/2 < \alpha \leq 1$, we see that ψ is regularly varying with index $\alpha \in (1, 2)$. On the other hand, since ψ is regularly varying at infinity with index $2\alpha\beta < 2\alpha$, it cannot be a stable mixture (see [7], Lemma 7.1). This example is not included in [5].

We describe now a class of examples that satisfy the assumptions of Proposition 2.3. This class of examples is also not included in [5] in general. Let μ be a Lévy measure on $(0, \infty)$, i.e.,

$$\int_0^\infty (1 \wedge t) \mu(dt) < \infty.$$

Define $g : (0, \infty) \rightarrow (0, \infty)$ by

$$g(x) := \int_0^\infty (1 - e^{-tx}) \mu(dt).$$

Clearly, g is a Bernstein function. An easy calculation shows that

$$\mathcal{L}g(\lambda) = \int_0^\infty \frac{1}{\lambda(\lambda + t)} t \mu(dt).$$

Therefore

$$\lambda^2 \mathcal{L}g(\lambda) = \lambda \int_0^\infty \frac{1}{\lambda + t} t \mu(dt).$$

Define $k : (0, \infty) \rightarrow (0, \infty)$ by

$$k(\lambda) := \lambda \int_0^\infty \frac{1}{\lambda + t} t \mu(dt).$$

Then

$$\frac{k(\lambda)}{\lambda} = \int_0^\infty \frac{1}{\lambda + t} t \mu(dt),$$

is a Stieltjes function. By Theorem 3.9.29 in [4], k is a complete Bernstein function. Define $f : (0, \infty) \rightarrow (0, \infty)$ by $f(\lambda) := \lambda/k(\lambda)$. By the same theorem, f is a complete Bernstein function. But,

$$\frac{\lambda}{f(\lambda)} = k(\lambda) = \lambda^2 \mathcal{L}g(\lambda)$$

for g of the form in Proposition 2.3. This shows that for any Lévy measure μ and g defined as above, the function $f(\lambda)$ defined by $\lambda/f(\lambda) := \lambda^2 \mathcal{L}g(\lambda)$ is a complete Bernstein function.

Suppose, additionally, that $\mu(dt) = \rho(t) dt$ where $\rho : (0, \infty) \rightarrow (0, \infty)$ is such that $t\rho(t)$ is decreasing. By Proposition 2.3, the corresponding ϕ is a complete Bernstein function.

Example 3.2 Let ξ be a finite measure on $(1, 2)$ with compact support. Define

$$\rho(t) = \int_1^2 t^{-\beta} \xi(d\beta).$$

Clearly, $t\rho(t)$ is decreasing. Since

$$\int_0^\infty \frac{t^{1-\beta}}{t+x} dt = \left(-\frac{\pi}{\sin \beta\pi} \right) x^{1-\beta},$$

it follows that

$$\begin{aligned} \int_0^\infty \frac{1}{t+\lambda} t \rho(t) dt &= \int_1^2 \int_0^\infty \frac{t^{1-\beta}}{t+\lambda} dt \xi(d\beta) \\ &= \int_1^2 \left(-\frac{\pi}{\sin \beta \pi} \right) \lambda^{1-\beta} \xi(d\beta). \end{aligned}$$

Therefore,

$$k(\lambda) = \lambda \int_0^\infty \frac{1}{t+\lambda} t \rho(t) dt = \int_1^2 \left(-\frac{\pi}{\sin \beta \pi} \right) \lambda^{2-\beta} \xi(d\beta)$$

and

$$f(\lambda) = \frac{\lambda}{k(\lambda)} = \frac{\lambda}{\int_1^2 \left(-\frac{\pi}{\sin \beta \pi} \right) \lambda^{2-\beta} \xi(d\beta)}.$$

The corresponding $\psi(\lambda) = f(\lambda^2)$ is of the form

$$\psi(\lambda) = \frac{\lambda^2}{\int_1^2 \left(-\frac{\pi}{\sin \beta \pi} \right) \lambda^{4-2\beta} \xi(d\beta)}.$$

Moreover, if the support of the measure ξ is contained in $(3/2, 2)$, then ψ is regularly varying at zero with index $\alpha \in (1, 2)$.

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