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Sharp Estimates on the Green Functions of Perturbations of Subordinate Brownian Motions in Bounded κ-Fat Open Sets

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Abstract In this paper we study perturbations of a large class of subordinate Brownian motions in bounded κ-fat open sets, which include bounded John domains. Suppose that X is such a subordinate Brownian motion and that J is the Lévy density of X. The main result of this paper implies, in particular, that if Y is a symmetric Lévy process with Lévy density J^Y satisfying $|J^Y(x) - J(x)| \le c \max\{|x|^{-d+\rho}, 1\}$ for some $c > 0$, $\rho \in (0, d)$, then for any bounded John domain D the Green function G_D^Y of Y in D is comparable to the Green function G_D of X in D. One of the main tools of this paper is the drift transform introduced in Chen and Song (J Funct Anal 201:262–281, 2003). To apply the drift transform, we first establish a generalized 3G theorem for X.

Keywords Green functions · Subordinate Brownian motion · 3G theorem · Generalized 3G theorem · Harmonic functions · Harnack inequality · Boundary Harnack principle

Mathematics Subject Classifications (2010) Primary 60J45; Secondary 60J25 · 60J51

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1 Introduction

This paper is a natural continuation of [21–23]. In [21, 22], a boundary Harnack principle for a large class of subordinate Brownian motions was proved, and in [23], sharp two-sided estimates on the Green functions of these subordinate Brownian motions in bounded κ-fat sets were established. The goal of this paper is, first, to extend and prove the generalized 3G theorem (first proved in [19] for symmetric stable processes) and, second, to establish sharp two-sided Green function estimates for a large class of Lévy processes which can be considered as perturbations of the subordinate Brownian motions considered in [21–23].

The 3G theorem is a very important tool in studying (local) Schrödinger operators. It was established for Brownian motion in bounded Lipschitz domains for $d \geq 3$ in [5]. Later it was extended to bounded uniformly John domains for $d \geq 3$ in [1] (See [2, 15, 27, 32] for $d = 2$). For symmetric α-stable processes, $\alpha \in (0, 2)$, it was proved for bounded $C^{1,1}$ domains in [7, 8, 24]. More precisely, it was proved in [7, 8, 24] that for every $d > \alpha$ and any bounded $C^{1,1}$ domain D there exists a positive constant $c = c(D, \alpha)$ such that

$$\frac{\widetilde{G}_D(x, y)\widetilde{G}_D(y, z)}{\widetilde{G}_D(x, z)} \leq c\frac{|x - z|^{d-\alpha}}{|x - y|^{d-\alpha}|y - z|^{d-\alpha}}, \quad x, y, z \in D, \tag{1.1}$$

where $\widetilde{G}_D$ is the Green function of the symmetric α-stable process in D. Later Eq. 1.1 was extended to bounded Lipschitz domains for symmetric α-stable processes ($0 < \alpha < 2$) in [16] and even to bounded κ-fat open sets in [31].

When the process is discontinuous, there is a large class of additive functionals which are not continuous. Such additive functionals give rise to a large family of non-local Schrödinger operators. In order to deal with non-local Schrödinger operators, one needs a generalized 3G theorem, which gives an upper bound on $\widetilde{G}(x, y, z, w) := \widetilde{G}_D(x, y)\widetilde{G}_D(z, w)/\widetilde{G}_D(x, w)$ where y and z can be different (see Theorem 3.16). The generalized 3G theorem was proved in [19] for symmetric stable processes in bounded κ-fat open sets (see also [16]) and it can be stated as that there exist constants $c = c(D, \alpha)$ and $\eta < \alpha$ such that for all $x, y, z, w \in D$

$$\widetilde{G}(x, y, z, w) \leq c\left(\frac{|x - w| \wedge |y - z|}{|x - y|} \vee 1\right)^{\eta}\left(\frac{|x - w| \wedge |y - z|}{|z - w|} \vee 1\right)^{\eta} \times \frac{|x - w|^{d-\alpha}}{|x - y|^{d-\alpha}|z - w|^{d-\alpha}}. \tag{1.2}$$

We first extend Eq. 1.2 to the subordinate Brownian motions considered in [21–23] in bounded κ-fat open sets D. Then we use this generalized 3G theorem to find concrete sufficient conditions for the Kato classes of the subordinate Brownian motions considered in [21–23] (See Theorems 4.4 and 4.5).

Sharp two-sided Green function estimates for a large class of subordinate Brownian motions X in κ-fat sets D were established in [23]. The main goal of this paper is to extend this result to more general Lévy processes. We prove that, for any symmetric Lévy process Y which can be considered as a perturbation of the process X considered in [23], the Green function $G_D(\cdot, \cdot)$ of X in D and its counterpart $G^Y_D(\cdot, \cdot)$ are comparable for any bounded κ-fat domain D. Let J be the Lévy density of X, then the process Y above is a symmetric purely discontinuous Lévy process with Lévy density $J^Y(x) = J(x) + \sigma(x)$ such that $|\sigma(x)| \le c \max\{|x|^{-d+\rho}, 1\}$ for some constants $c > 0$, $\rho \in (0, d)$. Note that our main assumption is on the behavior of the Lévy density of Y near 0 and we do not impose any restriction outside of the unit ball other than that σ is bounded there. The Lévy density of Y may vanish outside of the unit ball. In this case Y only has jumps of size less than 1 and it is a generalization of the truncated stable processes studied in [17, 18]. One of the main tools used in this paper is the drift transform introduced in [11]. We first use the drift transform and our generalized 3G theorem to show that, under the additional assumption that $J^Y(x) \ge J(x)$ for all $x \in \mathbb{R}^d$, G^Y_D is comparable to G_D for any bounded κ-fat (not necessarily connected) open sets D (Theorem 5.6). Then we deal with the general case where σ can take both signs (Theorem 5.13).

The organization of this paper is as follows. In Section 2, we collect some preliminary results on subordinate Brownian motions. Section 3 contains the proof of the generalized 3G theorem. In Section 4 we use the generalized 3G theorem to give sufficient conditions for the Kato classes. The main result is proved in the last section.

In this paper we always assume that $\alpha \in (0, 2)$ and d is a positive integer with $d > \alpha$. We will use the following convention: The values of the constants C_0, C_1, M, $r_0, r_1, r_2, \cdots$ and ε_1 will remain the same throughout this paper, while $c, c_1, c_2, \cdots$ stand for constants whose values are unimportant and which may change from location to location. The labeling of the constants $c_0, c_1, c_2, \cdots$ starts anew in the statement of each result. We use ":=" to denote a definition, which is read as "is defined to be". We denote $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$. $f(t) \asymp g(t)$, $t \to 0$ ($f(t) \asymp g(t)$, $t \to \infty$, respectively) means that the quotient $f(t)/g(t)$ stays bounded between two positive constants as $t \to 0$ (as $t \to \infty$, respectively). For any open set U, we denote by $\delta_U(x)$ the distance of a point x to the boundary of U, i.e., $\delta_U(x) = \text{dist}(x, \partial U)$.

2 Preliminaries

In this section, we define the subordinate Brownian motions we are going to work with and recall some preliminary results about them from [21, 23].

Suppose that $S = (S_t : t \ge 0)$ is a subordinator with Laplace exponent ϕ, that is, S is a nonnegative Lévy process with $S_0 = 0$ and

$$\mathbb{E}\left[e^{-\lambda S_t}\right] = e^{-t\phi(\lambda)}, \qquad \forall\, t, \lambda > 0.$$

In this paper we will always assume that ϕ is a complete Bernstein function, that is, the Lévy measure μ of S has a completely monotone density $\mu(t)$, i.e., $(-1)^n D^n \mu \ge 0$ for every non-negative integer n. For basic results on complete Bernstein functions, we refer our readers to [29]. Recall that a function $\ell : (0, \infty) \to (0, \infty)$ is slowly varying at infinity if

$$\lim_{t\to\infty} \frac{\ell(\lambda t)}{\ell(t)} = 1\,, \quad \text{for every } \lambda > 0\,.$$

We will also always assume that ϕ satisfies the following asymptotic behavior at infinity:

Assumption (H1) There exist $\alpha \in (0, 2 \wedge d)$ and a function $\ell : (0,\infty) \to (0,\infty)$ which is measurable, locally bounded above and below by positive constants and slowly varying at infinity such that

$$\phi(\lambda) \asymp \lambda^{\alpha/2}\ell(\lambda)\,, \quad \lambda \to \infty\,. \tag{2.1}$$

Using [30, Corollary 2.3] or [25, Theorem 2.3] we know that the potential measure of S has a completely monotone density u.

Suppose that $W = (W_t : t \geq 0)$ is a Brownian motion in $\mathbb{R}^d$ with

$$\mathbb{E}\left[e^{i\xi\cdot(W_t - W_0)}\right] = e^{-t|\xi|^2}, \qquad \forall\, \xi \in \mathbb{R}^d, t > 0\,,$$

and that W is independent of S. The process $X = (X_t : t \geq 0)$ defined by $X_t = W_{S_t}$ is called a subordinate Brownian motion. The process X is a (rotationally) symmetric Lévy process with characteristic exponent $\Psi(\xi) = \phi(|\xi|^2)$, $\xi \in \mathcal{R}^d$. It is easy to check that when $d \geq 3$ the process X is transient. In the case $\alpha < d \leq 2$, we will always assume the following:

Assumption (H2) There exists $\gamma \in [0, d/2)$ such that

$$\liminf_{\lambda \to 0} \frac{\phi(\lambda)}{\lambda^\gamma} > 0. \tag{2.2}$$

An immediate consequence of **(H2)** and [22, Corollary 2.6] is that the potential density u of S satisfies $u(t) \leq ct^{\gamma-1}$ for all $t \geq 1$, where $c > 0$ is some positive constant.

Under the assumption **(H2)** the process X is also transient for $d \leq 2$. This ensures that the Green function $G(x, y)$, $x, y \in \mathbb{R}^d$, of X is well defined. By spatial homogeneity we may write $G(x, y) = G(x - y)$, where the function G is radial and given by the following formula

$$G(x) = \int_0^\infty (4\pi t)^{-d/2} e^{-|x|^2/(4t)} u(t)dt, \qquad x \in \mathbb{R}^d.$$

Using this formula we see that G is radially decreasing and continuous in $\mathbb{R}^d \setminus \{0\}$.

The Lévy measure of the process X has a density J, called the Lévy density of X, given by $J(x) = j(|x|)$ where

$$j(r) := \int_0^\infty (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t)dt, \qquad r > 0$$

and $\mu(t)$ is the Lévy density of S. Note that the function $r \mapsto j(r)$ is continuous and decreasing on $(0,\infty)$. We will sometimes use the notation $J(x, y)$ for $J(x - y)$.

The following theorem establishes the asymptotic behaviors of G and j near the origin (see [23, Theorem 2.9, 2.11]).

Theorem 2.1

(i)

$$G(x) \asymp \frac{1}{|x|^d \phi(|x|^{-2})} \asymp \frac{1}{|x|^{d-\alpha}\ell(|x|^{-2})}, \qquad |x| \to 0.$$

(ii)

$$J(x) = j(|x|) \asymp \frac{\phi(|x|^{-2})}{|x|^d} \asymp \frac{\ell(|x|^{-2})}{|x|^{d+\alpha}}, \qquad |x| \to 0.$$

For any open set D, we use τ_D to denote the first exit time of D, i.e., $\tau_D = \inf\{t > 0 : X_t \notin D\}$. Given an open set $D \subset \mathbb{R}^d$, we define $X_t^D(\omega) = X_t(\omega)$ if $t < \tau_D(\omega)$ and $X_t^D(\omega) = \partial$ if $t \geq \tau_D(\omega)$, where ∂ is a cemetery state. X^D is called the killed subordinate Brownian motion X in D. We now recall the definition of harmonic functions with respect to X.

Definition 2.2 Let D be an open subset of $\mathbb{R}^d$. A function u defined on $\mathbb{R}^d$ is said to be

(1) harmonic in D with respect to X if

$$\mathbb{E}_x\left[|u(X_{\tau_B})|\right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x\left[u(X_{\tau_B})\right], \qquad x \in B$$

for every open set B whose closure is a compact subset of D;

(2) regular harmonic in D with respect to X if it is harmonic in D with respect to X and for each $x \in D$,

$$u(x) = \mathbb{E}_x\left[u(X_{\tau_D})\right].$$

The following version of Harnack inequality is [23, Theorem 2.14].

Theorem 2.3 *For any $L > 0$, there exists a positive constant $c = c(d, \phi, L) > 0$ such that the following is true: If $x_1, x_2 \in \mathbb{R}^d$ and $r \in (0, 1)$ are such that $|x_1 - x_2| < Lr$, then for every nonnegative function u which is harmonic with respect to X in $B(x_1, r) \cup B(x_2, r)$, we have*

$$c^{-1}u(x_2) \leq u(x_1) \leq cu(x_2).$$

For any open set D in $\mathbb{R}^d$, we will use $G_D(x, y)$ to denote the Green function of X^D. Using the continuity and the radial decreasing property of G, we can easily check that G_D is continuous in $(D \times D) \setminus \{(x, x) : x \in D\}$. We will frequently use the well-known fact that $G_D(\cdot, y)$ is harmonic in $D \setminus \{y\}$, and regular harmonic in $D \setminus \overline{B(y, \varepsilon)}$ for every $\varepsilon > 0$.

The following concept was introduced in [31].

Definition 2.4 Let $\kappa \in (0, 1/2]$. We say that an open set D in $\mathbb{R}^d$ is κ-fat if there exists $r_0 > 0$ such that for each $Q \in \partial D$ and $r \in (0, r_0)$, $D \cap B(Q, r)$ contains a ball $B(A_r(Q), \kappa r)$. The pair (r_0, κ) is called the characteristics of the κ-fat open set D.

The following boundary Harnack principle is [22, Theorem 4.22].

Theorem 2.5 ([21, Theorem 4.8], [22, Theorem 4.22]) *Suppose that D is a κ-fat open set with characteristics (r_0, κ). There exists a constant $c = c(d, r_0, \kappa, \phi) > 1$ such that, if $r \in (0, r_0 \wedge \frac{1}{4}]$ and $Q \in \partial D$, then for any nonnegative functions u, v in $\mathbb{R}^d$ which are regular harmonic in $D \cap B(Q, 2r)$ with respect to X and vanish in $D^c \cap B(Q, 2r)$, we have*

$$c^{-1} \frac{u(A_r(Q))}{v(A_r(Q))} \le \frac{u(x)}{v(x)} \le c \frac{u(A_r(Q))}{v(A_r(Q))}, \qquad x \in D \cap B\left(Q, \frac{r}{2}\right).$$

3 Generalized 3G Theorem

In this section, we prove a generalized 3G theorem for X in a bounded κ-fat open set D. This theorem will play an important role later in this paper.

We first present some preliminary results which are valid for any bounded open set D. The following proposition is a combination of [23, Proposition 3.2 and Lemma 3.3].

Proposition 3.1 *Suppose D is a bounded open set in $\mathbb{R}^d$. (i) There exists a positive constant $C_0 = C_0(\text{diam}(D), \phi, d)$ such that*

$$G_D(x, y) \le C_0 \frac{1}{|x-y|^{d-\alpha} \ell(|x-y|^{-2})}, \qquad x, y \in D. \tag{3.1}$$

(ii) For every $L > 0$, there exists $c = c(\text{diam}(D), \phi, L, d) > 0$ such that for every $|x - y| \le L(\delta_D(x) \wedge \delta_D(y))$,

$$G_D(x, y) \ge c \frac{1}{|x-y|^{d-\alpha} \ell(|x-y|^{-2})}.$$

In the remainder of this section, we assume D is a bounded κ-fat open set with characteristics (r_0, κ). Without loss of generality we may assume that $r_0 \le 1/4$. Recall that for each $Q \in \partial D$ and $r \in (0, r_0)$, $A_r(Q)$ is a point in $D \cap B(Q, r)$ satisfying $B(A_r(Q), \kappa r) \subset D \cap B(Q, r)$. Since $G_D(z, \cdot)$ is regular harmonic in $D \setminus \overline{B(z, \varepsilon)}$ for every $\varepsilon > 0$ and vanishes outside D, the following result follows easily from Theorem 2.5.

Theorem 3.2 *There exists a constant $c = c(d, r_0, \kappa, \phi) > 1$ such that for any $Q \in \partial D$, $r \in (0, r_0]$ and $z, w \in D \setminus B(Q, 2r)$, we have*

$$c^{-1} \frac{G_D(z, A_r(Q))}{G_D(w, A_r(Q))} \le \frac{G_D(z, x)}{G_D(w, x)} \le c \frac{G_D(z, A_r(Q))}{G_D(w, A_r(Q))}, \quad x \in D \cap B\left(Q, \frac{r}{2}\right).$$

Using the uniform convergence theorem [3, Theorem 1.2.1], we can choose $r_1 \le r_0$ such that if $r \le r_1$ then

$$\frac{1}{2} \le \min_{\frac{1}{6} \le \lambda \le 2\kappa^{-1}} \frac{\ell((\lambda r)^{-2})}{\ell(r^{-2})} \le \max_{\frac{1}{6} \le \lambda \le 2\kappa^{-1}} \frac{\ell((\lambda r)^{-2})}{\ell(r^{-2})} \le 2. \tag{3.2}$$

Fix $z_0 \in D$ with $\kappa r_1 < \delta_D(z_0) < r_1$ and let $\varepsilon_1 := \kappa r_1/24$. For $x, y \in D$, we define $r(x, y) := \delta_D(x) \vee \delta_D(y) \vee |x - y|$ and

$$\mathcal{B}(x, y) := \begin{cases} \left\{A \in D : \delta_D(A) > \frac{\kappa}{2} r(x, y),\ |x - A| \vee |y - A| < 5r(x, y)\right\} & \text{if } r(x, y) < \varepsilon_1 \\ \{z_0\} & \text{if } r(x, y) \geq \varepsilon_1. \end{cases} \tag{3.3}$$

Note that if $r(x, y) < \varepsilon_1$

$$\frac{1}{6}\delta_D(A) \leq \delta_D(x) \vee \delta_D(y) \vee |x - y| \leq 2\kappa^{-1}\delta_D(A), \qquad A \in \mathcal{B}(x, y). \tag{3.4}$$

Thus by Eq. 3.2, if $r(x, y) < \varepsilon_1$,

$$\frac{1}{2} \leq \frac{\ell((\delta_D(A))^{-2})}{\ell((r(x, y))^{-2})} \leq 2, \qquad A \in \mathcal{B}(x, y). \tag{3.5}$$

Let

$$C_1 := C_0 2^{d-\alpha} \delta_D(z_0)^{-d+\alpha} \cdot \sup_{\delta_D(z_0)/2 \leq r \leq \mathrm{diam}(D)} \ell(r^{-2})^{-1}$$

so that, by Proposition 3.1(i), $G_D(\cdot, z_0)$ is bounded from above by C_1 on the set $D \setminus B(z_0, \delta_D(z_0)/2)$. Now we define

$$g(x) := G_D(x, z_0) \wedge C_1.$$

Note that if $\delta_D(z) \leq 6\varepsilon_1$, then $|z - z_0| \geq \delta_D(z_0) - 6\varepsilon_1 \geq \delta_D(z_0)/2$ since $6\varepsilon_1 < \delta_D(z_0)/4$, and therefore $g(z) = G_D(z, z_0)$.

The following result is established in [23].

Theorem 3.3 [23, Theorem 1.2] *There exists $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ such that for every $x, y \in D$*

$$c^{-1} \frac{g(x)g(y)}{g(A)^2 |x - y|^d \phi(|x - y|^{-2})} \leq G_D(x, y) \leq c \frac{g(x)g(y)}{g(A)^2 |x - y|^d \phi(|x - y|^{-2})}, \qquad A \in \mathcal{B}(x, y). \tag{3.6}$$

Lemma 3.4 *There exist positive constants $c = c(d, r_0, \kappa, \phi)$, $\beta = \beta(d, r_0, \kappa, \phi) < \alpha$ and $r_2 \in (0, r_1]$ such that for any $Q \in \partial D$, $r \in (0, r_2)$, and nonnegative function u on $\mathbb{R}^d$ which is harmonic with respect to X in $D \cap B(Q, r)$ we have*

$$u(A_r(Q)) \leq c \left(\frac{r}{s}\right)^{\beta} \frac{\ell(s^{-2})}{\ell(r^{-2})} u(A_s(Q)), \qquad s \in (0, r). \tag{3.7}$$

Proof Without loss of generality, we assume $Q = 0$. Let $a_k := \left(\frac{\kappa}{2}\right)^k$ for $k = 0, 1, \cdots$. By using [22, Proposition 4.10] instead of [21, Proposition 3.8] and repeating the proof of [21, Lemma 5.2], we easily see that [21, Lemma 5.2] is valid in the present case. Thus there exist positive constants $c = c(d, r_0, \kappa, \phi)$, $\beta = \beta(d, r_0, \kappa, \phi) < \alpha$, and $R_1 \in (0, r_1]$ such that for every $k = 0, 1, \cdots$,

$$u(A_r(0)) \le c_1 \left(\frac{r}{a_k r}\right)^{\beta} \frac{\ell((a_k r)^{-2})}{\ell(r^{-2})} u(A_{a_k r}(0)), \qquad r \in (0, R_1].$$

Since ℓ is slowly varying at ∞, there exist $R_2 = R_2(d, \beta, \ell) \in (0, R_1]$ and $c_2 = c_2(d, \beta, \ell) > 0$ such that

$$\frac{s^{\beta}}{\ell(s^{-2})} \le c_2 \frac{r^{\beta}}{\ell(r^{-2})}, \qquad \forall\, 0 < s < r \le R_2. \tag{3.8}$$

Thus if $r \le R_2$ and $a_{k+1} r < s \le a_k r$, by Eq. 3.8 and Theorem 2.3,

$$u(A_r(0)) \le c_3 \frac{r^{\beta}}{\ell(r^{-2})} \frac{\ell((a_k r)^{-2})}{(a_k r)^{\beta}} u(A_s(0)) \le c_4 \frac{r^{\beta}}{\ell(r^{-2})} \frac{\ell(s^{-2})}{s^{\beta}} u(A_s(0))$$

for some positive constants c_3, c_4 independent of s. □

Applying [22, Lemma 4.19] to Green functions, we have the following.

Lemma 3.5 (Carleson's Estimate) *There exists* $c = c(d, r_0, \kappa, \phi) > 1$ *such that for every* $Q \in \partial D$, $r \in (0, 1/4)$, *and* $y \in D \setminus \overline{B(Q, 4r)}$

$$G_D(x, y) \le c\, G_D(A_r(Q), y), \quad x \in D \cap B(Q, r). \tag{3.9}$$

For every $x, y \in D$, let Q_x and Q_y be points on ∂D such that $\delta_D(x) = |x - Q_x|$ and $\delta_D(y) = |y - Q_y|$ respectively. It is easy to check that if $r(x, y) < \varepsilon_1$, $A_{r(x,y)}(Q_x),\ A_{r(x,y)}(Q_y) \in \mathcal{B}(x, y)$. (For example, see [19, p. 123].) Moreover, since $g(A_1) \asymp g(A_2)$ for all $A_1, A_2 \in \mathcal{B}(x, y)$ by Theorem 2.3, we have in particular

$$g(A_{r(x,y)}(Q_x)) \asymp g(A_{r(x,y)}(Q_y)) \asymp g(A_{x,y}) \quad \text{for all } A_{x,y} \in \mathcal{B}(x, y). \tag{3.10}$$

This simple but useful fact will be used later in this section.

Using our Theorem 2.3 and Lemma 3.5, the proofs of the next four lemmas are the same as those of [19, Lemmas 3.8–3.11], so we omit the proofs.

Lemma 3.6 *There exists* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y \in D$ *with* $r(x, y) < \varepsilon_1$,

$$g(z) \le c\, g(A_{r(x,y)}(Q_x)), \quad z \in D \cap B(Q_x, r(x, y)). \tag{3.11}$$

Lemma 3.7 *There exists* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y \in D$

$$g(x) \vee g(y) \le c\, g(A), \quad A \in \mathcal{B}(x, y).$$

Lemma 3.8 *If* $x, y, z \in D$ *satisfy* $r(x, z) \le r(y, z)$, *then there exists* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that*

$$g(A_{x,y}) \le c\, g(A_{y,z}) \qquad \text{for every } (A_{x,y}, A_{y,z}) \in \mathcal{B}(x, y) \times \mathcal{B}(y, z).$$

Lemma 3.9 *There exists* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y, z, w \in D$ *and* $(A_{x,y}, A_{y,z}, A_{z,w}, A_{x,w}) \in \mathcal{B}(x, y) \times \mathcal{B}(y, z) \times \mathcal{B}(z, w) \times \mathcal{B}(x, w)$,

$$g(A_{x,w})^2 \le c \left(g(A_{x,y})^2 + g(A_{y,z})^2 + g(A_{z,w})^2 \right). \tag{3.12}$$

Combining Theorem 3.3, Lemmas 3.7 and 3.8, and applying Theorem 2.1(i), we have the following 3G Theorem.

Theorem 3.10 (3G theorem) *There exists* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y, z \in D$

$$\begin{aligned} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} &\le c \frac{G(x,y)G(y,z)}{G(x,z)} \\ &\asymp \frac{\phi(|x-z|^{-2})}{\phi(|x-y|^{-2})\phi(|y-z|^{-2})} \frac{|x-z|^d}{|x-y|^d|y-z|^d}. \end{aligned} \tag{3.13}$$

In the remainder of this paper, β will always stand for the constant from Lemma 3.4.

Lemma 3.11 *There exists* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y \in D$ *with* $r(x, y) < \varepsilon_1$,

$$g(A_{x,y}) \ge c \frac{r(x,y)^\beta}{\ell((r(x,y))^{-2})}, \quad \textit{for all } A_{x,y} \in \mathcal{B}(x, y).$$

Proof Let $A := A_{r(x,y)}(Q_x)$. Note that $g(\cdot) = G_D(\cdot, z_0)$ is harmonic in $D \cap B(Q_x, 2\varepsilon_1)$. Since $r(x, y) < \varepsilon_1$, by Lemma 3.4 (recall $\varepsilon_1 = \kappa r_1/24$),

$$g(A) = G_D(A, z_0) \ge c \left(\frac{r(x,y)}{2\varepsilon_1} \right)^\beta \frac{\ell((2\varepsilon_1)^{-2}}{\ell((r(x,y))^{-2})} G_D(A_{2\varepsilon_1}(Q_x), z_0).$$

Note that $\delta_D(z_0) \ge r_1\kappa = 24\varepsilon_1$ and $\delta_D(A_{2\varepsilon_1}(Q_x)) > 2\kappa\varepsilon_1$. Thus by Proposition 3.1(ii) we have $G_D(A_{2\varepsilon_1}(Q_x), z_0) > c_1 > 0$. This completes the proof of Eq. 3.10. □

Lemma 3.12 *There exists* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y, z \in D$ *and* $(A_{x,y}, A_{y,z}) \in \mathcal{B}(x, y) \times \mathcal{B}(y, z)$

$$\frac{g(A_{y,z})}{g(A_{x,y})} \le c \left(\frac{r(y,z)^\beta}{r(x,y)^\beta} \frac{\ell((r(x,y))^{-2})}{\ell((r(y,z))^{-2})} \vee 1 \right).$$

Proof Note that if $r(x, y) \ge \varepsilon_1$, $g(A_{y,z}) \le C_1 = g(A_{x,y})$. We will consider three cases separately:

(a) $r(x, y) < \varepsilon_1$ and $r(y, z) \ge \varepsilon_1$: By Lemma 3.11, we have

$$\begin{aligned} \frac{g(A_{y,z})}{g(A_{r(x,y)}(Q_y))} &\le c\, C_1 \frac{\ell((r(x,y))^{-2})}{r(x,y)^\beta} \\ &\le c\, C_1 \varepsilon_1^{-\beta} \left(\sup_{\varepsilon_1 \le s \le \mathrm{diam}(D)} \ell(s^{-2}) \right) \frac{r(y,z)^\beta}{r(x,y)^\beta} \frac{\ell((r(x,y))^{-2})}{\ell((r(y,z))^{-2})}. \end{aligned}$$

(b) $r(y, z) \leq r(x, y) < \varepsilon_1$: Then $A_{r(y,z)}(Q_y) \in D \cap B(Q_y, r(x, y))$. Thus by Lemma 3.5 we have $g(A_{r(y,z)}(Q_y)) \leq c g(A_{r(x,y)}(Q_y))$.

(c) $r(x, y) < r(y, z) < \varepsilon_1$: By Lemma 3.4,

$$\frac{g(A_{r(y,z)}(Q_y))}{g(A_{r(x,y)}(Q_y))} \leq c \frac{r(y, z)^\beta}{r(x, y)^\beta} \frac{\ell((r(x, y))^{-2})}{\ell((r(y, z))^{-2})}.$$

Now the conclusion of the lemma follows immediately from Eq. 3.10. □

Thus, by Lemmas 3.7 and 3.12, we get the following lemma.

Lemma 3.13 *There exists a constant* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y, z, w \in D$ *and* $(A_{x,y}, A_{z,w}, A_{x,w}) \in \mathcal{B}(x, y) \times \mathcal{B}(z, w) \times \mathcal{B}(x, w)$*,*

$$\frac{g(y)g(z)g(A_{x,w})^2}{g(A_{x,y})^2 g(A_{z,w})^2} \leq c \left(\frac{r(x, w)^\beta}{r(x, y)^\beta} \frac{\ell((r(x, y))^{-2})}{\ell((r(x, w))^{-2})} \vee 1 \right) \left(\frac{r(x, w)^\beta}{r(z, w)^\beta} \frac{\ell((r(z, w))^{-2})}{\ell((r(x, w))^{-2})} \vee 1 \right).$$

Lemma 3.14 *There exists a constant* $c = c(\mathrm{diam}(D), d, r_0, \kappa, \phi) > 0$ *such that for every* $x, y, z, w \in D$ *and* $(A_{x,y}, A_{z,w}, A_{x,w}) \in \mathcal{B}(x, y) \times \mathcal{B}(z, w) \times \mathcal{B}(x, w)$*,*

$$\frac{g(y)g(z)g(A_{x,w})^2}{g(A_{x,y})^2 g(A_{z,w})^2} \leq c \left(\frac{r(y, z)^\beta}{r(x, y)^\beta} \frac{\ell((r(x, y))^{-2})}{\ell((r(y, z))^{-2})} \vee 1 \right) \left(\frac{r(y, z)^\beta}{r(z, w)^\beta} \frac{\ell((r(z, w))^{-2})}{\ell((r(y, z))^{-2})} \vee 1 \right).$$

Proof From Lemma 3.9, we get

$$\begin{aligned}\frac{g(y)g(z)g(A_{x,w})^2}{g(A_{x,y})^2 g(A_{z,w})^2} &\leq c_1 \frac{g(y)g(z)}{g(A_{x,y})^2 g(A_{z,w})^2} (g(A_{x,y})^2 + g(A_{y,z})^2 + g(A_{z,w})^2) \\ &= c_1 \left(\frac{g(y)g(z)}{g(A_{z,w})^2} + \frac{g(y)g(z)}{g(A_{x,y})^2} + \frac{g(y)g(z)g(A_{y,z})^2}{g(A_{x,y})^2 g(A_{z,w})^2} \right). \end{aligned} \tag{3.14}$$

By applying Lemma 3.7 to both y and z, we have that Eq. 3.14 is less than or equal to

$$\begin{aligned} &c_2 \frac{g(y)}{g(A_{z,w})} + c_2 \frac{g(z)}{g(A_{x,y})} + c_3 \left(\frac{g(A_{y,z})}{g(A_{x,y})} \right) \left(\frac{g(A_{y,z})}{g(A_{z,w})} \right) \\ &\leq c_2 \frac{g(y)}{g(A_{z,w})} + c_2 \frac{g(z)}{g(A_{x,y})} + c_4 \left(\frac{r(y, z)^\beta}{r(x, y)^\beta} \frac{\ell((r(x, y))^{-2})}{\ell((r(y, z))^{-2})} \vee 1 \right) \\ &\quad \times \left(\frac{r(y, z)^\beta}{r(z, w)^\beta} \frac{\ell((r(z, w))^{-2})}{\ell((r(y, z))^{-2})} \vee 1 \right), \end{aligned}$$

where we used Lemma 3.12 in the last inequality above. Moreover, by Lemmas 3.7 and 3.12,

$$\frac{g(y)}{g(A_{z,w})} = \left(\frac{g(y)}{g(A_{y,z})} \right) \left(\frac{g(A_{y,z})}{g(A_{z,w})} \right) \leq c \left(\frac{r(y, z)^\beta}{r(z, w)^\beta} \frac{\ell((r(z, w))^{-2})}{\ell((r(y, z))^{-2})} \vee 1 \right)$$

and

$$\frac{g(z)}{g(A_{x,y})} = \left(\frac{g(z)}{g(A_{y,z})} \right) \left(\frac{g(A_{y,z})}{g(A_{x,y})} \right) \leq c \left(\frac{r(y, z)^\beta}{r(x, y)^\beta} \frac{\ell((r(x, y))^{-2})}{\ell((r(y, z))^{-2})} \vee 1 \right).$$

Combining these, Eq, 3.14 and the inequality $(\frac{a}{b} \vee 1) + (\frac{a}{c} \vee 1) \le 2(\frac{a}{b} \vee 1)(\frac{a}{c} \vee 1)$, valid for all $a, b, c > 0$, we have finished the proof. □

Lemma 3.15 *Let $\psi(r) = \frac{r^\beta}{\ell(r^{-2})}$ and $M \in (0, \infty)$. Then there exists a constant $c = c(M, \ell, \beta) > 0$ such that*

$$\frac{\psi(a_2)}{\psi(b_2)} \le c\left(\frac{\psi(a_1)}{\psi(b_1)} \vee 1\right) \quad \text{for every } 0 < a_1 \le a_2 \le 2a_1 \le M \text{ and } 0 < b_1 \le b_2 \le M.$$

Proof Since ℓ is slowly varying at ∞, by [3, Theorem 1.5.3] there exists $R_1 < M/2$ such that

$$\frac{s^\beta}{\ell(s^{-2})} \le 2\frac{r^\beta}{\ell(r^{-2})} \quad \text{and} \quad \frac{\ell(r^{-2})}{\ell((2r)^{-2})} \le 2 \quad \forall s < r \le R_1. \tag{3.15}$$

Note that $\psi : (0, \infty) \to (0, \infty)$ is locally bounded from above and below by positive constants.

If $a_1 \le R_1/2$, since $a_2 < 2a_1 \le R_1$, by Eq. 3.15, $\psi(a_2) \le 2^{\beta+2}\psi(a_1)$. If $a_1 > R_1/2$, by the local boundedness of ψ, $\psi(a_2) \asymp \psi(a_1)$.

Similarly, if $b_2 \le R_1$, since $b_1 \le b_2 \le R_1$, by Eq. 3.15, $2\psi(b_2) \ge \psi(b_1)$. If $b_2 > R_1$, by the local boundedness of ψ and Eq. 3.15, there exists a c_1 such that $\psi(b_2) \ge c_1\psi(b_1)$. The lemma clearly follows from these observations. □

Now we are ready to prove the main result of this section, which is a generalization of the main result in [19].

Theorem 3.16 (Generalized 3G Theorem) *Let $\psi(r) := \frac{r^\beta}{\ell(r^{-2})}$. Suppose that D is a bounded κ-fat open set with characteristics (r_0, κ). Then there exists a positive constant $c = c(\text{diam}(D), d, r_0, \kappa, \phi)$ such that for every $x, y, z, w \in D$*

$$\begin{aligned}\frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} &\le c\left(\frac{\psi(|x-w|) \wedge \psi(|y-z|)}{\psi(|x-y|)} \vee 1\right) \\ &\quad \times \left(\frac{\psi(|x-w|) \wedge \psi(|y-z|)}{\psi(|z-w|)} \vee 1\right)\frac{G(x,y)G(z,w)}{G(x,w)}. \end{aligned} \tag{3.16}$$

Proof Let

$$G(x,y,z,w) := \frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} \quad \text{and} \quad H(x,y,z,w) := \frac{G(x,y)G(z,w)}{G(x,w)}.$$

If $|x-w| \le \delta_D(x) \wedge \delta_D(w)$, by Proposition 3.1(ii) and Theorem 2.1(i), $G_D(x,w) \ge cG(x,w)$. Thus by Eq. 3.1 and Theorem 2.1(i) we have $G(x,y,z,w) \le cH(x,y,z,w)$.

On the other hand, if $|y-z| \le \delta_D(y) \wedge \delta_D(z)$, then by Proposition 3.1(ii) and Theorem 2.1(i), $G_D(y,z) \ge cG(y,z)$. Using this and Theorem 3.10, we have that there exists a constant $c = c(\text{diam}(D), d, r_0, \kappa, \phi) > 0$ such that

$$\begin{aligned} G(x,y,z,w) &= \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}\frac{G_D(x,z)G_D(z,w)}{G_D(x,w)}\frac{1}{G_D(y,z)} \\ &\le c\,\frac{G(x,y)G(y,z)}{G(x,z)}\frac{G(x,z)G(z,w)}{G(x,w)}\frac{1}{G(y,z)} = cH(x,y,z,w). \end{aligned}$$

Now we assume that $|x-w| > \delta_D(x) \wedge \delta_D(w)$ and $|y-z| > \delta_D(y) \wedge \delta_D(z)$. Since $\delta_D(x) \vee \delta_D(w) \le \delta_D(x) \wedge \delta_D(w) + |x-w|$, using the assumption $\delta_D(x) \wedge \delta_D(w) < |x-w|$, we obtain $r(x,w) < 2|x-w|$. Similarly, $r(y,z) < 2|y-z|$. Let $A_{x,w} \in \mathcal{B}(x,w)$, $A_{x,y} \in \mathcal{B}(x,y)$ and $A_{z,w} \in \mathcal{B}(z,w)$. Applying Lemmas 3.13 and 3.14 to Theorem 3.3, we have

$$\begin{aligned} G(x,y,z,w) &\le c\,\frac{g(y)g(z)g(A_{x,w})^2}{g(A_{x,y})^2 g(A_{z,w})^2} H(x,y,z,w) \\ &\le c\left[\left(\frac{\psi(r(x,w))}{\psi(r(x,y))} \wedge \frac{\psi(r(y,z))}{\psi(r(x,y))}\right) \vee 1\right] \\ &\quad \times \left[\left(\frac{\psi(r(x,w))}{\psi(r(z,w))} \wedge \frac{\psi(r(y,z))}{\psi(r(z,w))}\right) \vee 1\right] H(x,y,z,w). \end{aligned}$$

Now applying Lemma 3.15, we arrive at the conclusion of the theorem. □

4 Feynman-Kac Perturbations

Throughout this section D is a bounded κ-fat open set. In this section, we will first recall the Kato classes introduced in [4, 9, 10]. Then we apply the 3G theorem and generalized 3G theorem to establish some concrete sufficient conditions for these classes. Note that X^D is an irreducible transient symmetric Hunt process satisfying the assumption at the beginning of [4, Section 3.2].

Definition 4.1 A function q is said to be in the class $\mathbf{S}_\infty(X^D)$ if for any $\varepsilon > 0$ there are a Borel subset $K = K(\varepsilon)$ of finite Lebesgue measure and a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{(x,z)\in(D\times D)\setminus\{x=z\}} \int_{D\setminus K} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |q(y)|dy \le \varepsilon$$

and that, for all measurable set $B \subset K$ with $|B| < \delta$,

$$\sup_{(x,z)\in(D\times D)\setminus\{x=z\}} \int_{B} \frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} |q(y)|dy \le \varepsilon.$$

Definition 4.2 Suppose F is a bounded function on $D \times D$ vanishing on the diagonal. Let

$$q_{|F|}(x) := \int_D |F(x,y)|J(x,y)dy.$$

(1) F is said to be in the class $\mathbf{A}_\infty(X^D)$ if for any $\varepsilon > 0$ there are a Borel subset $K = K(\varepsilon)$ of finite Lebesgue measure and a constant $\delta = \delta(\varepsilon) > 0$ such that

$$\sup_{(x,w)\in(D\times D)\setminus\{x=w\}} \int_{(D\times D)\setminus(K\times K)} \frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} |F(y,z)|J(y,z)dzdy \le \varepsilon$$

and that, for all measurable sets $B \subset K$ with $|B| < \delta$,

$$\sup_{(x,w)\in(D\times D)\setminus\{x=w\}} \int_{(B\times D)\cup(D\times B)} \frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} |F(y,z)| J(y,z) dz dy \le \varepsilon.$$

(2) F is said to be in the class $\mathbf{A}_2(X^D)$ if $F \in \mathbf{A}_\infty(X^D)$ and if the function $q_{|F|}$ is in $\mathbf{S}_\infty(X^D)$.

Now we are going to use the 3G theorem and generalized 3G theorem to give some concrete sufficient conditions for $\mathbf{S}_\infty(X^D)$ and $\mathbf{A}_2(X^D)$. First we prove the following simple lemma.

Lemma 4.3 *There exists a positive constant $c = c(\alpha, d, \ell)$ such that*

$$\ell(|x-z|^{-2})|x-z|^{d-\alpha} \le c\left(\ell(|x-y|^{-2})|x-y|^{d-\alpha} + \ell(|y-z|^{-2})|y-z|^{d-\alpha}\right).$$

Proof By symmetry, without loss of generality, we assume $|x-y| \le |y-z|$. Since ℓ is slowly varying at ∞, by [3, Theorem 1.5.3] there exists $R_1 > 0$ such that

$$s^{d-\alpha}\ell(s^{-2}) \le 2r^{d-\alpha}\ell(r^{-2}) \quad \text{and} \quad \ell((2r)^{-2}) \le 2\,\ell(r^{-2}) \quad \forall s < r \le R_1. \tag{4.1}$$

From Eq. 4.1, we see that

$$\ell(|x-z|^{-2})|x-z|^{d-\alpha} < c_1. \tag{4.2}$$

If $|y-z| \le R_1$, then $|x-z| \le |x-y| + |y-z| \le 2|y-z| \le 2R_1$. Thus by Eq. 4.1,

$$\begin{aligned}\ell(|x-z|^{-2})|x-z|^{d-\alpha} &\le 2^{1+d-\alpha}\ell((2|y-z|)^{-2})|y-z|^{d-\alpha} \\ &\le 2^{2+d-\alpha}\ell(|y-z|^{-2})|y-z|^{d-\alpha}.\end{aligned}$$

If $|y-z| > R_1$, by the local boundedness of ℓ and Eq. 4.1, we have

$$\ell(|x-z|^{-2})|x-z|^{d-\alpha} < c_1 < c_2\ell(|y-z|^{-2})|y-z|^{d-\alpha}.$$

□

Theorem 4.4 *A function q on D is in $\mathbf{S}_\infty(X^D)$ if*

$$\lim_{r\downarrow 0}\sup_{x\in D}\int_{|x-y|\le r} \frac{|q(y)|dy}{|x-y|^d\phi(|x-y|^{-2})} = 0. \tag{4.3}$$

Proof Without loss of generality, we assume that q is a positive function on D. It follows from Theorem 3.10, (**H1**), Lemma 4.3, and the assumption on ℓ that for every $x, y, z \in D$ we have

$$\begin{aligned}\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)} &\le c_1 \frac{\phi(|x-z|^{-2})}{\phi(|x-y|^{-2})\phi(|y-z|^{-2})} \frac{|x-z|^d}{|x-y|^d|y-z|^d} \\ &\le c_2\left(\frac{1}{\phi(|x-y|^{-2})|x-y|^d} + \frac{1}{\phi(|y-z|^{-2})|y-z|^d}\right).\end{aligned} \tag{4.4}$$

We claim that a positive function q satisfying Eq. 4.3 is integrable on D. Let

$$M(r) := \sup_{w\in D}\int_{|w-y|\le r}\frac{q(y)dy}{|w-y|^d\phi(|w-y|^{-2})}.$$

By (**H1**) and [3, Theorem 1.5.3], there exists $s_0>0$ such that

$$u^d\phi(u^{-2}) \le 2s^d\phi(s^{-2}), \qquad u\le s\le s_0. \tag{4.5}$$

Then, using Eq. 4.3, we can choose $s_1\le s_0$ such that $M(s_1)<\infty$. Now by Eq. 4.5,

$$\sup_{x\in D}\int_{|x-y|\le s_1}q(y)dy \le \sup_{x\in D}\int_{|x-y|\le s_1}\frac{2s_1^d\phi(s_1^{-2})q(y)dy}{|x-y|^d\phi(|x-y|^{-2})}\le 2s_1^d\phi(s_1^{-2})M(s_1)<\infty,$$

which implies that q is integrable on D.

By Eq. 4.4, we have for every Borel subset A of D and every $(x,z)\in D\times D$,

$$\begin{aligned}\int_A\frac{G_D(x,y)G_D(y,z)}{G_D(x,z)}q(y)dy &\le 2c_2M(r)+2c_2\sup_{w\in D}\int_{A\cap B(w,r)^c}\frac{q(y)dy}{\phi(|w-y|^{-2})|w-y|^d}\\ &\le 2c_2M(r)+\int_A q(y)dy\left(\sup_{s\in[r,\mathrm{diam(D)}]}\frac{2c_2}{\phi(s^{-2})s^d}\right)\\ &=: 2c_2M(r)+\left(\int_A q(y)dy\right)a(r).\end{aligned}$$

Given ε, choose $r_1=r_1(\varepsilon)\in(0,\mathrm{diam}(D))$ such that $2c_2M(r_1)<\varepsilon/2$ and let $\delta:=2^{-1}\varepsilon/a(r_1)$. This completes the proof of the theorem. □

The proof of the following theorem is similar to that of [19, Theorem 4.3].

Theorem 4.5 *If D is a bounded κ-fat open set with characteristics (r_0,κ) and F is a function on $D\times D$ with*

$$|F(x,y)| \le c_1\frac{|x-y|^\epsilon}{\phi(|x-y|^{-2})} \tag{4.6}$$

for some $\epsilon>0$ and $c_1>0$, then $F\in\mathbf{A}_2(X^D)$ and

$$\int_D\int_D\frac{G_D(x,y)G_D(z,w)}{G_D(x,w)}|F(y,z)|J(y,z)dydz\le c_2|x-w|^{\alpha+\epsilon}\phi(|x-w|^{-2}) \tag{4.7}$$

for some $c_2>0$.

Proof We assume, without loss of generality, $\varepsilon < d-\alpha$. By the generalized 3G theorem (Theorem 3.16), there exists a positive constant $c = c(\text{diam}(D), d, r_0, \kappa, \phi)$ such that

$$\frac{G_D(x,y)G_D(z,w)}{G_D(x,w)} \le c_1 \left(\frac{\ell(|x-w|^{-2})}{\ell(|x-y|^{-2})\ell(|z-w|^{-2})} \frac{|x-w|^{d-\alpha}}{|x-y|^{d-\alpha}|z-w|^{d-\alpha}} + \frac{|x-w|^{d-\alpha+\beta}}{|x-y|^{d-\alpha+\beta}|z-w|^{d-\alpha}\ell(|z-w|^{-2})} + \frac{|x-w|^{d-\alpha+\beta}}{|x-y|^{d-\alpha}|z-w|^{d-\alpha+\beta}\ell(|x-y|^{-2})} + \frac{|x-w|^{d-\alpha+2\beta}}{|x-y|^{d-\alpha+\beta}|z-w|^{d-\alpha+\beta}\ell(|x-w|^{-2})} \right).$$

Thus, by Theorem 2.1(ii) and Eq. 4.6, we have

$$\frac{G_D(x,y)G_D(z,w)}{G_D(x,w)}|F(y,z)|J(y,z) \le c_2 \sum_{i=1}^{4} A_i(x,y,z,w)$$

where

$$A_1(x,y,z,w) := \frac{\ell(|x-w|^{-2})}{\ell(|x-y|^{-2})\ell(|z-w|^{-2})} \frac{|x-w|^{d-\alpha}}{|x-y|^{d-\alpha}|z-w|^{d-\alpha}|y-z|^{d-\epsilon}},$$

$$A_2(x,y,z,w) := \frac{|x-w|^{d-\alpha+\beta}\ell(|z-w|^{-2})^{-1}}{|x-y|^{d-\alpha+\beta}|z-w|^{d-\alpha}|y-z|^{d-\epsilon}},$$

$$A_3(x,y,z,w) := \frac{|x-w|^{d-\alpha+\beta}\ell(|x-y|^{-2})^{-1}}{|z-w|^{d-\alpha+\beta}|x-y|^{d-\alpha}|y-z|^{d-\epsilon}},$$

$$A_4(x,y,z,w) := \frac{|x-w|^{d-\alpha+2\beta}\ell(|x-w|^{-2})^{-1}}{|x-y|^{d-\alpha+\beta}|z-w|^{d-\alpha+\beta}|y-z|^{d-\epsilon}}.$$

First let

$$c_3 := \sup_{(\widetilde{x},\widetilde{y})\in D\times D} \frac{|\widetilde{x}-\widetilde{y}|^{\alpha/2}}{\ell(|\widetilde{x}-\widetilde{y}|^{-2})} < \infty.$$

Then we have

$$\begin{aligned}
&\int_D\int_D A_1(x,y,z,w)dydz \\
&= \int_D\int_D \frac{\ell(|x-w|^{-2})}{\ell(|x-y|^{-2})\ell(|z-w|^{-2})} \frac{|x-w|^{d-\alpha}}{|x-y|^{d-\alpha}|z-w|^{d-\alpha}|y-z|^{d-\epsilon}}dydz \\
&\le c_3^2|x-w|^{d-\alpha}\ell(|x-w|^{-2})\int_D\int_D |x-y|^{-d+\frac{\alpha}{2}}|z-w|^{-d+\frac{\alpha}{2}}|y-z|^{-d+\epsilon}dydz \\
&\le c_3^2|x-w|^{\epsilon}\ell(|x-w|^{-2}) \le c_4|x-w|^{\alpha+\epsilon}\phi(|x-w|^{-2}).
\end{aligned}$$

The second to last inequality comes from [14, Lemma 3.12], and the last follows from (**H1**). Similar techniques can be applied to the case A_2, A_3, A_4 and this proves Eq. 4.7.

Now using Lemma 4.3, we have

$$A_1(x, y, z, w) \le \frac{1}{\ell(|z-w|^{-2})|z-w|^{d-\alpha}} \frac{1}{|y-z|^{d-\epsilon}} + \frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}} \frac{1}{|y-z|^{d-\epsilon}} + \frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}} \frac{1}{\ell(|z-w|^{-2})|z-w|^{d-\alpha}} \frac{1}{|y-z|^{\alpha-\epsilon}}.$$

Since $\varepsilon > 0$ and ℓ is slowly varying at ∞, the following two families

$$\{(y, z) \mapsto \ell(|x-y|^{-2})^{-1}|x-y|^{\alpha-d}|y-z|^{\epsilon-d},\ x \in D\},$$
$$\{(y, z) \mapsto \ell(|z-w|^{-2})^{-1}|z-w|^{\alpha-d}|y-z|^{\epsilon-d},\ w \in D\}$$

are uniformly integrable over cylindrical sets of the form $B \times D$ and $D \times B$, for any Borel set $B \subset D$. Now let us show that the following family of functions are uniformly integrable over cylindrical sets of the form $B \times D$ and $D \times B$:

$$\left\{(y, z) \mapsto \frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}\ell(|z-w|^{-2})|z-w|^{d-\alpha}|y-z|^{\alpha-\epsilon}},\ x, w \in D\right\}. \tag{4.8}$$

Let us consider the family (4.8) when the exponent of $|y-z|$ is negative, i.e., $\epsilon < \alpha$. Otherwise the family (4.8) is uniformly integrable since $|y-z|^{\epsilon-\alpha} < c$.

Applying Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}\ell(|z-w|^{-2})|y-z|^{\alpha-\epsilon}|z-w|^{d-\alpha}} \\
&= \left(\frac{1}{\ell(|x-y|^{-2})|x-y|^{d-\alpha}\ell(|z-w|^{-2})|z-w|^{d-\alpha}}\right)\left(\frac{1}{|y-z|^{\alpha-\epsilon}}\right) \\
&\le \frac{1}{p}\left(\frac{1}{(\ell(|x-y|^{-2}))^p|x-y|^{(d-\alpha)p}(\ell(|z-w|^{-2}))^p|z-w|^{(d-\alpha)p}}\right) \\
&\quad + \frac{1}{q}\left(\frac{1}{|y-z|^{(\alpha-\epsilon)q}}\right).
\end{aligned}$$

Since ℓ is slowly varying at ∞, it suffices to find $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ and $(d-\alpha)p < d$, $(\alpha-\epsilon)q < d$. By choosing p in the interval

$$\left(\left(1 \vee \frac{d}{d-\alpha+\epsilon}\right), \frac{d}{d-\alpha}\right),$$

we get that the family (4.8) is uniformly integrable. Note that this interval is not empty since $\dfrac{d}{d-\alpha+\epsilon} < \dfrac{d}{d-\alpha}$ by $(\alpha+\epsilon) \wedge d > \alpha$ and $\dfrac{d}{d-\alpha} > 1$. Similar techniques can be applied to the case A_2, A_3, A_4 and this proves $F \in \mathcal{A}_\infty(X^D)$. (See [19, pp. 131–132].) Since

$$q_{|F|}(dx) = \int_D |F(x,y)| J(x,y) dy \le \int_D c\,|x-y|^{\epsilon - d} dy \le c,$$

it follows from Theorem 4.4 that $q_{|F|} \in \mathbf{S}_\infty(X^D)$ and therefore F is in $\mathbf{A}_2(X^D)$. □

For $w \in D$, we denote by $\mathbb{E}_x^w$ the expectation for the conditional process obtained from X^D through Doob's h-transform with $h(\cdot) = G_D(\cdot, w)$ starting from $x \in D$. For $q \in \mathbf{S}_\infty(X^D)$ and $F \in \mathbf{A}_2(X^D)$, we define

$$e_{q+F}(t) := \exp\left(\int_0^t q(X_s^D) ds + \sum_{0<s\le t} F(X_{s-}^D, X_s^D) \right).$$

It gives rise to a Schrödinger semigroup

$$Q_t f(x) := \mathbb{E}_x\left[e_{q+F}(t) f(X_t^D) \right]. \tag{4.9}$$

When $x \mapsto \mathbb{E}_x\left[e_{q+F}(\tau_D)\right]$ is bounded, it follows from [4, Theorem 3.9] that the Green function for the Schrödinger semigroup $\{Q_t,\ t \ge 0\}$ is

$$V_D(x,y) = \mathbb{E}_x^y\left[e_{q+F}(\tau_D)\right] G_D(x,y), \tag{4.10}$$

that is,

$$\int_D V_D(x,y) f(y)\, dy = \int_0^\infty Q_t f(x)\, dt = \mathbb{E}_x\left[\int_0^\infty e_{q+F}(t) f(X_t^D)\, dt \right] \tag{4.11}$$

for any Borel measurable function $f \ge 0$ on D.

Let $u(x,y) := \mathbb{E}_x^y\left[e_{q+F}(\tau_D)\right]$ for $y \in D$. Applying [4, Theorems 3.10] and [6, Theorems 3.4 and Section 6] (see also [9]) to our case, we get

Theorem 4.6 *Let $q \in \mathbf{S}_\infty(X^D)$ and $F \in \mathbf{A}_\infty(X^D)$ be such that the gauge function $x \mapsto \mathbb{E}_x\left[e_{q+F}(\tau_D)\right]$ is bounded. The following properties hold.*

(1) *The conditional gauge function $u(x,y)$ is continuous on $(D \times D) \setminus \{(x,x) : x \in D\}$, hence by Eq. 4.10 so is $V_D(x,y)$.*
(2) *There exists a positive constant $c = c(\phi, D)$ such that*

$$c^{-1} G_D(x,y) \le V_D(x,y) \le c\, G_D(x,y), \quad x, y \in D.$$

5 Green Function Estimate for Perturbation of Subordinated Brownian Motion

In this section, we consider Green function estimates for perturbations of subordinated Brownian motions. Throughout this section, Y is a symmetric Lévy process with a Lévy density $J^Y(x) := J(x) + \sigma(x)$ and we assume that there exist some constants $c > 0$, $\rho \in (0, d)$ such that

$$|\sigma(x)| \le c \max\{|x|^{-d+\rho}, 1\} \qquad \text{for } x \in \mathbb{R}^d. \tag{5.1}$$

Since $|\sigma(x)| \le J^Y(x) + J(x)$, clearly Eq. 5.1 implies that σ is integrable in $\mathbb{R}^d$. One particular example of Y is obtained with $J^Y(x) = J(x)1_{B(0,1)}(x)$.

First we show that the transition density of Y is in $C_b^\infty(\mathbb{R}^d)$, where $C_b^\infty(\mathbb{R}^d)$ is the set of smooth and bounded functions on $\mathbb{R}^d$.

Lemma 5.1 *The process Y has a transition density $p^Y(t, x, y) = p^Y(t, y - x)$ such that $x \to p^Y(t, x)$ is in $C_b^\infty(\mathbb{R}^d)$ for each $t > 0$.*

Proof The Lévy exponent of Y is given by

$$\Psi^Y(\xi) = \Psi(\xi) + \int_{\mathbb{R}^d\setminus\{0\}} (1 - \cos(\xi, x))\sigma(x)dx.$$

Since

$$\left|\int_{\mathbb{R}^d\setminus\{0\}} (1 - \cos(\xi, x))\sigma(x)dx\right| \le 2|\sigma|_{L^1(\mathbb{R}^d)}, \tag{5.2}$$

we have $\int |\exp(-t\Psi^Y(\xi))||\xi|^n d\xi < \infty$ for every $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Note that for $t > 0$

$$p^Y(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t\Psi^Y(\xi)} d\xi \le (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Psi^Y(\xi)} d\xi = p^Y(t, 0) < \infty.$$

Now the assertion of the lemma follows immediately. □

For any open set U, we will use τ_U^Y to denote the first time Y exits U, i.e., $\tau_U^Y = \inf\{t > 0 : Y_t \notin U\}$. The killed process of Y in U is denoted by Y^U. It follows easily from [28, Lemma 48.3] that for any bounded open subset U, there exists $t_1 > 0$ such that $\sup_{x\in\mathbb{R}^d} \mathbb{P}_x(Y_{t_1} \in U) < 1$. Put $\theta := \sup_{x\in\mathbb{R}^d} \mathbb{P}_x(\tau_U^Y > t_1) \le \sup_{x\in\mathbb{R}^d} \mathbb{P}_x(Y_{t_1} \in U) < 1$. Then by the Markov property and an induction argument,

$$\sup_{x\in\mathbb{R}^d} \mathbb{P}_x(\tau_U^Y > nt_1) \le \theta^n.$$

Thus

$$\sup_{x\in U} \mathbb{E}_x[\tau_U^Y] \le \frac{t_1}{1-\theta} < \infty. \tag{5.3}$$

Now we state some auxiliary properties of $p^X(t, x)$. We need these properties only when we prove the (killed) heat kernel $p_D^Y(t, x)$ is continuous and it will not be needed in the rest of the paper.

Lemma 5.2 *There exist constants $c > 0$ and $\zeta > 0$ such that $p^X(t, x) \le ct^{-\zeta}$ for every $t \in (0, 1]$.*

Proof The heat kernel $p^X(t, x)$ can be expressed in terms of Fourier transforms by $p^X(t, x) = (2\pi)^{-d} \int e^{-i\xi\cdot x} e^{-t\Psi(\xi)} d\xi$. Since ℓ is slowly varying at ∞ there is a constant c_1 such that $|\xi|^\alpha \ell(|\xi|^2) \geq c_1|\xi|^{\alpha/2}$ for $|\xi| \geq 1$. From this it follows that for $t \in (0, 1]$

$$
\begin{aligned}
p^X(t,x) \leq p^X(t,0) &= (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Psi(\xi)} d\xi \\
&\leq (2\pi)^{-d} \int_{|\xi|<1} 1 d\xi + (2\pi)^{-d} \int_{|\xi|\geq 1} e^{-tc_1|\xi|^{\frac{\alpha}{2}}} d\xi \\
&\leq (2\pi)^{-d} \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)} + c_2 t^{-\frac{2d}{\alpha}} \leq c_3 t^{-\frac{2d}{\alpha}}.
\end{aligned}
$$

□

Lemma 5.3 *For every $\delta > 0$ there exists a constant $c = c(\delta)$ such that for every $|x| \geq \delta$ and $t > 0$*

$$p^X(t,x) \leq c(\delta), \tag{5.4}$$

$$|\sigma(x) + \left(p^X(t,\cdot) * \sigma\right)(x)| \leq c(\delta). \tag{5.5}$$

Proof The heat kernel $p^X(t,x)$ can also be written as $p^X(t,x) = \int_0^\infty (4\pi s)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4s}} \mathbb{P}(S_t \in ds)$ and thus $p^X(t,x) < c_1(\delta)$ for all $|x| \geq \delta$ and $t > 0$. Next since σ is integrable on $\mathbb{R}^d$ and uniformly bounded away from 0, it follows from Eq. 5.4 that for $|x| \geq \delta$ and $t > 0$

$$
\begin{aligned}
p^X(t,\cdot) * \sigma(x) &= \int p^X(t,x-y)\sigma(y)dy \\
&= \int_{|x-y|\geq\delta/2} p^X(t,x-y)\sigma(y)dy + \int_{|x-y|<\delta/2} p^X(t,x-y)\sigma(y)dy \\
&\leq c_1(\delta)\|\sigma\|_{L^1(\mathbb{R}^d)} + \|\sigma\|_{L^\infty(B(0,\delta/2)^c)} \int_{|x-y|<\delta/2} p^X(t,x-y)dy \\
&\leq c_2(\delta) < \infty.
\end{aligned}
$$

□

In the remainder of this section ζ will stand for the constant in Lemma 5.2. Using Lemmas 5.2 and 5.3, the proof of the next lemma is the same as that of [14, Lemma 2.6], so we omit the proof.

Lemma 5.4 *For every δ there exists a constant $c = c(\delta, \zeta) > 0$ such that $p^Y(t,x) \leq c$ for $|x| \geq (1 \vee [\zeta])\delta$ and $t > 0$.*

Now we prove that $p^Y_D(t,\cdot,\cdot)$ is jointly continuous for any bounded open set D.

Lemma 5.5 *For any bounded open set D, $p^Y_D(t,\cdot,\cdot)$ is jointly continuous on $D \times D$.*

Proof By Lemmas 5.2, 5.3, and 5.4, we have for every $T, L > 0$

$$\sup_{|x-y|\geq L, 0<t\leq T} p^Y(t,x,y) < \infty. \tag{5.6}$$

By the strong Markov property and the continuity of $p^Y(t,\cdot,\cdot)$, the transition density $p^Y_D(t,x,y)$ of Y^D for any open set D can be written as

$$p^Y_D(t,x,y) := p^Y(t,x,y) - \mathbb{E}_x\left[\, p^Y(t-\tau^Y_D, Y_{\tau^Y_D}, y) : \tau^Y_D \leq t\right] \quad \text{for } t>0,\, x,y \in \mathbb{R}^d. \tag{5.7}$$

Now using Eqs. 5.6 and 5.7 and following the routine argument (see [12]), one can show that for any open set D, $p^Y_D(t,\cdot,\cdot)$ is jointly continuous in $D \times D$. □

In the remainder of this section we will show that, for any bounded κ-fat open domain D, G^Y_D is comparable to G_D, the Green function of X^D. We will accomplish this by first dealing with the case σ is positive, then the general case.

5.1 Positive σ Case

Assume Z is a symmetric Lévy process with a Lévy density $J^Z(x) := J(x) + \widetilde{\sigma}(x)$ and we assume that there exist some constants $c > 0$, $\rho \in (0,d)$ such that

$$0 \leq \widetilde{\sigma}(x) \leq c \max\{|x|^{-d+\rho}, 1\} \qquad \text{for } x \in \mathbb{R}^d. \tag{5.8}$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ of X is given by

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))J(x,y)dxdy,$$

$$\mathcal{F} = \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty\}.$$

Another expression for $\mathcal{E}$ is given by

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \hat{u}(\xi)\bar{\hat{v}}(\xi)\Psi(\xi)d\xi,$$

where $\hat{u}$ is the Fourier transform of u. The Dirichlet form $(\mathcal{E}^Z, \mathcal{F}^Z)$ of Z is given by

$$\mathcal{E}^Z(u, v) = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x) - u(y))(v(x) - v(y))J^Z(x, y)dxdy,$$

$$\mathcal{F}^Z = \{u \in L^2(\mathbb{R}^d) : \mathcal{E}^Z(u, u) < \infty\}.$$

Another expression for $\mathcal{E}^Z$ is given by

$$\mathcal{E}^Z(u, v) = \int_{\mathbb{R}^d} \hat{u}(\xi)\bar{\hat{v}}(\xi)\Psi^Z(\xi)d\xi$$

where $\Psi^Z(\xi) = \Psi(\xi) + \int_{\mathbb{R}^d\setminus\{0\}}(1 - \cos(\xi, x))\widetilde{\sigma}(dx)$. It follows from Eq. 5.2 that there exists $c > 0$ such that

$$c^{-1}\mathcal{E}_1(u, u) \le \mathcal{E}_1^Z(u, u) \le c\mathcal{E}_1(u, u).$$

Therefore we know that $\mathcal{F}^Z = \mathcal{F}$ and that a set is of zero capacity for X if and only if it is of zero capacity for Z.

In the remainder of this subsection, we always assume that D is a bounded κ-fat set. The Dirichlet forms of X^D and Z^D are given by $(\mathcal{E}, \mathcal{F}_D)$ and $(\mathcal{E}^Z, \mathcal{F}_D^Z)$ respectively, where

$$\mathcal{F}_D = \mathcal{F}_D^Z = \{u \in \mathcal{F} | u = 0 \text{ on } D^c \text{ except for a set of zero capacity}\}.$$

For $u, v \in \mathcal{F}_D$, we have

$$\mathcal{E}(u, v) = \frac{1}{2}\int_D\int_D(u(x) - u(y))(v(x) - v(y))J(y - x)dxdy + \int_D u(x)v(x)\kappa_D(x)dx,$$

$$\mathcal{E}^Z(u, v) = \frac{1}{2}\int_D\int_D(u(x) - u(y))(v(x) - v(y))J^Z(y - x)dxdy + \int_D u(x)v(x)\kappa_D^Z(x)dx,$$

where $\kappa_D(x) = \int_{D^c} J(y - x)dy$ and $\kappa_D^Z(x) = \int_{D^c} J^Z(y - x)dy = \kappa_D^X(x) + \int_{D^c}\widetilde{\sigma}(y - x)dy$. Define $F(x, y) := \frac{J^Z(y-x)}{J(y-x)} - 1 = \frac{\widetilde{\sigma}(y-x)}{J(y-x)}$ and $q(x) := \kappa_D(x) - \kappa_D^Z(x)$. Note that $\inf_{x,y\in D} F(x, y) \ge 0$. Now define

$$K_t = \exp\left(\sum_{0<s\le t}\ln(1 + F(X_{s-}^D, X_s^D)) - \int_0^t\int_D F(X_s^D, y)J(y - X_s^D)dyds + \int_0^t q(X_s^D)ds\right)$$

and

$$Q_t f(x) := \mathbb{E}_x[K_t f(X_t^D)], \quad x \in D.$$

By calculating the quadratic form of Q_t using techniques similar to those on [11, p. 275], one can see that Q_t is the semigroup associated with the Dirichlet form $(\mathcal{E}^Z, \mathcal{F}_D^Z)$.

By using Theorem 2.1 and the assumption on $\widetilde{\sigma}$, it is easy to see there exist $\epsilon > 0$ and $c' > 0$ such that $F(x, y) \le c' \frac{|x-y|^\epsilon}{\phi(|x-y|^{-2})}$ for all $x, y \in D$. (For example, we can take $\epsilon = \frac{\rho}{2}$.) Thus, by Theorem 4.5, the function $F(x, y) \in \mathbf{A}_2(X^D)$. Since $|q(x)| = |-\int_{D^c} \sigma(y-x)dy| \le \int_{\mathbb{R}^d} \sigma(z)dz < \infty$, we know that $q \in S_\infty(X^D)$ by Theorem 4.4.

Note that the killing intensity κ_D^Z of Z^D is bounded from below by a positive constant so it follows that

$$\inf\{\mathcal{E}^Z(u, u) : u \in \mathcal{F}_D^Z \text{ with } \int_D u(x)^2 dx = 1\} > 0.$$

This implies that $\int_0^\infty Q_t dt$ is a bounded operator in $L^2(D, dx)$ and so for any Borel subset $B \subset D$,

$$\int_0^\infty Q_t 1_B(x)dt = \mathbb{E}_x[\int_0^\infty K_t 1_B(X_t^D)dt] < \infty, \qquad \text{for all } x \in D. \tag{5.9}$$

It follows from Eq. 5.3 and [13, Proposition 2.2 (ii)] that the Green function $G_D^Z(\cdot, \cdot)$ of Z^D exists and strictly positive on $D \times D$ for any bounded open set D. Moreover, since Z satisfies the condition $(A1)$ in [20], it follows from [13, Proposition 2.1], [20, Theorem 3.11] and our Lemmas 5.2 and 5.5 that the semigroup of Z^D is intrinsically ultracontractive, that is there exits a constant $c_1 = c_1(D, t)$ such that $p_D^Z(t, x, y) \le c_1 \phi_1(x)\phi_1(y)$, where ϕ_1 is the eigenfunction of semigroup of Z^D associated with the largest eigenvalue $\lambda_1 < 0$ of the generator of Z^D and $\|\phi_1\|_{L^2(D)} = 1$. Furthermore it follows from [20, Theorem 3.13] there is a constant $c_2 > 0$ such that $p_D^Z(t, x, y) \le c_2 e^{\lambda_1 t}\phi_1(x)\phi_1(y)$ for all $t > 1$. Hence by Lemma 5.4, the dominated convergence theorem and the continuity of $p_D^Z(t, \cdot, \cdot)$, $G_D^Z(\cdot, \cdot)$ is continuous on $(D \times D) \setminus \{x = y\}$. Now, from Eq. 5.9, Theorems 4.4 and 4.5, we know that the assumptions of Theorem 4.6 are satisfied. Since the Green function of the semigroup Q_t is $G_D^Z(x, y) = G_D(x, y)\mathbb{E}_x^y[K_{\tau_D}]$, the following result is an immediate consequence of Theorem 4.6.

Theorem 5.6 *If Z is a purely discontinuous symmetric Lévy process with Lévy density $J^Z(x) = J(x) + \widetilde{\sigma}(x)$ satisfying Eq. 5.8 and D be a bounded κ-fat open set in $\mathbb{R}^d$. Then the Green function $G_D^Z(x, y)$ for Z in D is continuous on $(D \times D) \setminus \{(x, x) : x \in D\}$. Moreover, there is a constant $c = c(D, d, \phi) > 0$ such that*

$$c^{-1}\, G_D(x, y) \;\le\; G_D^Z(x, y) \;\le\; c\, G_D(x, y), \quad x, y \in D.$$

5.2 General Case

Now we return to the general case where σ can take both signs. From now on we assume D is a bounded κ-fat domain (connected open set). Let Z be the Lévy process with a Lévy density $J^Z(x) := J^Y(x) \vee J(x)$. Then $\widetilde{\sigma}(x) := J^Z(x) - J(x)$ satisfies Eq. 5.8. By Lemma 5.5, $p_D^Y(t, \cdot, \cdot)$ and $p_D^Z(t, \cdot, \cdot)$ are jointly continuous on $D \times D$. Note that [20, Condition (A1)(b)] is true for all three processes X, Y and Z. Since D is a domain, by following the argument in the proof of [13, Proposition 2.2], one can show that $p_D^X(t, \cdot, \cdot)$, $p_D^Y(t, \cdot, \cdot)$ and $p_D^Z(t, \cdot, \cdot)$ are strictly positive for all $t > 0$. Thus [14, Property A] is valid. (Also see [20, Corollary 3.12].) Using an argument similar to the one in the paragraph before Theorem 5.6, we see that $G_D^Y(\cdot, \cdot)$ and $G_D^Z(\cdot, \cdot)$ are strictly positive and jointly continuous on $D \times D$. Now it follows from [14, Theorem 3.1] and the joint continuity of $G^Y_D(x, y)$ that for every bounded κ-fat domain D

$$G^Y_D(x, y) \le c_1 G^Z_D(x, y) \le c_2 G_D(x, y), \tag{5.10}$$

for some constants $c_1 = c_1(d, D, \phi)$ and $c_2 = c_2(d, D, \phi)$.

In the remainder of this subsection we will show that $G^Y_D(x, y) \ge c_3 G_D(x, y)$ for some $c_3 > 0$. We will follow the argument in [14] closely.

By [14, Lemma 2.4], for any bounded open set D, $\mathbb{E}_x[\tau^Z_D] \asymp \mathbb{E}_x[\tau^Y_D]$ and $\mathbb{E}_x[\tau^Z_D] \asymp \mathbb{E}_x[\tau_D]$. Thus

Lemma 5.7 *For any bounded open set D, we have* $\mathbb{E}_x[\tau_D] \asymp \mathbb{E}_x[\tau^Y_D]$.

The following result is similar to [16, Lemma 17]. Recall that the function g is defined in Section 3.

Lemma 5.8 *Let D be a bounded κ-fat domain. Then*

$$g(x) \asymp \mathbb{E}_x[\tau_D].$$

Proof Pick a point $z \in D^c$ such that $\delta_D(z) = \text{diam}(D) + 1$ and let $B := B(z, 1)$. Consider the function $f(x) := \mathbb{P}_x(X_{\tau_D} \in B)$. By the Lévy system of X, we know that $f(x) = \int_B \int_D G_D(x, y) J(z - y) dy dz$. For $y \in D, z \in B$, $\text{diam}(D) < |y - z| < 2\text{diam}(D) + 2$, so by monotonicity of j, $j(2\text{diam}(D) + 2)|B| \cdot \mathbb{E}_x[\tau_D] \le f(x) \le j(\text{diam}(D))|B| \cdot \mathbb{E}_x[\tau_D]$. Since $g(x)$ is equal to $G_D(x, z_0)$ on $|x - z_0| > \frac{\delta_D(z_0)}{2}$, the assertion of this lemma now follows from Theorem 2.5. □

Lemma 5.9 *Let D be a bounded κ-fat domain and $\theta > 0$ a constant. If $x, y \in D$ satisfy $|x - y| \ge \theta$, then there is a constant $c = c(\theta, \phi, D, d)$ such that $G_D(x, y) \le c\,\mathbb{E}_x[\tau_D]\mathbb{E}_y[\tau_D]$.*

Proof The proof of this lemma is similar to that of [14, Corollary 3.11]. By Theorem 3.3, we have

$$G_D(x, y) \le c_1 \frac{g(x)g(y)}{g(A)^2 |x - y|^d \phi(|x - y|^{-2})},$$

where $A \in \mathcal{B}(x, y)$. Since $\delta_D(A) \ge \frac{\kappa}{2} r(x, y) \ge \frac{\kappa}{2}|x - y| \ge \frac{\kappa\theta}{2}$, it follows from [22, Lemma 4.2] that

$$g(A) \asymp \mathbb{E}_A[\tau_D] \ge \mathbb{E}_A[\tau_{B(A, \frac{\kappa\theta}{2})}] \ge c_2 \frac{1}{\phi((\frac{\kappa\theta}{4})^{-2})}.$$

Now the theorem follows from Lemma 5.8. □

Recall that Y also satisfies [14, Property A] for the bounded κ-fat domain D, i.e.,

$$c\,\mathbb{E}_x[\tau^Y_D]\,\mathbb{E}_y[\tau^Y_D] \le G^Y_D(x, y). \tag{5.11}$$

The following result says that the Green functions $G_D(x, y)$ and $G^Y_D(x, y)$ are comparable when the distance between x and y is not too small.

Theorem 5.10 *Let D be a bounded κ-fat domain and $\theta > 0$ a constant. If $x, y \in D$ satisfy $|x - y| \geq \theta$, there is a constant $c = c(\theta, \phi, D, d)$ such that $G_D(x, y) \leq c\, G^Y_D(x, y)$.*

Proof It follows from Eq. 5.11, Lemmas 5.9 and 5.7 that

$$G_D(x, y) \leq c_1 \mathbb{E}_x[\tau_D]\mathbb{E}_y[\tau_D] \leq c_2 \mathbb{E}_x[\tau^Y_D]\mathbb{E}_y[\tau^Y_D] \leq c_3 G^Y_D(x, y).$$

□

Now we are going to prove that $G_D(x, y) \leq c\, G^Y_D(x, y)$ for some $c = c(d, D, \phi) > 0$ when x and y are close to each other. The next lemma is adapted from [14, Lemma 3,5 and Corollary 3.6] which use the proofs of [26, Lemmas 7 and 9]. In fact, the proofs of [26, Lemmas 7 and 9] work for a large class of Lévy processes including our Y and Z. Thus, we omit the proof.

Lemma 5.11 *For any bounded open set D, we have for any $x, w \in D$,*

$$G^Z_D(x, w) \leq G^Y_D(x, w) + \int_D \int_D G^Y_D(x, y)\sigma(y - z)G^Z_D(z, w)dydz.$$

Theorem 5.12 *For every bounded κ-fat domain D, there are constants $\delta = \delta(d, \phi, D, \sigma, \rho) > 0$ and $c = c(d, \phi, D, \sigma, \rho) > 0$ such that for all $x, w \in D$ with $|x - w| < \delta$, we have*

$$G_D(x, w) \leq c\, G^Y_D(x, w).$$

Proof By Theorem 5.6, Lemma 5.11, and Eq. 5.10 there exist constants $c_i = c_i(d, \phi, D, \sigma, \rho), i = 1, 2$, such that

$$\begin{aligned} G_D(x, w) &\leq c_1 G^Z_D(x, w) \leq c_1 G^Y_D(x, w) + c_1 \int_D \int_D G^Y_D(x, y)\sigma(y - z)G^Z_D(z, w)dydz \\ &\leq c_1 G^Y_D(x, w) + c_2 \int_D \int_D G_D(x, y)\sigma(y - z)G_D(z, w)dydz \\ &= c_1 G^Y_D(x, w) + c_2 G_D(x, w) \int_D \int_D \frac{G_D(x, y)G_D(z, w)}{G_D(x, w)} \\ &\quad \times \frac{\sigma(y - z)}{J(y - z)} J(y - z)dydz. \end{aligned}$$

Since $\frac{\sigma(y-z)}{J(y-z)} \leq c_3 \frac{|y-z|^\rho}{\phi(|y-z|^{-2})}$, by Theorem 4.5, there exists a $c_4 > 0$ such that

$$G_D(x, w) \leq c_1 G^Y_D(x, w) + c_4 |x - w|^{\alpha+\rho} \phi(|x - w|^{-2}) G_D(x, w).$$

Now take δ small so that $c_4 |x - w|^{\alpha+\rho} \phi(|x - w|^{-2}) G_D(x, w) \leq \frac{1}{2} G_D(x, w)$ if $|x - w| < \delta$. □

Combining Eq. 5.10, Theorems 5.10 and 5.12, we have proved the next theorem which is the main result of this paper.

Theorem 5.13 *Suppose that $\alpha \in (0, 2 \wedge d)$ and D is a bounded κ-fat open domain. If Y is a symmetric Lévy process with a Lévy density $J^Y(x) := J(x) + \sigma(x)$ with σ satisfying the condition (5.1), then the Green function G^Y_D of Y^D is comparable to the Green function G^X_D of X^D, i.e., there exists a constant $c = c(D, d, \phi, \rho, \sigma)$ such that*

$$c^{-1} G^Y_D(x, y) \le G_D(x, y) \le c G^Y_D(x, y), \quad x, y \in D.$$

Remark 5.14 The condition that D is connected is crucial in Theorem 5.13. For example, if Y has a Lévy density $\nu^Y(x) = \nu(x) 1_{\{|x|<1\}}$ and $D = B(z, 1) \cup B(w, 1)$ where $z, w \in \mathbb{R}^d$, $|z - w| > 2$, then $G_D(x, y) > 0$ for $x, y \in D$ whereas $G^Y_D(x, y) = 0$ for $x \in B(z, 1)$ and $y \in B(w, 1)$.

Combining the above theorem with the main result in [23, Theorem 1.1], we immediately get the following.

Corollary 5.15 *Suppose that the assumptions of Theorem* 5.13 *are valid and further that D is a bounded $C^{1,1}$ domain, then the Green function $G^Y_D(x, y)$ satisfies*

$$G^Y_D(x, y) \asymp \left(1 \wedge \frac{\phi(|x-y|^{-2})}{\sqrt{\phi(\delta_D(x)^{-2})\phi(\delta_D(y)^{-2})}}\right) \frac{1}{|x-y|^d \phi(|x-y|^{-2})}.$$

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