

# Potential theory of truncated stable processes

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**Abstract** For any  $\alpha \in (0, 2)$ , a truncated symmetric  $\alpha$ -stable process is a symmetric Lévy process in  $\mathbb{R}^d$  with a Lévy density given by  $c|x|^{-d-\alpha} 1_{\{|x|<1\}}$  for some constant  $c$ . In this paper we study the potential theory of truncated symmetric stable processes in detail. We prove a Harnack inequality for nonnegative harmonic functions of these processes. We also establish a boundary Harnack principle for nonnegative functions which are harmonic with respect to these processes in bounded convex domains. We give an example of a non-convex domain for which the boundary Harnack principle fails.

**Keywords** Green functions · Poisson kernels · Truncated symmetric stable processes · Symmetric stable processes · Harmonic functions · Harnack inequality · Boundary Harnack principle · Martin boundary

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## 1 Introduction

Recently there have been a lot of interests in studying discontinuous stable processes due to their importance in theory as well as applications. Many important results have

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been established. These results include, among other things, sharp estimates on the Green functions and Poisson kernels [13,22], the boundary Harnack principle [4,29] and the identification of the Martin boundary for various domains [5,14,29]. See [9] for a survey of some of these results.

However in a lot of applications one needs to use discontinuous Markov processes which are not stable processes. Therefore we need to extend the known results on stable processes to other discontinuous Markov processes.

In [25,16], sharp estimates on the Green functions of killed relativistic stable processes in bounded  $C^{1,1}$  domains were established. These estimates can be used to establish various properties of relativistic stable processes.

Another discontinuous Markov process, the censored stable process, was introduced and studied in [6]. Roughly speaking, for  $\alpha \in (0, 2)$ , a censored  $\alpha$ -stable process in an open set  $D \subset \mathbb{R}^d$  is a process obtained from a symmetric  $\alpha$ -stable Lévy process by restricting its Lévy measure to  $D$ . The censored process is repelled from the complement of the open set  $D$  because it is prohibited to make jumps outside  $D$ . Some potential theoretic properties of censored stable processes, such as Green function estimates, Martin boundary, and Fatou type theorem, were established recently [11,12,21].

In this paper we study yet another type of discontinuous Markov processes which we call truncated symmetric stable processes. For  $\alpha \in (0, 2)$ , a truncated symmetric  $\alpha$ -stable process is a symmetric Lévy process in  $\mathbb{R}^d$  whose Lévy density  $l(x)$  coincides with the Lévy density of a symmetric  $\alpha$ -stable process for  $|x|$  small (say,  $|x| < 1$ ) and is equal to zero for  $|x|$  large (say,  $|x| \geq 1$ ). In other words, a truncated symmetric  $\alpha$ -stable process is a symmetric Lévy process in  $\mathbb{R}^d$  with a Lévy density given by  $c|x|^{-d-\alpha} 1_{\{|x|<1\}}$  for some constant  $c$ . Truncated stable processes are very natural and important in applications where only jumps up to a certain size are allowed. One expects that many properties of the truncated stable processes should be similar to those of the symmetric stable processes, but some properties are very different. For instance, the boundary Harnack principle for symmetric stable processes is valid on any  $\kappa$ -fat set, while we will show that on non-convex domains the boundary Harnack principle for truncated stable processes might fail.

In some aspects, truncated stable processes have nicer behaviors and are more preferable than symmetric stable processes, for instance, by Theorem 25.17 of [26] we know that truncated stable processes have finite exponential moments. However, as we shall see later, in some other respects, truncated stable processes are much more difficult and more delicate to study than symmetric stable processes.

The starting point of our research on truncated stable processes was our attempt to establish a Harnack inequality for nonnegative harmonic functions of truncated stable processes. The recent developments in Harnack inequalities for discontinuous Markov processes were initiated in [1]. The method of [1] was extended in [28] to cover a large class of Markov processes. Two other methods for proving the Harnack inequality for discontinuous Markov processes were contained in [8] and [10]. However, none of the methods above apply to truncated stable processes. This gives another indication that truncated stable processes are pretty delicate to deal with.

Our strategy for studying truncated stable processes is as follows. First, we consider killed truncated stable processes on small sets and show that its Green functions are comparable to the Green functions of the corresponding killed symmetric stable processes. Then we study Poisson kernels for truncated stable processes on small sets in detail. Finally we prove the Harnack inequality and boundary Harnack principle for

nonnegative harmonic functions of truncated stable processes by using properties of its Poisson kernels and some ideas in [4, 7, 29].

In this paper we will always assume that  $d \geq 2$ . The case of  $d = 1$  can also be considered, but some arguments need to be modified. We leave this case to the interested reader.

In this paper, we use “:=” as a way of definition, which is read as “is defined to be”. The letter  $c$ , with or without subscripts, signifies a constant whose value is unimportant and which may change from location to location, even within a line.

### 2 Stable processes and truncated stable processes

Throughout this paper we assume  $\alpha \in (0, 2)$  and  $d \geq 2$ . Recall that a symmetric  $\alpha$ -stable process  $X = (X_t, \mathbb{P}_x)$  in  $\mathbb{R}^d$  is a Lévy process such that

$$\mathbb{E}_x \left[ e^{i\xi \cdot (X_t - X_0)} \right] = e^{-t|\xi|^\alpha}, \quad \text{for every } x \in \mathbb{R}^d \quad \text{and} \quad \xi \in \mathbb{R}^d.$$

The Dirichlet form  $(\mathcal{E}, D(\mathcal{E}))$  associated with  $X$  is given by

$$\mathcal{E}(u, v) := \int_{\mathbb{R}^d} \hat{u}(\xi) \bar{\hat{v}}(\xi) |\xi|^\alpha d\xi, \quad D(\mathcal{E}) := \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 |\xi|^\alpha d\xi < \infty \right\},$$

where  $\hat{u}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi \cdot y} u(y) dy$  is the Fourier transform of  $u$ . As usual, we define  $\mathcal{E}_1(u, v) := \mathcal{E}(u, v) + (u, v)_{L^2(\mathbb{R}^d)}$  for  $u, v \in D(\mathcal{E})$ . Then we have

$$\mathcal{E}_1(u, u) = \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 (1 + |\xi|^\alpha) d\xi, \quad u \in D(\mathcal{E}). \tag{2.1}$$

Another expression for  $\mathcal{E}$  is as follows:

$$\mathcal{E}(u, v) = \frac{1}{2} \mathcal{A}(d, -\alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dx dy,$$

where  $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1} \pi^{-d/2} \Gamma(\frac{d+\alpha}{2}) \Gamma(1 - \frac{\alpha}{2})^{-1}$ . Here  $\Gamma$  is the Gamma function defined by  $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1} e^{-t} dt$  for every  $\lambda > 0$ . By a truncated symmetric  $\alpha$ -stable process in  $\mathbb{R}^d$  we mean a symmetric Lévy process  $Y = (Y_t, \mathbb{P}_x)$  in  $\mathbb{R}^d$  such that

$$\mathbb{E}_x \left[ e^{i\xi \cdot (Y_t - Y_0)} \right] = e^{-t\psi(\xi)}, \quad \text{for every } x \in \mathbb{R}^d \quad \text{and} \quad \xi \in \mathbb{R}^d,$$

with

$$\psi(\xi) = \mathcal{A}(d, -\alpha) \int_{\{|y|<1\}} \frac{1 - \cos(\xi \cdot y)}{|y|^{d+\alpha}} dy. \tag{2.2}$$

The Dirichlet form  $(\mathcal{Q}, D(\mathcal{Q}))$  of  $Y$  is given by

$$\mathcal{Q}(u, v) := \int_{\mathbb{R}^d} \hat{v}(\xi) \bar{\hat{u}}(\xi) \psi(\xi) d\xi, \quad D(\mathcal{Q}) := \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \psi(\xi) d\xi < \infty \right\}.$$

The Dirichlet form  $(\mathcal{Q}, D(\mathcal{Q}))$  of  $Y$  can also be written as follows:

$$\mathcal{Q}(u, v) = \frac{1}{2} \mathcal{A}(d, -\alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} 1_{\{|x-y|<1\}} dx dy$$

$$D(\mathcal{Q}) = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} 1_{\{|x-y|<1\}} dx dy < \infty \right\}.$$

Similar to  $\mathcal{E}_1$ , we can also define  $\mathcal{Q}_1$ . Then we have

$$\mathcal{Q}_1(u, u) = \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 (1 + \psi(\xi)) d\xi. \tag{2.3}$$

By the change of variable  $y = x/|\xi|$ , we have from (2.2)

$$\psi(\xi) = \mathcal{A}(d, -\alpha) |\xi|^\alpha \int_{\{|x|<|\xi|\}} \frac{1 - \cos(\frac{\xi}{|\xi|} \cdot x)}{|x|^{d+\alpha}} dx.$$

Since  $1 - \cos(\frac{\xi}{|\xi|} \cdot x)$  behaves like  $|x|^2$  for small  $|x|$ , it is easy to check that  $\psi(\xi)$  behaves like  $|\xi|^2$  near the origin. Also we see that as  $|\xi|$  goes to infinity, the integral in the above equation goes to a positive constant. So  $\psi(\xi)$  behaves like  $|\xi|^\alpha$  near infinity. Therefore by the definition of  $D(\mathcal{E})$  and  $D(\mathcal{Q})$  we see that  $D(\mathcal{E}) = D(\mathcal{Q})$ . From now on we will use  $\mathcal{F}$  to stand for  $D(\mathcal{E})$ . Using (2.1), (2.3) and the fact above, we see that there exist positive constants  $c_1$  and  $c_2$  such that

$$c_1 \mathcal{E}_1(u, u) \leq \mathcal{Q}_1(u, u) \leq c_2 \mathcal{E}_1(u, u), \quad u \in \mathcal{F}.$$

Therefore the capacities corresponding to the Dirichlet forms  $(\mathcal{E}, \mathcal{F})$  and  $(\mathcal{Q}, \mathcal{F})$  are comparable, hence we get that a set  $A$  has zero capacity with respect to  $(\mathcal{E}, \mathcal{F})$  if and only if it has zero capacity with respect to  $(\mathcal{Q}, \mathcal{F})$  and that a function  $u$  is quasi continuous with respect to the capacity of  $(\mathcal{E}, \mathcal{F})$  if and only if it is quasi continuous with respect to the capacity of  $(\mathcal{Q}, \mathcal{F})$ . So when we speak of quasi continuous functions or sets of zero capacity, we do not need to specify the Dirichlet forms. For concepts and results related to Dirichlet forms, we refer our readers to [18].

It is well known that any function  $u \in \mathcal{F}$  admits a quasi continuous version. From now on, whenever we talk about a function  $u \in \mathcal{F}$ , we always use the quasi continuous version.

Using the asymptotic behavior of  $\psi$  and Proposition 28.1 in [26] we know that the process  $Y$  has a smooth density  $p^Y(t, x, y)$ . Since  $\psi(\xi)$  behaves like  $|\xi|^2$  near the origin, it follows from Corollary 37.6 of [26] that  $Y$  is recurrent when  $d = 2$  and transient when  $d \geq 3$ . By using the smoothness of the density, one can easily check that, when  $d \geq 3$ , the Green function of  $Y$

$$G^Y(x, y) = \int_0^\infty p^Y(t, x, y) dt$$

is continuous off the set  $\{(x, x) : x \in \mathbb{R}^d\}$ .

For any open set  $D$ , we use  $\tau_D^X$  to denote the first exit time from  $D$  by the process  $X$  and use  $X^D$  to denote the process obtained by killing the symmetric  $\alpha$ -stable process

upon leaving  $D$ . The process  $X^D$  is usually called the killed symmetric  $\alpha$ -stable process in  $D$ . The Dirichlet form of  $X^D$  is  $(\mathcal{E}, \mathcal{F}^D)$ , where

$$\mathcal{F}^D = \{u \in \mathcal{F} : u = 0 \text{ on } D^c \text{ except for a set of zero capacity}\}.$$

For any  $u, v \in \mathcal{F}^D$ ,

$$\mathcal{E}(u, v) = \frac{1}{2} \int_D \int_D (u(x) - u(y))(v(x) - v(y))J(x, y)dx dy + \int_D u(x)v(x)\kappa_D(x)dx,$$

where

$$J(x, y) := A(d, -\alpha)|x - y|^{-(d+\alpha)} \quad \text{and} \quad \kappa_D(x) := A(d, -\alpha) \int_{D^c} |x - y|^{-(d+\alpha)}dy.$$

Similarly, for any open set  $D$ , we use  $\tau_D^Y$  to denote the first exit time from  $D$  by the process  $Y$  and use  $Y^D$  to denote the process obtained by killing the process  $Y$  upon exiting  $D$ . The Dirichlet form of  $Y^D$  is  $(\mathcal{Q}, \mathcal{F}^D)$ . For any  $u, v \in \mathcal{F}^D$ ,

$$\mathcal{Q}(u, v) = \frac{1}{2} \int_D \int_D (u(x) - u(y))(v(x) - v(y))J^Y(x, y)dx dy + \int_D u(x)v(x)\kappa_D^Y(x)dx,$$

where

$$J^Y(x, y) := J(x, y)1_{\{|x-y|<1\}} \quad \text{and} \quad \kappa_D^Y(x) := \int_{D^c} J^Y(x, y)dy. \tag{2.4}$$

Note that

$$0 \leq \kappa_D(x) - \kappa_D^Y(x) = \int_{D^c \cap \{|x-y|\geq 1\}} J(x, y)dy \leq \int_{\{|x-y|\geq 1\}} J(x, y)dy =: \mathcal{B}(d, \alpha), \quad \forall x \in D. \tag{2.5}$$

Using the continuity of  $p^Y$ , it is routine (see, for instance, the proof of Theorem 2.4 [17]) to show that  $Y^D$  has a continuous and symmetric density  $p_D^Y(t, x, y)$ . From this one can easily show that the Green function  $G_D^Y$  of  $Y^D$  is continuous on  $(D \times D) \setminus \{(x, x) : x \in D\}$ .

### 3 Comparability between Green functions of stable and truncated stable processes in small sets

In this section we take an open set  $D$  with  $\text{diam}(D) \leq \frac{1}{2}$ , where  $\text{diam}(D)$  stands for the diameter of  $D$ . In this case, we have from (2.4) that  $J(x, y) = J^Y(x, y)$  for all  $x, y \in D$ . So we can regard  $Y^D$  as a Feynman–Kac transform of the process  $X^D$  with the potential  $q_D(x) := \kappa_D(x) - \kappa_D^Y(x)$ , that is, the Feynman–Kac semigroup  $(Q_t^D)$  defined by

$$Q_t^D f(x) = \mathbb{E}_x \left[ \exp \left( \int_0^t q_D(X_s^D)ds \right) f(X_t^D) \right]$$

is the semigroup of  $Y^D$ . Recall that  $G_D^Y$  is the Green function of  $Y^D$ . Let  $G_D$  be the Green function of  $X^D$ . Since  $q_D$  is nonnegative, we see that  $G_D(x, y) \leq G_D^Y(x, y)$  for

all  $x, y \in D$ . To get an upper bound for  $G_D^Y$ , we need to assume that  $D$  is  $\kappa$ -fat. We first recall the definition of  $\kappa$ -fat set from [29].

**Definition 3.1** *Let  $\kappa \in (0, 1/2]$ . We say that an open set  $D$  in  $\mathbb{R}^d$  is  $\kappa$ -fat if there exists  $R > 0$  such that for each  $Q \in \partial D$  and  $r \in (0, R)$ ,  $D \cap B(Q, r)$  contains a ball  $B(A_r(Q), \kappa r)$ . The pair  $(R, \kappa)$  is called the characteristics of the  $\kappa$ -fat open set  $D$ .*

Note that all Lipschitz domain and all non-tangentially accessible domain (see [20] for the definition) are  $\kappa$ -fat. Moreover, every *John domain* is  $\kappa$ -fat (see Lemma 6.3 in [24]). The boundary of a  $\kappa$ -fat open set can be highly nonrectifiable and, in general, no regularity of its boundary can be inferred. Bounded  $\kappa$ -fat open set may be disconnected.

Suppose further that  $D$  is a  $\kappa$ -fat set. Since  $q_D$  is bounded, one can use the 3G-type estimates for symmetric stable processes in [29] to check that  $q_D$  belongs to the class  $\mathbf{S}_\infty(X^D)$  there, thus it follows from [15] or [16] that  $G_D^Y(x, y)$  is continuous and there exists a positive constant  $C$  such that

$$G_D(x, y) \leq G_D^Y(x, y) \leq CG_D(x, y), \quad \forall x, y \in D. \tag{3.1}$$

But the constant  $C$  above might depend on  $D$ . For later applications, we will need the constant  $C$  to be invariant under scaling and translation. First we consider the case of balls.

**Proposition 3.2** *There exists a positive constant  $r_0 \leq \frac{1}{4}$  such that for all  $r \in (0, r_0]$  and  $a \in \mathbb{R}^d$ , we have*

$$G_{B(a,r)}(x, y) \leq G_{B(a,r)}^Y(x, y) \leq 2 G_{B(a,r)}(x, y), \quad x, y \in B(a, r).$$

*Proof* Let  $B_r := B(0, r)$  with  $r \leq \frac{1}{4}$ . For any  $z \in B_r$ , let  $(\mathbb{P}_x^z, X_t^{B_r})$  be the  $G_{B_r}(\cdot, z)$ -transform of  $(\mathbb{P}_x, X_t^{B_r})$ , that is, for any nonnegative Borel functions  $f$  in  $B_r$ ,

$$\mathbb{E}_x^z \left[ f(X_t^{B_r}) \right] = \mathbb{E}_x \left[ \frac{G_{B_r}(X_t^{B_r}, z)}{G_{B_r}(x, z)} f(X_t^{B_r}) \right].$$

It is well known that there exists a positive constant  $C$  independent of  $r$  such that

$$\frac{G_{B_r}(x, y)G_{B_r}(y, z)}{G_{B_r}(x, z)} \leq C (|x - y|^{\alpha-d} + |y - z|^{\alpha-d}), \quad \forall x, y, z \in B_r. \tag{3.2}$$

So there exists a positive constant  $r_0$  such that for any  $r \in (0, r_0]$  and all  $x, z \in B_r$ ,

$$B(d, \alpha) \mathbb{E}_x^z \tau_{B_r}^X = B(d, \alpha) \int_{B_r} \frac{G_{B_r}(x, y)G_{B_r}(y, z)}{G_{B_r}(x, z)} dy < \frac{1}{2}, \tag{3.3}$$

where  $B(d, \alpha)$  is the constant in (2.5). Hence by (2.5) and Khasminskii’s lemma (see, for instance, Lemma 3.7 in [17]) we get that for  $r \in (0, r_0 \wedge \frac{1}{4}]$

$$\mathbb{E}_x^z \left[ \exp \left( \int_0^{\tau_{B_r}^X} q(X_s^{B_r}) ds \right) \right] \leq \mathbb{E}_x^z \left[ \exp \left( B(d, \alpha) \tau_{B_r}^X \right) \right] \leq 2.$$

Since

$$G_{B_r}^Y(x, z) = G_{B_r}(x, z) \mathbb{E}_x^z \left[ \exp \left( \int_0^{\tau_{B_r}^X} q(X_s^{B_r}) ds \right) \right], \quad x, z \in B(0, r),$$

using the translation invariance property of our Green functions, we arrive at our desired result. □

The 3G-type estimate for symmetric stable processes on  $\kappa$ -fat open sets was proved in [29]. It is easy to see that the constant appearing in the 3G estimate depends only on the characteristics of the  $\kappa$ -fat open set and the diameter of the set. Moreover, by the scaling and translation invariant property of  $X$ , the constant is invariant under scaling and translation of  $D$

**Theorem 3.3** (Theorem 6.1 in [29]) *For a bounded  $\kappa$ -fat open set  $D$  in  $\mathbb{R}^d$ , there exists a constant  $c$  depending only on the characteristics  $(\kappa, R)$  of  $D$  and  $\text{diam}(D)$  such that for  $x, y, z \in D_r^a := a + rD$ ,*

$$\frac{G_{D_r^a}(x, y)G_{D_r^a}(y, z)}{G_{D_r^a}(x, z)} \leq c \frac{|x - z|^{d-\alpha}}{|x - y|^{d-\alpha}|y - z|^{d-\alpha}}. \tag{3.4}$$

By using (3.4) instead of (3.2) in the proof of Proposition 3.2, we immediately get the following result.

**Proposition 3.4** *Assume that  $D$  is a bounded  $\kappa$ -fat open set in  $\mathbb{R}^d$  with the characteristics  $(\kappa, R)$ . Then there exists constant  $r_1 = r_1(\kappa, R, \alpha, d, \text{diam}(D)) \leq \frac{1}{2} \text{diam}(D)$  such that for all  $r \in (0, r_1]$  and  $a \in \mathbb{R}^d$ , we have*

$$G_{D_r^a}(x, y) \leq G_{D_r^a}^Y(x, y) \leq 2 G_{D_r^a}(x, y), \quad x, y \in D_r^a := a + rD.$$

*Proof* We omit the details. □

### 4 Harnack inequality for truncated stable processes

In this section we will prove a Harnack inequality for truncated stable processes. It is well-known (see Lemma 6 of [4]) that for any bounded Lipschitz domain  $D$  in  $\mathbb{R}^d$  (see Sect. 5 for the definition),

$$\mathbb{P}_x(X_{\tau_D} \in \partial D) = 0, \quad x \in D. \tag{4.1}$$

The process  $X$  has a Lévy system  $(N, H)$  with  $N(x, dy) = \mathcal{A}(d, -\alpha)|x - y|^{-(d+\alpha)}dy$  and  $H_t = t$  (see [18]). Using this and (4.1) we know that for every bounded Lipschitz domain  $D$  and  $f \geq 0$ , we have

$$\mathbb{E}_x [f(X_{\tau_D})] = \int_{\overline{D}^c} K_D(x, z)f(z)dz, \quad x \in D, \tag{4.2}$$

where

$$K_D(x, z) = \mathcal{A}(d, -\alpha) \int_D \frac{G_D(x, y)}{|y - z|^{d+\alpha}} dy. \tag{4.3}$$

Recall that for any ball  $B(x, r)$ , we use  $\tau_{B(x,r)}^X$  and  $\tau_{B(x,r)}^Y$  to denote the first exit times from the ball  $B(x, r)$  by the processes  $X$  and  $Y$ , respectively. Using Proposition 3.2, we can easily see that, for  $r \leq r_0$ ,

$$\mathbb{E}_0 \tau_{B(x,r)}^X \leq \mathbb{E}_0 \tau_{B(x,r)}^Y \leq 2 \mathbb{E}_0 \tau_{B(x,r)}^X.$$

Thus it follows from Theorem 1 of [30] that for any bounded Lipschitz domain  $D$  in  $\mathbb{R}^d$  we have

$$\mathbb{P}_x(Y_{\tau_D} \in \partial D) = 0, \quad x \in D. \tag{4.4}$$

The process  $Y$  has a Lévy system  $(N^Y, H^Y)$  with  $N^Y(x, dy) = \mathcal{A}(d, -\alpha)|x - y|^{-(d+\alpha)} 1_{|x-y|<1} dy$  and  $H_t^Y = t$  (see [18]). Using this and (4.4) we have the following result.

**Theorem 4.1** *Suppose that  $D$  is a bounded Lipschitz domain in  $\mathbb{R}^d$ . Then there is a Poisson kernel  $K_D^Y(x, z)$  defined on  $D \times \overline{D}_1^c$  such that*

$$\mathbb{E}_x [f(Y_{\tau_D})] = \int_{\overline{D}_1^c} K_D^Y(x, z) f(z) dz, \quad x \in D$$

for every  $f \geq 0$  on  $\overline{D}_1^c$ , where  $\overline{D}_1^c := \{y \in \overline{D}^c : \text{dist}(y, D) < 1\}$ . Moreover,

$$K_D^Y(x, z) = \mathcal{A}(d, -\alpha) \int_{D \cap \{|y-z|<1\}} \frac{G_D^Y(x, y)}{|y - z|^{d+\alpha}} dy, \quad (x, z) \in D \times \overline{D}_1^c.$$

Using the Lévy system for  $Y$  again, we know that for every bounded open subset  $D$  and every  $f \geq 0$  and  $x \in D$ ,

$$\mathbb{E}_x [f(Y_{\tau_D}); Y_{\tau_D-} \neq Y_{\tau_D}] = \int_{\overline{D}_1^c} \mathcal{A}(d, -\alpha) \int_{D \cap \{|y-z|<1\}} \frac{G_D^Y(x, y)}{|y - z|^{d+\alpha}} dy f(z) dz. \tag{4.5}$$

For notational convenience, we define

$$K_D^Y(x, z) := \mathcal{A}(d, -\alpha) \int_{D \cap \{|y-z|<1\}} \frac{G_D^Y(x, y)}{|y - z|^{d+\alpha}} dy, \quad (x, z) \in D \times \overline{D}_1^c \tag{4.6}$$

even if  $D$  is not a bounded Lipschitz domain in  $\mathbb{R}^d$ , so (4.5) can be simply written as

$$\mathbb{E}_x [f(Y_{\tau_D}); Y_{\tau_D-} \neq Y_{\tau_D}] = \int_{\overline{D}_1^c} K_D^Y(x, z) f(z) dz.$$

Recall that  $r_0$  is the constant from Proposition 3.2 and  $K_{B(x_0,r)}(x, z)$  is the Poisson kernel of  $B(x_0, r)$  with respect to  $X$ . Let  $A(x, r, R) := \{y \in \mathbb{R}^d : r \leq |y - x| < R\}$ .

**Lemma 4.2** *Suppose that  $x_0 \in \mathbb{R}^d$ . Then for every  $r < \frac{1}{4}$  and  $z \in A(x_0, r, 1 - r)$ ,*

$$K_{B(x_0,r)}(x, z) \leq K_{B(x_0,r)}^Y(x, z), \quad x \in B(x_0, r). \tag{4.7}$$

*If  $r < r_0$  and  $z \in A(x_0, r, \infty)$ , then*

$$K_{B(x_0,r)}^Y(x, z) \leq 2K_{B(x_0,r)}(x, z), \quad x \in B(x_0, r). \tag{4.8}$$

*Proof* Note that if  $z \in A(x_0, r, 1 - r)$  and  $y \in B(x_0, r)$ , then  $|y - z| \leq |x_0 - y| + |x_0 - z| < r + 1 - r = 1$ . Thus by Theorem 4.1,

$$K_{B(x_0,r)}^Y(x, z) = \mathcal{A}(d, -\alpha) \int_{B(x_0,r)} \frac{G_{B(x_0,r)}^Y(x, y)}{|y - z|^{d+\alpha}} dy.$$

So (4.7) follows from (3.1) and (4.2).

On the other hand, if  $r < r_0$ , by Theorem 4.1

$$\begin{aligned} K_{B(x_0,r)}^Y(x, z) &= \mathcal{A}(d, -\alpha) \int_{B(x_0,r) \cap \{|y-z|<1\}} \frac{G_{B(x_0,r)}^Y(x, y)}{|y - z|^{d+\alpha}} dy \\ &\leq \mathcal{A}(d, -\alpha) \int_{B(x_0,r)} \frac{G_{B(x_0,r)}^Y(x, y)}{|y - z|^{d+\alpha}} dy. \end{aligned}$$

for every  $z \in A(x_0, r, \infty)$  and  $x \in B(x_0, r)$ . Thus (4.8) follows from Proposition 3.2 and (4.2). □

It is well-known that

$$K_{B(x_0,r)}(x, z) = c_1 \frac{(r^2 - |x - x_0|^2)^{\frac{\alpha}{2}}}{(|z - x_0|^2 - r^2)^{\frac{\alpha}{2}}} \frac{1}{|x - z|^d} \tag{4.9}$$

for some constant  $c_1 = c_1(d, \alpha) > 0$ .

**Lemma 4.3** *Suppose that  $r < r_0$ . Then there exists a constant  $c = c(d, \alpha) > 0$  such that for any  $z \in A(x_0, r, 1 - r)$  and  $x_1, x_2 \in B(x_0, \frac{r}{2})$ ,*

$$c^{-1} K_{B(x_0,r)}(x_2, z) \leq K_{B(x_0,r)}^Y(x_1, z) \leq c K_{B(x_0,r)}(x_2, z).$$

*Proof* By the previous lemma, we have for every  $z \in A(x_0, r, 1 - r)$ ,

$$K_{B(x_0,r)}(x, z) \leq K_{B(x_0,r)}^Y(x, z) \leq 2 K_{B(x_0,r)}(x, z), \quad x \in B(x_0, r). \tag{4.10}$$

By the explicit formula for  $K_{B(x_0,r)}(x, z)$  in (4.9), we see that there exists a constant  $c_1 = c_1(d, \alpha)$  such that for  $x \in B(x_0, \frac{r}{2})$  and  $z \in A(x_0, r, 1 - r)$

$$c_1^{-1} K_{B(x_0,r)}(x, z) \leq K_{B(x_0,r)}(x_0, z) \leq c_1 K_{B(x_0,r)}(x, z). \tag{4.11}$$

The inequalities (4.10) and (4.11) imply the lemma. □

**Lemma 4.4** *Suppose that  $r < r_0$ . Then there exists a constant  $c = c(\alpha, d) > 0$  such that for any  $z \in A(x_0, 1 - r, 1 + r)$  and  $x \in B(x_0, r)$  we have  $K_{B(x_0,r)}^Y(x, z) \leq cr^\alpha$ .*

*Proof* Without loss of generality, we may assume  $x_0 = 0$ . Fix  $r < r_0, z \in A(0, 1 - r, 1 + r)$  and  $x \in B(0, r)$ . By (4.3) and (4.8), we have

$$K_{B(0,r)}^Y(x, z) \leq c_1 \int_{B(0,r)} \frac{G_{B(0,r)}(x, y)}{|y - z|^{d+\alpha}} dy$$

for some constant  $c_1 = c_1(d, \alpha)$ . Note that for any  $y \in B(0, r)$ ,  $|y - z| \geq |z| - |y| \geq 1 - r - r \geq \frac{1}{2}$ . Since  $G_{B(0,r)}(x, y) \leq c_2|x - y|^{-d+\alpha}$  for some constant  $c_2 = c_2(d, \alpha)$  (see, for instance, [13]), we have

$$K_{B(0,r)}^Y(x, z) \leq c_1 c_2 \int_{B(0,r)} \frac{dy}{|x - y|^{d-\alpha}} \leq c_1 c_2 \int_{B(0,2r)} \frac{dw}{|w|^{d-\alpha}} \leq c_3 r^\alpha$$

for some constant  $c_3 = c_3(d, \alpha)$ . □

**Lemma 4.5** *Suppose that  $r < r_0$ . Then there exists  $c = c(\alpha, d) > 0$  such that for any  $z \in A(x_0, 1 - r, 1 + \frac{1}{2}r)$  and  $x \in B(x_0, \frac{r}{4})$ ,  $c^{-1}r^\alpha \leq K_{B(x_0,r)}^Y(x, z) \leq cr^\alpha$ .*

*Proof* Without loss of generality, we may assume  $x_0 = 0$ . Fix  $r < r_0$ ,  $z \in A(0, 1 - r, 1 + \frac{1}{2}r)$  and  $x \in B(0, \frac{r}{4})$ . Let  $B_r := B(0, r)$ . By Lemma 4.4, we only need to prove the lower bound. Note that for any  $y \in B_r$ ,  $|y - z| \leq |y| + |z| < 1 + \frac{1}{2}r + r < 2$ . So by Theorem 4.1 and (3.1), we have

$$K_{B_r}^Y(x, z) \geq c_1 \int_{B_r \cap \{|y-z|<1\}} G_{B_r}(x, y) dy \geq c_1 \int_{\{|y-z|<1, |y|<\frac{7r}{8}\}} G_{B_r}(x, y) dy$$

for some constant  $c_1 = c_1(d, \alpha)$ . Using the Green function estimates in [13], there exists  $c_2 = c_2(d, \alpha)$  such that

$$G_{B_r}(x, y) \geq c_2|x - y|^{-d+\alpha}, \quad y \in B\left(0, \frac{7r}{8}\right).$$

So we have

$$K_{B_r}^Y(x, z) \geq c_1 c_2 \int_{\{|y-z|<1, |y|<\frac{7r}{8}\}} |x - y|^{-d+\alpha} dy.$$

In the above integral, we will consider the smallest possible open set to integrate on. Let  $z_r := (0, \dots, 0, 1 + \frac{r}{2})$ . The above integral is larger than or equal to

$$\int_{\{|y-z_r|<1, |y|<\frac{7r}{8}\}} |x - y|^{-d+\alpha} dy.$$

Since  $|y| \geq |z_r| - |y - z_r| > (1 + \frac{r}{2}) - 1 = \frac{r}{2}$  for  $|y - z_r| < 1$ , we have

$$|x - y| \leq |x| + |y| < \frac{1}{4}r + |y| < 2|y|, \tag{4.12}$$

and

$$\left\{ |y - z_r| < 1, |y| < \frac{7r}{8} \right\} = \left\{ |y - z_r| < 1, \frac{r}{2} < |y| < \frac{7r}{8} \right\}. \tag{4.13}$$

By a direct computation, one can show that

$$\left\{ \frac{9r}{16} < y_d < \frac{11r}{16}, |(y_1, \dots, y_{d-1})| < \frac{r}{2} \right\} \subset \left\{ |y - z_r| < 1, \frac{r}{2} < |y| < \frac{7r}{8} \right\}. \tag{4.14}$$

Putting (4.12) and (4.14) together, we get

$$K_{B_r}^Y(x, z) \geq c_1 c_2 \int_{\frac{9r}{16}}^{\frac{11r}{16}} \int_{\{|(y_1, \dots, y_{d-1})| < \frac{r}{2}\}} |y|^{-d+\alpha} dy_1 \cdots dy_d \geq c_3 r^\alpha$$

for some constant  $c_3 = c_3(d, \alpha)$ . We have proved the lower bound. □

Combining Lemmas 4.3–4.5, we have proved the following.

**Lemma 4.6** *Suppose that  $r < r_0$ . Then there exists a constant  $c = c(d, \alpha) > 0$  such that for any  $z \in A(x_0, r, 1 + \frac{r}{2})$  and  $x_1, x_2 \in B(x_0, \frac{r}{4})$ ,*

$$c^{-1} K_{B(x_0, r)}(x_2, z) \leq K_{B(x_0, r)}^Y(x_1, z) \leq c K_{B(x_0, r)}(x_2, z).$$

**Definition 4.7** *Let  $D$  be an open subset of  $\mathbb{R}^d$ . A function  $u$  defined on  $\mathbb{R}^d$  is said to be*

1. *harmonic in  $D$  with respect to  $Y$  if*

$$\mathbb{E}_x [|u(Y_{\tau_B})|] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x [u(Y_{\tau_B})], \quad x \in B,$$

*for every open set  $B$  whose closure is a compact subset of  $D$ ;*

2. *regular harmonic in  $D$  with respect to  $Y$  if it is harmonic in  $D$  with respect to  $Y$  and for each  $x \in D$ ,*

$$u(x) = \mathbb{E}_x [u(Y_{\tau_D})].$$

We define (regular) harmonic function with respect to  $X$  similarly. The next lemma is a preliminary version of the Harnack inequality for  $Y$  and it is an immediate consequence of Lemma 4.3.

**Lemma 4.8** *Suppose that  $r \leq r_0$ . There exists a constant  $c = c(d, \alpha)$  such that*

$$c^{-1} u(y) \leq u(x) \leq c u(y), \quad y \in B\left(x, \frac{r}{2}\right)$$

*for any nonnegative function  $u$  which is regular harmonic in  $B(x, r)$  and zero in  $B(x, 2r)^c$ .*

Now we are ready to prove a (scale-invariant) Harnack inequality for  $Y$ .

**Theorem 4.9** *Suppose  $x_1, x_2 \in \mathbb{R}^d, r < r_0$  are such that  $|x_1 - x_2| < Mr$  for some  $M \leq \frac{1}{r} - \frac{1}{2}$ . Then there exists a constant  $J > 0$  depending only on  $d$  and  $\alpha$ , such that*

$$J^{-1} M^{-(d+\alpha)} u(x_2) \leq u(x_1) \leq J M^{d+\alpha} u(x_2)$$

*for every nonnegative function  $u$  which is regular harmonic with respect to  $Y$  in  $B(x_1, r) \cup B(x_2, r)$ .*

*Proof* Fix  $r < r_0, x_1, x_2 \in \mathbb{R}^d$  and a nonnegative regular harmonic function  $u$  in  $B(x_1, r) \cup B(x_2, r)$  with respect to  $Y$ . Let  $B^i = B(x_i, \frac{1}{4}r), i = 1, 2$ .

We split into two cases. First we deal with the case  $|x_1 - x_2| < \frac{1}{4}r$ . In this case we have  $\emptyset \neq B^1 \cap B^2 \supset \{x_1, x_2\}$ . By Theorem 4.1 we have for any  $y \in B^1 \cap B^2$  and  $i = 1, 2$ ,

$$\begin{aligned} u(y) &= \mathbb{E}_y [u(Y_{\tau_{B(x_i, r)}})] \\ &= \int_{A(x_i, r, 1+\frac{r}{2})} K_{B(x_i, r)}^Y(y, z) u(z) dz + \mathbb{E}_y [u(Y_{\tau_{B(x_i, r)}}); Y_{\tau_{B(x_i, r)}} \in A(x_i, 1 + \frac{r}{2}, 1+r)]. \end{aligned}$$

By Lemma 4.6,

$$\begin{aligned} \int_{A(x_i, r, 1 + \frac{r}{2})} K_{B(x_i, r)}^Y(y, z)u(z)dz &\leq c_1 \int_{A(x_i, r, 1 + \frac{r}{2})} K_{B(x_i, r)}^Y(x_i, z)u(z)dz \\ &= c_1 \mathbb{E}_{x_i} \left[ u(Y_{\tau_{B(x_i, r)}}); Y_{\tau_{B(x_i, r)}} \in A \left( x_i, r, 1 + \frac{r}{2} \right) \right], \end{aligned}$$

for some constant  $c_1 = c_1(d, \alpha)$ . Note that by Theorem 4.1

$$\begin{aligned} &\mathbb{P}_y \left( Y_{\tau_{B(x_i, r)}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right), \tau_{B(x_i, r)} = \tau_{B(x_i, \frac{r}{2})} \right) \\ &= \mathbb{P}_y \left( Y_{\tau_{B(x_i, \frac{r}{2})}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right) \right) = \int_{A(x_i, 1 + \frac{r}{2}, 1 + r)} K_{B(x_i, \frac{r}{2})}^Y(y, z)dz = 0. \end{aligned}$$

Thus by the strong Markov property, we have

$$\begin{aligned} &\mathbb{E}_y \left[ u \left( Y_{\tau_{B(x_i, r)}} \right); Y_{\tau_{B(x_i, r)}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right) \right] \\ &= \mathbb{E}_y \left[ u \left( Y_{\tau_{B(x_i, r)}} \right); Y_{\tau_{B(x_i, r)}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right), \tau_{B(x_i, r)} > \tau_{B(x_i, \frac{r}{2})} \right] \\ &= \mathbb{E}_y \left[ \mathbb{E}_{Y_{\tau_{B(x_i, \frac{r}{2})}}} \left[ u \left( Y_{\tau_{B(x_i, r)}} \right); Y_{\tau_{B(x_i, r)}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right) \right] 1_{A(x_i, \frac{r}{2}, r)} \left( Y_{\tau_{B(x_i, \frac{r}{2})}} \right) \right]. \end{aligned}$$

For  $i = 1, 2$ , let

$$g_i(z) := \mathbb{E}_z \left[ u \left( Y_{\tau_{B(x_i, r)}} \right); Y_{\tau_{B(x_i, r)}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right) \right]$$

for  $z \in A(x_i, \frac{r}{2}, r)$ , and zero otherwise. Then we have from the above argument that

$$\mathbb{E}_y \left[ u \left( Y_{\tau_{B(x_i, r)}} \right); Y_{\tau_{B(x_i, r)}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right) \right] = \mathbb{E}_y \left[ g_i \left( Y_{\tau_{B(x_i, \frac{r}{2})}} \right) \right].$$

Since, for  $i = 1, 2$ , the function  $y \mapsto \mathbb{E}_y \left[ g_i \left( Y_{\tau_{B(x_i, \frac{r}{2})}} \right) \right]$  is regular harmonic on  $B(x_i, \frac{r}{2})$  with respect to  $Y$ , and is zero on  $\overline{B(x_i, r)}^c$ , we get by Lemma 4.8 that for  $y \in B^1 \cap B^2$ ,

$$\begin{aligned} &\mathbb{E}_y \left[ u \left( Y_{\tau_{B(x_i, r)}} \right); Y_{\tau_{B(x_i, r)}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right) \right] \\ &\leq c_2 \mathbb{E}_{x_i} \left[ \mathbb{E}_{Y_{\tau_{B(x_i, \frac{r}{2})}}} \left[ u \left( Y_{\tau_{B(x_i, r)}} \right); Y_{\tau_{B(x_i, r)}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right) \right] 1_{A(x_i, \frac{r}{2}, r)} \left( Y_{\tau_{B(x_i, \frac{r}{2})}} \right) \right] \\ &= c_2 \mathbb{E}_{x_i} \left[ u \left( Y_{\tau_{B(x_i, r)}} \right); Y_{\tau_{B(x_i, r)}} \in A \left( x_i, 1 + \frac{r}{2}, 1 + r \right) \right], \end{aligned}$$

for some constant  $c_2 = c_2(d, \alpha)$ . Combining the two parts together, we get that

$$u(y) \leq c_3 u(x_i), \quad y \in B^1 \cap B^2 \tag{4.15}$$

for some constant  $c_3 = c_3(d, \alpha)$ . Therefore  $c_3^{-1}u(x_2) \leq u(x_1) \leq c_3u(x_2)$ .

Now we consider the case when  $\frac{1}{4}r \leq |x_1 - x_2| \leq Mr$  with  $M \leq \frac{1}{r} - \frac{1}{2}$ . Since  $M \leq \frac{1}{r} - \frac{1}{2}$ , we have  $|x_1 - x_2| < 1 - \frac{1}{2}r$ , and  $|y - w| \leq |y - x_2| + |w - x_1| + |x_1 - x_2| < 1$  for  $(y, w) \in B(x_2, \frac{r}{8}) \times B(x_1, \frac{r}{8})$ . Thus, by Proposition 3.2 and Theorem 4.1, we have for  $w \in B(x_1, \frac{r}{8})$ ,

$$\begin{aligned}
 K_{B(x_2, \frac{r}{8})}^Y(x_2, w) &= \mathcal{A}(d, -\alpha) \int_{B(x_2, \frac{r}{8})} \frac{G_{B(x_2, \frac{r}{8})}^Y(x_2, y)}{|y - w|^{d+\alpha}} dy \\
 &\geq \mathcal{A}(d, -\alpha) \int_{B(x_2, \frac{r}{8})} \frac{G_{B(x_2, \frac{r}{8})}(x_2, y)}{|y - w|^{d+\alpha}} dy = K_{B(x_2, \frac{r}{8})}(x_2, w).
 \end{aligned}$$

From (4.9), we have for  $w \in B(x_1, \frac{r}{8})$ ,

$$K_{B(x_2, \frac{r}{8})}^Y(x_2, w) \geq c_4 \frac{r^\alpha}{|x_2 - w|^{d+\alpha}} \geq c_4 \frac{r^{-d}}{(2M)^{d+\alpha}}$$

for some constant  $c_4 = c_4(d, \alpha)$ , because  $|x_2 - w| \leq |x_1 - x_2| + |w - x_1| < (M + \frac{1}{8})r \leq 2Mr$ . For any  $y \in B(x_1, \frac{r}{8})$ ,  $u$  is regular harmonic in  $B(y, \frac{7r}{8}) \cup B(x_1, \frac{7r}{8})$ . Since  $|y - x_1| < \frac{r}{8}$ , we can apply the conclusion of the first case with  $x_2 = y$  and  $r$  replaced by  $\frac{7r}{8}$  to get that

$$u(y) \geq c_5 u(x_1), \quad y \in B(x_1, \frac{r}{8}),$$

for some constant  $c_5 = c_5(d, \alpha)$ . Therefore

$$\begin{aligned}
 u(x_2) &= \mathbb{E}_{x_2} \left[ u \left( Y_{\tau_{B(x_2, \frac{r}{8})}} \right) \right] \geq \mathbb{E}_{x_2} \left[ u \left( Y_{\tau_{B(x_2, \frac{r}{8})}} \right); Y_{\tau_{B(x_2, \frac{r}{8})}} \in B \left( x_1, \frac{r}{8} \right) \right] \\
 &\geq c_5 u(x_1) \mathbb{P}_{x_2} \left( Y_{\tau_{B(x_2, \frac{r}{8})}} \in B \left( x_1, \frac{r}{8} \right) \right) = c_5 u(x_1) \int_{B(x_1, \frac{r}{8})} K_{B(x_2, \frac{r}{8})}^Y(x_2, w) dw \\
 &\geq c_4 c_5 u(x_1) \int_{B(x_1, \frac{r}{8})} \frac{r^{-d}}{(2M)^{d+\alpha}} dw \geq c_6 u(x_1) M^{-(d+\alpha)},
 \end{aligned}$$

for some constant  $c_6 = c_6(d, \alpha)$ . We have thus proved the right hand side inequality in the conclusion of the theorem. The inequality on the left hand side can be proved similarly.  $\square$

The Harnack inequality above is similar to the Harnack inequality (Lemma 2) for symmetric stable processes in [4], the difference is that we have to require that the two balls are not too far apart. Because our process can only make jumps of size at most 1, one can easily see that, without the assumption above, the Harnack inequality fails.

As a consequence of the theorem above we immediately get the following

**Corollary 4.10** *Suppose that  $r \leq r_0$ . There exists a constant  $c = c(d, \alpha) > 0$  such that*

$$c^{-1} u(y) \leq u(x) \leq c u(y), \quad y \in B \left( x, \frac{r}{2} \right)$$

for any nonnegative function  $u$  which is harmonic in  $B(x, r)$ .

Using this and a standard chain argument, we can get the following:

**Corollary 4.11** *Suppose that  $D$  is a domain (i.e., a connected open set) in  $\mathbb{R}^d$  and  $K$  is a compact subset of  $D$ . There exists a constant  $c = c(D, K, \alpha) > 0$  such that*

$$c^{-1} u(y) \leq u(x) \leq c u(y), \quad x, y \in K$$

for any nonnegative function  $u$  which is harmonic in  $D$

### 5 Boundary Harnack principle for truncated stable processes

In this section we will prove two versions of the boundary Harnack principle for truncated stable processes. Throughout this section,  $r_0$  is the constant in Proposition 3.2.

We will use  $\mathcal{A}_\alpha$  to denote the  $L_2$ -generator of  $Y$ , and  $C_c^\infty(\mathbb{R}^d)$  to denote the space of continuous function with compact support. It is well-known that  $C_c^\infty(\mathbb{R}^d)$  is in the domain of  $\mathcal{A}_\alpha$  and, for every  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,

$$\mathcal{A}_\alpha \phi(x) = A(d, -\alpha) \int_{|y|<1} \frac{\phi(x+y) - \phi(x) - (\nabla \phi(x) \cdot y) 1_{B(0,\varepsilon)}(y)}{|y|^{d+\alpha}} dy, \tag{5.1}$$

(see Sect. 4.1 in [27]).

For any  $\lambda > 0$ , let  $G^{Y,\lambda}(x, y)$  be the  $\lambda$ -Green function of  $Y$ . We have  $(\mathcal{A}_\alpha - \lambda)G^{Y,\lambda}(x, y) = -\delta_x(y)$  in the weak sense. For any bounded open subset  $D$  of  $\mathbb{R}^d$ , let  $G_D^{Y,\lambda}(x, y)$  be the  $\lambda$ -Green function of  $Y^D$ . Since  $G_D^{Y,\lambda}(x, y) = G^{Y,\lambda}(x, y) - \mathbb{E}_x[e^{-\lambda\tau_D} G^{Y,\lambda}(Y_{\tau_D}, y)]$ , we have, by the symmetry of  $\mathcal{A}_\alpha$ , for any  $x \in D$  and any nonnegative  $\phi \in C_c^\infty(\mathbb{R}^d)$ ,

$$\begin{aligned} \int_D G_D^{Y,\lambda}(x, y)(\mathcal{A}_\alpha - \lambda)\phi(y)dy &= \int_{\mathbb{R}^d} G_D^{Y,\lambda}(x, y)(\mathcal{A}_\alpha - \lambda)\phi(y)dy \\ &= \int_{\mathbb{R}^d} G^{Y,\lambda}(x, y)(\mathcal{A}_\alpha - \lambda)\phi(y)dy - \int_{\mathbb{R}^d} \mathbb{E}_x[e^{-\lambda\tau_D} G^{Y,\lambda}(Y_{\tau_D}, y)](\mathcal{A}_\alpha - \lambda)\phi(y)dy \\ &= \int_{\mathbb{R}^d} G^{Y,\lambda}(z, y)(\mathcal{A}_\alpha - \lambda)\phi(y)dy \\ &\quad - \int_0^\infty e^{-\lambda t} \int_{D^c} \int_{\mathbb{R}^d} G^{Y,\lambda}(z, y)(\mathcal{A}_\alpha - \lambda)\phi(y)dy \mathbb{P}_x(Y_{\tau_D} \in dz, \tau_D \in dt) \\ &= -\phi(x) + \int_0^\infty e^{-\lambda t} \int_{D^c} \phi(z) \mathbb{P}_x(Y_{\tau_D} \in dz, \tau_D \in dt) = -\phi(x) + \mathbb{E}_x[e^{-\lambda\tau_D} \phi(Y_{\tau_D})]. \end{aligned}$$

In particular, if  $\phi(x) = 0$  for  $x \in D$ , we have

$$\mathbb{E}_x[e^{-\lambda\tau_D} \phi(Y_{\tau_D})] = \int_D G_D^{Y,\lambda}(x, y)(\mathcal{A}_\alpha - \lambda)\phi(y)dy. \tag{5.2}$$

Since  $G_D^{Y,\lambda}(x, y)$  increases to  $G_D^Y(x, y)$  as  $\lambda \downarrow 0$  and  $(\mathcal{A}_\alpha - \lambda)\phi$  is bounded for small  $\lambda$ , by letting  $\lambda \downarrow 0$  in the equation above, the dominated convergence theorem gives

$$\mathbb{E}_x [\phi(Y_{\tau_D})] = \int_D G_D^Y(x, y) \mathcal{A}_\alpha \phi(y) dy \tag{5.3}$$

for any  $x \in D$  satisfying  $\phi(x) = 0$ . Take a sequence of radial functions  $\phi_m$  in  $C_c^\infty(\mathbb{R}^d)$  such that  $0 \leq \phi_m \leq 1$ ,

$$\phi_m(y) = \begin{cases} 0, & |y| < 1/2, \\ 1, & 1 \leq |y| \leq m + 1, \\ 0, & |y| > m + 2, \end{cases}$$

and that  $\sum_{i,j} |\frac{\partial^2}{\partial y_i \partial y_j} \phi_m|$  is uniformly bounded. Define  $\phi_{m,r}(y) = \phi_m(\frac{y}{r})$  so that  $0 \leq \phi_{m,r} \leq 1$ ,

$$\phi_{m,r}(y) = \begin{cases} 0, & |y| < r/2, \\ 1, & r \leq |y| \leq r(m + 1), \\ 0, & |y| > r(m + 2), \end{cases}$$

and

$$\sup_{y \in \mathbb{R}^d} \sum_{i,j} \left| \frac{\partial^2}{\partial y_i \partial y_j} \phi_{m,r}(y) \right| < c r^{-2}.$$

We claim that there exists a constant  $C > 0$  such that for all  $r \in (0, 1)$ ,

$$\sup_{m \geq 1} \sup_{y \in \mathbb{R}^d} |\mathcal{A}_\alpha \phi_{m,r}(y)| \leq C r^{-\alpha}. \tag{5.4}$$

In fact, we have

$$\begin{aligned} |\mathcal{A}_\alpha \phi_{m,r}(x)| &\leq \mathcal{A}(d, -\alpha) \left| \int_{\{|y| < 1\}} \frac{\phi_{m,r}(x + y) - \phi_{m,r}(x) - (\nabla \phi_{m,r}(x) \cdot y) 1_{B(0,r)}(y)}{|y|^{d+\alpha}} dy \right| \\ &= \mathcal{A}(d, -\alpha) \left( \left| \int_{\{|y| \leq r\}} \frac{\phi_{m,r}(x + y) - \phi_{m,r}(x) - (\nabla \phi_{m,r}(x) \cdot y)}{|y|^{d+\alpha}} dy \right| + \int_{\{r < |y| < 1\}} \frac{1}{|y|^{d+\alpha}} dy \right) \\ &\leq \mathcal{A}(d, -\alpha) \left( \frac{c}{r^2} \int_{\{|y| \leq r\}} \frac{|y|^2}{|y|^{d+\alpha}} dy + \int_{\{r < |y| < 1\}} \frac{1}{|y|^{d+\alpha}} dy \right) \leq C_1 r^{-\alpha}, \end{aligned}$$

for some constant  $C_1 = C_1(d, \alpha) > 0$ . When  $D \subset B(0, r)$  for some  $r \in (0, 1)$ , we get, by combining (5.3) and (5.4), that for any  $x \in D \cap B(0, \frac{r}{2})$ ,

$$\mathbb{P}_x (Y_{\tau_D} \in B(0, r)^c) = \lim_{m \rightarrow \infty} \mathbb{P}_x (Y_{\tau_D} \in A(0, r, (m + 1)r)) \leq C r^{-\alpha} \int_D G_D^Y(x, y) dy.$$

We have proved the following.

**Lemma 5.1** *Let  $r \in (0, 1)$  and  $D$  be an open subset with  $D \subset B(0, r)$ . Then*

$$\mathbb{P}_x (Y_{\tau_D} \in B(0, r)^c) \leq C r^{-\alpha} \int_D G_D^Y(x, y) dy, \quad x \in D \cap B\left(0, \frac{r}{2}\right)$$

for some constant  $C = C(d, \alpha) > 0$ .

Recall that  $r_0$  is the constant from Proposition 3.2.

**Lemma 5.2** *Let  $D$  be an open set such that  $B(A, \kappa r) \subset D \subset B(0, r)$  for some  $r > 0$  and  $\kappa \in (0, 1)$ . If  $r < r_0$ , then*

$$\mathbb{P}_x (Y_{\tau_{D \setminus B(A, \kappa r)}} \in B(A, \kappa r)) \geq C r^{-\alpha} \kappa^d \int_D G_D^Y(x, y) dy, \quad x \in D \setminus B(A, \kappa r)$$

for some constant  $C = C(d, \alpha) > 0$ .

*Proof* Fix a point  $x \in D \setminus B(A, \kappa r)$  and let  $B := B(A, \frac{\kappa r}{2})$ . Since  $G_D^Y(x, \cdot)$  is harmonic in  $D \setminus \{x\}$  with respect to  $Y$ ,

$$G_D^Y(x, A) = \int_{D \cap B^c} K_B^Y(A, y) G_D^Y(x, y) dy \geq \int_{D \cap B\left(A, \frac{3\kappa r}{4}\right)^c} K_B^Y(A, y) G_D^Y(x, y) dy.$$

Since  $r < \frac{1}{4}$ , by (4.7), we have

$$G_D^Y(x, A) \geq \int_{D \cap B\left(A, \frac{3\kappa r}{4}\right)^c} K_B(A, y) G_D^Y(x, y) dy.$$

Since  $\frac{3\kappa r}{4} \leq |y - A| \leq 2r$  for  $y \in B(A, \frac{3\kappa r}{4})^c \cap D$ , it follows from (4.9) that

$$\begin{aligned} G_D^Y(x, A) &\geq c_1 \int_{D \cap B\left(A, \frac{3\kappa r}{4}\right)^c} \frac{(\kappa r)^\alpha}{|y - A|^{d+\alpha}} G_D^Y(x, y) dy \\ &\geq c_2 \kappa^\alpha r^{-d} \int_{D \cap B\left(A, \frac{3\kappa r}{4}\right)^c} G_D^Y(x, y) dy, \end{aligned}$$

for some constants  $c_1 = c_1(d, \alpha)$  and  $c_2 = c_2(d, \alpha)$ . Applying Theorem 4.9 we get

$$\int_{B\left(A, \frac{3\kappa r}{4}\right)} G_D^Y(x, y) dy \leq c_3 \int_{B\left(A, \frac{3\kappa r}{4}\right)} G_D^Y(x, A) dy \leq c_4 \kappa^d r^d G_D^Y(x, A),$$

for some constants  $c_3 = c_3(d, \alpha)$  and  $c_4 = c_4(d, \alpha)$ . Combining these two estimates we get that

$$\int_D G_D^Y(x, y) dy \leq c_5 \kappa^{-\alpha} r^d G_D^Y(x, A)$$

for some constant  $c_5 = c_5(d, \alpha)$ .

Let  $\Omega = D \setminus B(A, \frac{\kappa r}{2})$ . Since  $\text{diam}(D) < 1$ , from (4.6) we have for  $z \in B(A, \frac{\kappa r}{4})$

$$K_{\Omega}^Y(x, z) = \mathcal{A}(d, -\alpha) \int_{\Omega} \frac{G_{\Omega}^Y(x, y)}{|y - z|^{d+\alpha}} dy. \tag{5.5}$$

Note that for any  $z \in B(A, \frac{\kappa r}{4})$  and  $y \in \Omega$ ,  $2^{-1}|y - z| \leq |y - A| \leq 2|y - z|$ . Thus we get from (5.5) that for  $z \in B(A, \frac{\kappa r}{4})$ ,

$$2^{-d-\alpha} K_{\Omega}^Y(x, A) \leq K_{\Omega}^Y(x, z) \leq 2^{d+\alpha} K_{\Omega}^Y(x, A). \tag{5.6}$$

Using the harmonicity of  $G_D^Y(\cdot, A)$  in  $D \setminus \{A\}$  with respect to  $Y$ , we can split  $G_D^Y(\cdot, A)$  into two parts:

$$\begin{aligned} G_D^Y(x, A) &= \mathbb{E}_y \left[ G_D^Y(Y_{\tau_{\Omega}}, A) \right] \\ &= \mathbb{E}_y \left[ G_D^Y(Y_{\tau_{\Omega}}, A) : Y_{\tau_{\Omega}} \in B \left( A, \frac{\kappa r}{4} \right) \right] \\ &\quad + \mathbb{E}_y \left[ G_D^Y(Y_{\tau_{\Omega}}, A) : Y_{\tau_{\Omega}} \in \left\{ \frac{\kappa r}{4} \leq |y - A| \leq \frac{\kappa r}{2} \right\} \right] \\ &:= I_1 + I_2. \end{aligned}$$

By the monotonicity of Green functions and Proposition 3.2,

$$G_D^Y(y, A) \leq G_{B(0,r)}^Y(y, A) \leq 2 G_{B(0,r)}(y, A) \leq 2 G(y, A), \quad y \in B(0, r), \tag{5.7}$$

where  $G(\cdot, \cdot)$  is the Green function of  $X$ . So using (5.6) twice and the explicit formula for  $G(\cdot, \cdot)$ , we have

$$\begin{aligned} I_1 &\leq 2^{d+\alpha} K_{\Omega}^Y(x, A) \int_{B(A, \frac{\kappa r}{4})} G_D^Y(y, A) dy \leq c_6 K_{\Omega}^Y(x, A) \int_{B(A, \frac{\kappa r}{4})} \frac{dy}{|y - A|^{d-\alpha}} \\ &\leq c_7 \kappa^{\alpha} r^{\alpha} K_{\Omega}^Y(x, A) \leq c_8 \kappa^{\alpha-d} r^{\alpha-d} \int_{B(A, \frac{\kappa r}{4})} K_{\Omega}^Y(x, z) dz, \end{aligned}$$

for some constants  $c_i = c_i(d, \alpha), i = 6, 7, 8$ . On the other hand, by (5.7)

$$\begin{aligned} I_2 &\leq \int_{\{\frac{\kappa r}{4} \leq |y-A| \leq \frac{\kappa r}{2}\}} G_{B(0,r)}^Y(y, A) \mathbb{P}_x(Y_{\tau_{\Omega}} \in dy) \\ &\leq c_9 \int_{\{\frac{\kappa r}{4} \leq |y-A| \leq \frac{\kappa r}{2}\}} \frac{1}{|y - A|^{d-\alpha}} \mathbb{P}_x(Y_{\tau_{\Omega}} \in dy) \\ &\leq c_{10} \kappa^{\alpha-d} r^{\alpha-d} \mathbb{P}_x \left( Y_{\tau_{\Omega}} \in \left\{ \frac{\kappa r}{4} \leq |y - A| \leq \frac{\kappa r}{2} \right\} \right), \end{aligned}$$

for some constants  $c_i = c_i(d, \alpha), i = 9, 10$ . Therefore

$$G_D^Y(x, A) \leq c_{11} \kappa^{\alpha-d} r^{\alpha-d} \mathbb{P}_x \left( Y_{\tau_{\Omega}} \in B \left( A, \frac{\kappa r}{2} \right) \right).$$

for some constant  $c_{11} = c_{11}(d, \alpha)$ . This implies that

$$\int_D G_D^Y(x, y) dy \leq c_{12} \kappa^{-d} r^{\alpha} \mathbb{P}_x \left( Y_{\tau_{D \setminus B(A, \frac{\kappa r}{2})}} \in B \left( A, \frac{\kappa r}{2} \right) \right),$$

for some constant  $c_{12} = c_{12}(d, \alpha)$ . It follows immediately that

$$\int_D G_D^Y(x, y) dy \leq c_{12} \kappa^{-d} r^\alpha \mathbb{P}_x(Y_{\tau_{D \setminus B(A, \kappa r)}} \in B(A, \kappa r)).$$

□

Combining Lemmas 5.1 and 5.2 and using the translation invariant property, we have the following:

**Lemma 5.3** *Let  $D$  be an open set such that  $B(A, \kappa r) \subset D \subset B(Z_0, r)$  for some  $r > 0$  and  $\kappa \in (0, 1)$ . If  $r < r_0$ , then*

$$\mathbb{P}_x(Y_{\tau_D} \in B(Z_0, r)^c) \leq C \kappa^{-d} \mathbb{P}_x(Y_{\tau_{D \setminus B(A, \kappa r)}} \in B(A, \kappa r)), \quad x \in D \cap B\left(Z_0, \frac{r}{2}\right),$$

for some constant  $C = C(d, \alpha) > 0$ .

The next lemma is adapted from [4] (see pp. 54–55 in [4]).

**Lemma 5.4** *Let  $D$  be an open set and  $0 < 2r < r_0$ . For any positive function  $u$ , there is a  $\sigma \in (\frac{10}{6}r, \frac{11}{6}r)$  such that for any  $a \in (-2, \frac{3}{2}]$ ,  $z_0 \in \mathbb{R}^d$  and  $x \in D \cap B(z_0, \frac{3}{2}r)$ ,*

$$\mathbb{E}_x \left[ u(Y_{\tau_{D \cap B(z_0, \sigma)}}); Y_{\tau_{D \cap B(z_0, \sigma)}} \in A(z_0, \sigma, 1 - ar) \right] \leq C r^\alpha \int_{A(z_0, \frac{10r}{6}, 1 - ar)} \frac{u(y)}{|y|^{d+\alpha}} dy$$

for some constant  $C = C(d, \alpha)$ .

*Proof* Without loss of generality, we may assume  $z_0 = 0$ . Note that

$$\begin{aligned} \int_{\frac{10}{6}r}^{\frac{11}{6}r} \int_{A(0, \sigma, 2r)} (|y| - \sigma)^{-\frac{\alpha}{2}} u(y) dy d\sigma &= \int_{A(0, \frac{10}{6}r, 2r)} \int_{\frac{10}{6}r}^{|y| \wedge \frac{11}{6}r} (|y| - \sigma)^{-\frac{\alpha}{2}} d\sigma u(y) dy \\ &\leq c_1 \int_{A(0, \frac{10r}{6}, 2r)} \left( |y| - \frac{10r}{6} \right)^{-\frac{\alpha}{2} + 1} u(y) dy \leq c_1 \int_{A(0, \frac{10r}{6}, 2r)} |y|^{1 - \frac{\alpha}{2}} u(y) dy, \end{aligned}$$

for some constant  $c_1 = c_1(\alpha)$ . Thus there is a  $\sigma \in (\frac{10}{6}r, \frac{11}{6}r)$  such that

$$\int_{A(0, \sigma, 2r)} (|y| - \sigma)^{-\frac{\alpha}{2}} u(y) dy \leq c_2 r^{-1} \int_{A(0, \frac{10r}{6}, 2r)} u(y) |y|^{1 - \frac{\alpha}{2}} dy \tag{5.8}$$

for some constant  $c_2 = c_2(\alpha)$ . Let  $x \in D \cap B(0, \frac{3}{2}r)$ . Note that, by Theorem 4.1

$$\begin{aligned} &\mathbb{E}_x [u(Y_{\tau_{D \cap B(0, \sigma)}}); Y_{\tau_{D \cap B(0, \sigma)}} \in A(0, \sigma, 1 - ar)] \\ &= \mathbb{E}_x [u(Y_{\tau_{D \cap B(0, \sigma)}}); Y_{\tau_{D \cap B(0, \sigma)}} \in A(0, \sigma, 1 - ar), \tau_{D \cap B(0, \sigma)} = \tau_{B(0, \sigma)}] \\ &= \mathbb{E}_x [u(Y_{\tau_{B(0, \sigma)}}); Y_{\tau_{B(0, \sigma)}} \in A(0, \sigma, 1 - ar), \tau_{D \cap B(0, \sigma)} = \tau_{B(0, \sigma)}] \\ &\leq \mathbb{E}_x [u(Y_{\tau_{B(0, \sigma)}}); Y_{\tau_{B(0, \sigma)}} \in A(0, \sigma, 1 - ar)] = \int_{A(0, \sigma, 1 - ar)} K_{B(0, \sigma)}^Y(x, y) u(y) dy. \end{aligned}$$

Since  $\sigma < 2r < r_0$ , by (4.8) and (4.9) we have

$$\begin{aligned} \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B(0,\sigma)}}); Y_{\tau_{D \cap B(0,\sigma)}} \in A(0, \sigma, 1 - ar) \right] &\leq 2 \int_{A(0,\sigma,1-ar)} K_{B(0,\sigma)}(x, y) u(y) dy \\ &\leq c_3 \left( \int_{A(0,\sigma,2r)} + \int_{A(0,2r,1-ar)} \right) \frac{(\sigma^2 - |x|^2)^{\frac{\alpha}{2}}}{(|y|^2 - \sigma^2)^{\frac{\alpha}{2}}} \frac{1}{|y - x|^d} u(y) dy \end{aligned}$$

for some constant  $c_3 = c_3(d, \alpha)$ . For  $y \in A(0, 2r, 1 - ar)$ ,  $|y|^2 - \sigma^2 \geq \frac{1}{12}|y|^2$  and  $\sigma^2 - |x|^2 \leq cr^2$ . So

$$\frac{(\sigma^2 - |x|^2)^{\frac{\alpha}{2}}}{(|y|^2 - \sigma^2)^{\frac{\alpha}{2}}} \frac{1}{|y - x|^d} \leq c_4 r^\alpha |y|^{-d-\alpha}$$

for some constant  $c_4 = c_4(d, \alpha)$ . On the other hand, by (5.8),

$$\begin{aligned} \int_{A(0,\sigma,2r)} \frac{(\sigma^2 - |x|^2)^{\frac{\alpha}{2}}}{(|y|^2 - \sigma^2)^{\frac{\alpha}{2}}} \frac{1}{|y - x|^d} u(y) dy &\leq c_5 \int_{A(0,\sigma,2r)} \frac{r^\alpha}{(|y| - \sigma)^{\frac{\alpha}{2}} |y|^{\frac{\alpha}{2}}} \frac{1}{r^d} u(y) dy \\ &\leq c_5 r^{\frac{\alpha}{2}-d} \int_{A(0,\sigma,2r)} (|y| - \sigma)^{-\frac{\alpha}{2}} u(y) dy \leq c_6 r^{-1+\frac{\alpha}{2}-d} \int_{A(0, \frac{10r}{6}, 2r)} u(y) |y|^{-\frac{\alpha}{2}+1} dy \end{aligned}$$

for some constants  $c_i = c_i(d, \alpha), i = 5, 6$ . Hence

$$\mathbb{E}_x \left[ u(Y_{\tau_{D \cap B(z_0,\sigma)}}); Y_{\tau_{D \cap B(z_0,\sigma)}} \in A(z_0, \sigma, 1 - ar) \right] \leq c_7 r^\alpha \int_{A(0, \frac{10r}{6}, 1-ar)} \frac{u(y)}{|y|^{d+\alpha}} dy$$

for some constant  $c_7 = c_7(d, \alpha)$ . □

**Lemma 5.5** *Let  $D$  be an open set. Assume that  $B(A, \kappa r) \subset D \cap B(Q, r)$  for some  $0 < r < r_0$  and  $\kappa \in (0, \frac{1}{2}]$ . Suppose that  $u \geq 0$  is regular harmonic in  $D \cap B(Q, 2r)$  with respect to  $Y$  and  $u = 0$  in  $(D^c \cap B(Q, 2r)) \cup B(Q, 1 - r)^c$ . If  $w$  is a regular harmonic function with respect to  $Y$  in  $D \cap B(Q, r)$  such that*

$$w(x) = \begin{cases} u(x), & x \in B\left(Q, \frac{3r}{2}\right)^c \cup (D^c \cap B(Q, r)), \\ 0, & x \in A\left(Q, r, \frac{3r}{2}\right), \end{cases}$$

then

$$u(A) \geq w(A) \geq C \kappa^\alpha u(x), \quad x \in D \cap B\left(Q, \frac{3}{2}r\right)$$

for some constant  $C = C(d, \alpha) > 0$ .

*Proof* Without loss of generality, we may assume  $Q = 0$  and  $x \in D \cap B(0, \frac{3}{2}r)$ . The left hand side inequality in the conclusion of the lemma is obvious, so we only need to prove the right hand side inequality. Since  $u$  is regular harmonic in  $D \cap B(0, 2r)$  with

respect to  $Y$  and  $u = 0$  on  $B(0, 1 - r)^c$ , we know from Lemma 5.4 that there exists  $\sigma \in (\frac{10r}{6}, \frac{11r}{6})$  such that

$$u(x) = \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B(0, \sigma)}}); Y_{\tau_{D \cap B(0, \sigma)}} \in A(0, \sigma, 1 - r) \right] \leq c_1 r^\alpha \int_{A(0, \frac{3r}{2}, 1-r)} \frac{u(y)}{|y|^{d+\alpha}} dy$$

for some constant  $c_1 = c_1(d, \alpha)$ . On the other hand, by (4.7), we have that

$$\begin{aligned} w(A) &= \int_{A(0, \frac{3r}{2}, 1-r)} K_{D \cap B(0, r)}^Y(A, y) u(y) dy \geq \int_{A(0, \frac{3r}{2}, 1-r)} K_{B(A, \kappa r)}^Y(A, y) u(y) dy \\ &\geq \int_{A(0, \frac{3r}{2}, 1-r)} K_{B(A, \kappa r)}(A, y) u(y) dy \\ &= c_2 \int_{A(0, \frac{3r}{2}, 1-r)} \frac{(\kappa r)^\alpha}{(|y - A|^2 - (\kappa r)^2)^{\frac{\alpha}{2}}} \frac{1}{|y - A|^d} u(y) dy \end{aligned}$$

for some constant  $c_2 = c_2(d, \alpha)$ . Note that  $|y - A| \leq 2|y|$  on  $A(0, \frac{3r}{2}, 1 - r)$ . Hence

$$w(A) \geq c_3 \kappa^\alpha r^\alpha \int_{A(0, \frac{3r}{2}, 1-r)} \frac{u(y)}{|y|^{d+\alpha}} dy$$

for some constant  $c_3 = c_3(d, \alpha)$ . Therefore  $w(A) \geq c_4 \kappa^\alpha u(x)$  for some constant  $c_4 = c_4(d, \alpha)$ . □

The following result is a boundary Harnack principle for nonnegative functions which are harmonic with respect to  $Y$  and vanish outside a small ball. The proof is similar to the proof in [29] but we spell out the details for the reader’s convenience.

**Theorem 5.6** *Suppose that  $D$  is an open set,  $Q \in \partial D, r > 0$  and that  $B(A, \kappa r)$  is a ball in  $D \cap B(Q, r)$ . If  $2r < r_0$ , then for any nonnegative functions  $u, v$  in  $\mathbb{R}^d$  which are regular harmonic in  $D \cap B(Q, 2r)$  with respect to  $Y$  and vanish in  $(D^c \cap B(Q, 2r)) \cup B(Q, 1 - r)^c$ , we have*

$$C^{-1} \kappa^{d+\alpha} \frac{u(A)}{v(A)} \leq \frac{u(x)}{v(x)} \leq C \kappa^{-d-\alpha} \frac{u(A)}{v(A)}, \quad x \in D \cap B\left(Q, \frac{r}{2}\right),$$

for some constant  $C = C(d, \alpha) > 1$ .

*Proof* Without loss of generality we may assume that  $Q = 0$  and  $u(A) = v(A)$ . Define  $u_1$  and  $u_2$  to be regular harmonic functions in  $D \cap B(0, r)$  with respect to  $Y$  such that

$$u_1(x) = \begin{cases} u(x), & r \leq |x| < \frac{3r}{2}, \\ 0, & x \in B\left(0, \frac{3r}{2}\right)^c \cup (D^c \cap B(0, r)), \end{cases}$$

and

$$u_2(x) = \begin{cases} u(x), & x \in B\left(0, \frac{3r}{2}\right)^c \cup (D^c \cap B(0, r)), \\ 0, & r \leq |x| < \frac{3r}{2}, \end{cases}$$

and note that  $u = u_1 + u_2$ . If  $D \cap \{r \leq |y| < \frac{3r}{2}\}$  is empty, then  $u_1 = 0$  and the inequality (5.11) below holds trivially. So we assume  $D \cap \{r \leq |y| < \frac{3r}{2}\}$  is not empty. Then by Lemma 5.5,

$$u(y) \leq c_1 \kappa^{-\alpha} u(A), \quad y \in D \cap B\left(0, \frac{3r}{2}\right),$$

for some constant  $c_1 = c_1(d, \alpha)$ . For  $x \in D \cap B(0, \frac{r}{2})$ , we have

$$\begin{aligned} u_1(x) &= \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B(0,r)}}) : Y_{\tau_{D \cap B(0,r)}} \in D \cap \left\{ r \leq |y| < \frac{3r}{2} \right\} \right] \\ &\leq \left( \sup_{D \cap \{r \leq |y| < \frac{3r}{2}\}} u(y) \right) \mathbb{P}_x \left( Y_{\tau_{D \cap B(0,r)}} \in D \cap \left\{ r \leq |y| < \frac{3r}{2} \right\} \right) \\ &\leq \left( \sup_{D \cap \{r \leq |y| < \frac{3r}{2}\}} u(y) \right) \mathbb{P}_x (Y_{\tau_{D \cap B(0,r)}} \in B(0, r)^c) \\ &\leq c_1 \kappa^{-\alpha} u(A) \mathbb{P}_x (Y_{\tau_{D \cap B(0,r)}} \in B(0, r)^c). \end{aligned}$$

Now using Lemma 5.3 we get

$$u_1(x) \leq c_2 \kappa^{-d-\alpha} u(A) \mathbb{P}_x \left( Y_{\tau_{(D \cap B(0,r)) \setminus B(A, \frac{\kappa r}{2})}} \in B\left(A, \frac{\kappa r}{2}\right) \right), \quad x \in D \cap B\left(0, \frac{r}{2}\right) \tag{5.9}$$

for some constant  $c_2 = c_2(d, \alpha)$ . Since  $2r < r_0$ , Corollary 4.10 implies that

$$u(y) \geq c_3 u(A), \quad y \in B\left(A, \frac{\kappa r}{2}\right)$$

for some constant  $c_3 = c_3(d, \alpha)$ . Therefore for  $x \in D \cap B(0, \frac{r}{2})$

$$u(x) = \mathbb{E}_x \left[ u \left( Y_{\tau_{(D \cap B(0,r)) \setminus B(A, \frac{\kappa r}{2})}} \right) \right] \geq c_3 u(A) \mathbb{P}_x \left( Y_{\tau_{(D \cap B(0,r)) \setminus B(A, \frac{\kappa r}{2})}} \in B\left(A, \frac{\kappa r}{2}\right) \right). \tag{5.10}$$

From (5.9), the analogue of (5.10) for  $v$  and the assumption that  $u(A) = v(A)$ , it follows that for  $x \in D \cap B(0, \frac{r}{2})$ ,

$$u_1(x) \leq c_2 \kappa^{-d-\alpha} v(A) \mathbb{P}_x \left( Y_{\tau_{(D \cap B(0,r)) \setminus B(A, \frac{\kappa r}{2})}} \in B\left(A, \frac{\kappa r}{2}\right) \right) \leq c_4 \kappa^{-d-\alpha} v(x) \tag{5.11}$$

for some constant  $c_4 = c_4(d, \alpha)$ . Since  $u = 0$  on  $B(0, 1 - r)^c$ , we have that for  $x \in D \cap B(0, r)$ ,

$$\begin{aligned} u_2(x) &= \int_{A(0, \frac{3r}{2}, 1-r)} K_{D \cap B(0,r)}^Y(x, z) u(z) dz \\ &= \mathcal{A}(d, -\alpha) \int_{A(0, \frac{3r}{2}, 1-r)} \left( \int_{D \cap B(0,r)} \frac{G_{D \cap B(0,r)}^Y(x, y)}{|y - z|^{d+\alpha}} dy \right) u(z) dz. \end{aligned}$$

Let

$$s(x) := \mathcal{A}(d, -\alpha) \int_{D \cap B(0,r)} G_{D \cap B(0,r)}^Y(x, y) dy,$$

then we have

$$c_5^{-1} \leq \frac{u_2(x)}{u_2(A)} \bigg/ \frac{s(x)}{s(A)} \leq c_5, \tag{5.12}$$

for some constant  $c_5 = c_5(d, \alpha)$ . Applying (5.12) to  $u$  and  $v$  and Lemma 5.5 to  $v$  and  $v_2$ , we obtain for  $x \in D \cap B(0, \frac{r}{2})$ ,

$$u_2(x) \leq c_5 u_2(A) \frac{s(x)}{s(A)} \leq c_5^2 \frac{u_2(A)}{v_2(A)} v_2(x) \leq c_6 \frac{u(A)}{\kappa^\alpha v(A)} v_2(x) = c_6 \kappa^{-\alpha} v_2(x), \tag{5.13}$$

for some constant  $c_6 = c_6(d, \alpha)$ . Combining (5.11) and (5.13), we have

$$u(x) \leq c_7 \kappa^{-d-\alpha} v(x), \quad x \in D \cap B\left(0, \frac{r}{2}\right),$$

for some constant  $c_7 = c_7(d, \alpha)$ . □

The theorem above applies to any  $\kappa$ -fat  $D$ , but the harmonic functions there are assumed to vanish outside a small ball. Thus the theorem above is very useful in studying properties of positive functions which are harmonic with respect to  $Y$  in  $\kappa$ -fat sets with diameters less than 1, and not very useful in the case when the diameters of the  $\kappa$ -fat sets are large.

Comparing the boundary Harnack principle above with the boundary Harnack principle for symmetric stable processes in [29], we notice that in the boundary Harnack principle above we assumed an extra condition that the functions vanish in  $B(Q, 1 - r)^c$ . This extra condition is not purely technical. In the next section, we will give an example of a bounded non-convex domain showing that, without this extra condition, the boundary Harnack principle for  $Y$  fails.

In the remainder of this section, we will prove a boundary Harnack principle for nonnegative functions which are harmonic with respect to  $Y$  in bounded convex domains without assuming that they vanish outside small balls.

It is well-known that every convex domain is Lipschitz. Recall that a bounded domain  $D$  is said to be Lipschitz if there exist a localization radius  $R_0 > 0$  and a constant  $\Lambda > 0$  such that for every  $Q \in \partial D$ , there exist a Lipschitz function  $\phi_Q : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  satisfying  $\phi_Q(0) = 0$ ,  $|\phi_Q(x) - \phi_Q(z)| \leq \Lambda|x - z|$ , and an orthonormal coordinate system  $y = (y_1, \dots, y_{d-1}, y_d) := (\tilde{y}, y_d)$  such that  $B(Q, R_0) \cap D = B(Q, R_0) \cap \{y : y_d > \phi_Q(\tilde{y})\}$ . The pair  $(R_0, \Lambda)$  is called the characteristics of the Lipschitz domain  $D$ . It is easy to see that  $D$  is  $\kappa$ -fat with characteristics  $(R_0, \kappa_0)$  with some  $\kappa_0 = \kappa_0(D)$ .

In the remainder of this section we assume  $D$  is a bounded convex domain with the Lipschitz characteristics  $(R_0, \Lambda)$  and the  $\kappa$ -fat characteristics  $(R_0, \kappa_0)$ .

For every  $Q \in \partial D$  and  $x \in B(Q, R_0) \cap \{y : y_d > \phi_Q(\tilde{y})\}$ , let  $\delta_Q(x) := x_d - \phi_Q(\tilde{x})$ . Since  $D$  is bounded Lipschitz, there exists a constant  $c = c(d, \Lambda) \geq 1$  such for every  $Q \in \partial D$  and  $x \in B(Q, R_0) \cap \{y : y_d > \phi_Q(\tilde{y})\}$  we have

$$c^{-1} \delta_Q(x) \leq \rho(x) \leq \delta_Q(x). \tag{5.14}$$

The next result is well-known (see Lemma 6.7 in [17] for the Brownian motion case).

**Lemma 5.7** *Let  $Q \in \partial D$ . Assume that  $B(A, \kappa r) \subset D \cap B(Q, r)$  for some positive  $r < \frac{1}{4}R_0$  and  $\kappa \in (0, \frac{1}{2}]$ . Then there exists  $c = c(\alpha, d, D) > 0$  such that for every  $y \in B(Q, (4 - \frac{1}{2}\kappa)r)$  with  $\delta_Q(y) > \frac{1}{2}\kappa r$ , we have*

$$G_{B(Q, 4r) \cap D}(A, y) \geq c|A - y|^{-d+\alpha}.$$

*Proof* The proof of this result is standard and we omit the details. □

**Lemma 5.8** *Let  $Q \in \partial D$ . Assume that  $B(A, \kappa r) \subset D \cap B(Q, r)$  for some  $0 < r < \frac{1}{4}(r_0 \wedge R_0)$  and  $\kappa \in (0, \frac{1}{2}]$ . Suppose that  $u \geq 0$  is regular harmonic in  $D \cap B(Q, 4r)$  with respect to  $Y$  and  $u = 0$  in  $D^c$ , then*

$$u(A) \geq C r^\alpha \int_{A(Q,r,1+2r)} |z|^{-d-\alpha} u(z) dz$$

for some constant  $C = C(d, \alpha, D) > 0$ .

*Proof* Without loss of generality, we may assume  $Q = 0$ . Let  $\phi(\tilde{x}) := \phi_0(\tilde{x})$  and  $\delta(x) := \delta_0(x)$ . Since  $u$  is regular harmonic in  $D \cap B(0, 4r)$  with respect to  $Y$  and  $D \cap B(0, r)$  is bounded Lipschitz, by Theorem 4.1 we have

$$\begin{aligned} u(A) &= \mathbb{E}_A [u(Y_{\tau_{D \cap B(0,r)}})] \geq \int_{A(0,r,1-r)} K_{D \cap B(0,r)}^Y(A, z) u(z) dz \\ &= \int_{A(0,r,1-r)} A(d, -\alpha) \int_{D \cap B(0,r) \cap \{|y-z| < 1\}} \frac{G_{D \cap B(0,r)}^Y(A, y)}{|y-z|^{d+\alpha}} dy u(z) dz \\ &= \int_{A(0,r,1-r)} A(d, -\alpha) \int_{D \cap B(0,r)} \frac{G_{D \cap B(0,r)}^Y(A, y)}{|y-z|^{d+\alpha}} dy u(z) dz \end{aligned}$$

Since  $B(A, \kappa r) \subset D \cap B(0, r)$ , by the monotonicity of the Green functions and (3.1),

$$G_{D \cap B(0,r)}^Y(A, y) \geq G_{D \cap B(0,r)}(A, y) \geq G_{B(A, \kappa r)}(A, y), \quad y \in B(A, \kappa r).$$

Thus

$$\begin{aligned} u(A) &\geq \int_{A(0,r,1-r)} A(d, -\alpha) \int_{B(A, \kappa r)} \frac{G_{B(A, \kappa r)}(A, y)}{|y-z|^{d+\alpha}} dy u(z) dz \\ &= \int_{A(0,r,1-r)} K_{B(A, \kappa r)}(A, z) u(z) dz, \end{aligned}$$

which is equal to

$$c_1 \int_{A(0,r,1-r)} \frac{(\kappa r)^\alpha}{(|z-A|^2 - (\kappa r)^2)^{\frac{\alpha}{2}}} \frac{1}{|z-A|^d} u(z) dz$$

for some constant  $c_1 = c_1(d, \alpha)$  by (4.9). Note that  $|z - A| \leq 2|z|$  for  $z \in A(0, r, 1 - r)$ . Hence

$$u(A) \geq c_2 \kappa^\alpha r^\alpha \int_{A(0,r,1-r)} \frac{u(z)}{|z|^{d+\alpha}} dz \tag{5.15}$$

for some constant  $c_2 = c_2(d, \alpha)$ . Now we will establish a different lower bound for  $u(A)$ . Since  $u$  is regular harmonic in  $D \cap B(0, 4r)$  with respect to  $Y$  and is zero outside of  $D$ , we have

$$\begin{aligned}
 u(A) &\geq \mathbb{E}_A \left[ u(Y_{\tau_{D \cap B(0,4r)}}); Y_{\tau_{D \cap B(0,4r)}} \in A(0, 1 - r, 1 + 2r) \right] \\
 &= \int_{A(0,1-r,1+2r) \cap D} K_{B(0,4r) \cap D}^Y(A, z) u(z) dz.
 \end{aligned}$$

Let

$$\Omega_{\kappa r} := \left\{ y \in D \cap B \left( 0, \left( 4 - \frac{1}{2} \kappa \right) r \right) : \delta(y) > \frac{1}{2} \kappa r \right\}.$$

Since  $|y - z| \leq |y| + |z| < 4r + 1 + 2r < 2$  for  $z \in A(0, 1 - r, 1 + 2r)$  and  $y \in B(0, 4r)$ , Theorem 4.1 and (3.1) imply that

$$\begin{aligned}
 K_{B(0,4r) \cap D}^Y(A, z) &\geq c_3 \int_{B(0,4r) \cap D \cap \{|y-z|<1\}} G_{B(0,4r) \cap D}^Y(A, y) dy \\
 &\geq c_3 \int_{B(0,4r) \cap D \cap \{|y-z|<1\}} G_{B(0,4r) \cap D}(A, y) dy \\
 &\geq c_3 \int_{\Omega_{\kappa r} \cap \{|y-z|<1\}} G_{B(0,4r) \cap D}(A, y) dy
 \end{aligned}$$

for some constant  $c_3 = c_3(d, \alpha)$ . By Lemma 5.7, there exists  $c_4 = c_4(d, \alpha, D)$  such that

$$G_{B(0,4r) \cap D}(A, y) \geq c_4 |A - y|^{-d+\alpha} \geq c_4 8^{-d+\alpha} r^{-d+\alpha}, \quad y \in \Omega_{\kappa r}.$$

So we have

$$\inf_{z \in A(0,1-r,1+2r) \cap D} K_{B(0,4r) \cap D}^Y(A, z) \geq c_5 r^{-d+\alpha} \inf_{z \in A(0,1-r,1+2r) \cap D} |\Omega_{\kappa r} \cap \{|y - z| < 1\}|$$

for some constant  $c_5 = c_5(d, \alpha, D)$ . For each  $z \in A(0, 1 - r, 1 + 2r) \cap D$ , let  $b^z$  be the point on the line segment between  $z$  and the origin such that

$$|b^z| = \left( 3 - \frac{\kappa}{4} \right) r \quad \text{and} \quad |b^z - z| = |z| - \left( 3 - \frac{\kappa}{4} \right) r.$$

Note that since  $D$  is convex and  $z \in D$ ,  $b^z$  is in  $D$ . Let

$$S_z := \left\{ (\tilde{y}, y_d) \in B(0, R_0) \cap D : |\tilde{y} - \tilde{b}^z| < \frac{r}{8(1 + \Lambda)}, \delta(b^z) + \frac{1}{2} \kappa r < \delta(y) < \delta(b^z) + \frac{3}{8} r \right\}.$$

We claim that for every  $z \in A(0, 1 - r, 1 + 2r) \cap D$ ,  $S_z \subset \Omega_{\kappa r} \cap \{|y - z| < 1\}$ .

For every  $y \in S_z$ ,

$$\begin{aligned}
 |y - b^z| &\leq |\tilde{y} - \tilde{b}^z| + |y_d - b_d^z| < \frac{r}{8(1 + \Lambda)} + |\delta(y) - \delta(b^z)| + |\phi(\tilde{y}) - \phi(\tilde{b}^z)| \\
 &< \frac{r}{8(1 + \Lambda)} + \frac{3}{8} r + \Lambda |\tilde{y} - \tilde{b}^z| < \frac{r}{8(1 + \Lambda)} + \frac{3}{8} r + \frac{r\Lambda}{8(1 + \Lambda)} = \frac{r}{2}.
 \end{aligned}$$

Thus for every  $z \in A(0, 1 - r, 1 + 2r) \cap D$  and  $y \in S_z$ , we have

$$|y - z| \leq |y - b^z| + |b^z - z| < \frac{r}{2} + |z| - \left( 3 - \frac{\kappa}{4} \right) r = |z| - 2r - \frac{1}{4} (2 - \kappa) r < 1,$$

and

$$|y| \leq |y - b^z| + |b^z| < \frac{r}{2} + \left( 3 - \frac{\kappa}{4} \right) r = \left( \frac{7}{2} - \frac{\kappa}{4} \right) r < \left( 4 - \frac{\kappa}{2} \right) r.$$

Thus the claim is proved. Let  $\varphi^z(\cdot) := \phi(\cdot + \tilde{b}^z) - \phi(\tilde{b}^z)$ . By the change of variable  $w = y - b^z$ ,

$$\begin{aligned} |S_z| &= \int_{\left\{|\tilde{y}-\tilde{b}^z| < \frac{r}{8(1+\Lambda)}\right\}} \int_{\left\{b_d^z-\phi(\tilde{b}^z)+\frac{1}{2}\kappa r < y_d-\phi(\tilde{y}) < b_d^z-\phi(\tilde{b}^z)+\frac{3}{8}r\right\}} dy_d d\tilde{y} \\ &= \int_{\left\{|\tilde{w}| < \frac{r}{8(1+\Lambda)}\right\}} \int_{\left\{\frac{1}{2}\kappa r < w_d-\varphi^z(\tilde{w}) < \frac{3}{8}r\right\}} dw_d d\tilde{w} \geq \int_{\left\{|\tilde{w}| < \frac{r}{8(1+\Lambda)}\right\}} \int_{\left\{\frac{1}{4}r < w_d-\varphi^z(\tilde{w}) < \frac{3}{8}r\right\}} dw_d d\tilde{w}. \end{aligned}$$

Since  $\varphi^z(\cdot)$  is Lipschitz with Lipschitz constant  $\Lambda$  for  $z \in A(0, 1 - r, 1 + 2r) \cap D$ , the last quantity above is bounded below by  $c_6 r^d$  for some positive constant  $c_6 = c_6(D)$  for every  $z \in A(0, 1 - r, 1 + 2r) \cap D$ . Thus we get

$$\inf_{z \in A(0, 1 - r, 1 + 2r) \cap D} K_{B(0, 4r) \cap D}^Y(A, z) \geq c_7 r^\alpha$$

for some constant  $c_7 = c_7(d, \alpha, D)$ . Therefore

$$u(A) \geq c_8 r^\alpha \int_{A(0, 1 - r, 1 + 2r) \cap D} |z|^{-d-\alpha} u(z) dz = c_8 r^\alpha \int_{A(0, 1 - r, 1 + 2r)} |z|^{-d-\alpha} u(z) dz \tag{5.16}$$

for some constant  $c_8 = c_8(d, \alpha, D)$ . (5.15) and (5.16) imply the lemma. □

The next lemma is a Carleson type estimates for truncated stable processes.

**Lemma 5.9** *Let  $Q \in \partial D$  and assume that  $B(A, \kappa r) \subset D \cap B(Q, r)$  for some  $0 < r < \frac{1}{4}(r_0 \wedge R_0)$  and  $\kappa \in (0, \frac{1}{2}]$ . If  $u \geq 0$  is regular harmonic in  $D \cap B(Q, 4r)$  with respect to  $Y$  and  $u = 0$  in  $D^c$ , then*

$$u(A) \geq C u(x), \quad x \in D \cap B\left(Q, \frac{3}{2}r\right)$$

for some constant  $C = C(D, d, \alpha, \kappa)$ .

*Proof* Without loss of generality, we may assume  $Q = 0$ . Let  $x \in D \cap B(0, \frac{3}{2}r)$ . Since  $u$  is regular harmonic in  $D \cap B(0, 4r)$  with respect to  $Y$ , we have

$$\begin{aligned} u(x) &= \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B(0, \sigma)}}); Y_{\tau_{D \cap B(0, \sigma)}} \in A \left( 0, \sigma, 1 - \frac{3}{2}r \right) \right] \\ &\quad + \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B(0, \sigma)}}); Y_{\tau_{D \cap B(0, \sigma)}} \in A \left( 0, 1 - \frac{3}{2}r, 1 + \sigma \right) \right] =: u_1(x) + u_2(x), \end{aligned}$$

where  $\sigma$  is the constant from Lemma 5.4. By Lemma 5.4, We have

$$u_1(x) \leq c_1 r^\alpha \int_{A\left(0, \frac{10r}{6}, 1 - \frac{3}{2}r\right)} \frac{u(z)}{|z|^{d+\alpha}} dz \tag{5.17}$$

for some constant  $c_1 = c_1(d, \alpha)$ .

Now we consider  $u_2(x)$ . We have

$$\begin{aligned}
 u_2(x) &= \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B(0,\sigma)}}); Y_{\tau_{D \cap B(0,\sigma)}} \in A \left( 0, 1 - \frac{3}{2}r, 1 + \sigma \right) \right] \\
 &\leq \int_{A(0, 1 - \frac{3}{2}r, 1 + \sigma)} K_{B(0,\sigma)}^Y(x, z) u(z) dz.
 \end{aligned}$$

We know from Lemma 4.4 that  $K_{B(0,\sigma)}^Y(x, z) \leq c_2 r^\alpha$  for some constant  $c_2 = c_2(d, \alpha)$ . Therefore

$$u_2(x) \leq c_2 r^\alpha \int_{A(0, 1 - \frac{3}{2}r, 1 + \sigma)} u(z) dz \leq c_3 r^\alpha \int_{A(0, 1 - \frac{3}{2}r, 1 + \sigma)} \frac{u(z)}{|z|^{d+\alpha}} dz \tag{5.18}$$

for some constant  $c_3 = c_3(d, \alpha)$  since  $\sigma \in (\frac{10r}{6}, \frac{11r}{6})$ . Combining (5.17), (5.18) and Lemma 5.8, we have proved the lemma.  $\square$

We shall follow the ‘‘box method’’ of [7], originally developed by Bass and Burdzy ([2] and [3]). Since we are going to use results of [7], we will closely follow their notations for the reader’s convenience. Recall that for every  $Q \in \partial D$  and  $x \in B(Q, R_0) \cap \{y : y_d > \phi_Q(\tilde{y})\}$ ,  $\delta_Q(x) := x_d - \phi_Q(\tilde{x})$ . For  $x \in B(Q, R_0) \cap \{y : y_d \geq \phi_Q(\tilde{y})\}$ , we define

$$\begin{aligned}
 \Delta_Q(x, ar, br) &:= \{y : ar > \delta_Q(x) > 0, |\tilde{y} - \tilde{x}| < br\}, \\
 \nabla_Q(x, ar, br) &:= \{y : 0 > \delta_Q(x) > -ar, |\tilde{y} - \tilde{x}| < br\}, \\
 F_{1,Q}^r &:= \left\{ X_{\tau_{\Delta_Q(Q,r,3r)}} \in \mathbb{R}^d \setminus (\Delta_Q(Q, r, 3r) \cup \nabla_Q(Q, 3r, 5r)) \right\}, \\
 F_{2,Q}^r &:= \left\{ X_{\tau_{\Delta_Q(Q,r,3r)}} \in \Delta_Q(Q, 2r, 3r) \right\}.
 \end{aligned}$$

We start with the following simple lemma.

**Lemma 5.10** *For any positive constants  $a, b$  and  $r$  with  $((a + b) + b\Lambda)r < R_0$ ,*

$$\Delta_Q(Q, ar, br) \cup \nabla_Q(Q, ar, br) \subset B(Q, ((a + b) + b\Lambda)r).$$

*Proof* For  $y \in \Delta_Q(Q, ar, br) \cup \nabla_Q(Q, ar, br)$ ,

$$\begin{aligned}
 |y - Q| &\leq |\tilde{y} - \tilde{Q}| + |y_d - Q_d| < br + |\delta_Q(y)| + |\phi_Q(\tilde{y}) - \phi_Q(\tilde{Q})| \\
 &< (a + b)r + \Lambda|\tilde{y} - \tilde{Q}| < ((a + b) + b\Lambda)r.
 \end{aligned}$$

$\square$

Since the above lemma implies that

$$\Delta_Q(Q, 3r, 5r) \cup \nabla_Q(Q, 3r, 5r) \subset B(Q, 3(3 + 2\Lambda)r),$$

the next lemma follows from Lemma 7 and Remark 2 of [7] and the scaling property.

**Lemma 5.11** *Suppose  $Q \in \partial D$  and  $r < \frac{R_0}{3(3+2\Lambda)r}$ . If  $x \in \Delta_Q(Q, r, 3r)$ , then  $\mathbb{P}_x$  distribution of  $X_{\tau_{\Delta_Q(Q,r,3r)}}$  is absolutely continuous on  $\mathbb{R}^d \setminus (\Delta_Q(Q, 2r, 4r) \cup \nabla_Q(Q, 2r, 4r))$  with respect to the  $d$ -dimensional Lebesgue measure and has a density function  $f_Q^{x,r}$  satisfying*

$$\begin{aligned}
 f_Q^{x,r}(y) &\leq c r^\alpha \mathbb{P}_x(F_{1,Q}^r) (\text{dist}(y, \Delta_Q(Q, r, 3r))^{-d-\alpha}, \\
 y &\in \mathbb{R}^d \setminus (\Delta_Q(Q, 2r, 4r) \cup \nabla_Q(Q, 2r, 4r)),
 \end{aligned}
 \tag{5.19}$$

where  $c = c(D)$  is independent of  $Q \in \partial D$ .

For any Lipschitz function  $\psi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  with Lipschitz constant  $\Lambda$ , let

$$\Delta^\psi := \left\{ y : \frac{R_0}{2(4 + 3\Lambda)} > y_d - \psi(\tilde{y}) > 0, \quad |\tilde{y}| < \frac{3R_0}{2(4 + 3\Lambda)} \right\}.$$

We observe that, for any Lipschitz function  $\varphi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  with Lipschitz constant  $\Lambda$ , its dilation  $\varphi_r(x) := r\varphi(x/r)$  is also Lipschitz with the same Lipschitz constant  $\Lambda$ . For any  $Q \in \partial D$ , let  $\phi_Q$  be the function in the definition of a Lipschitz domain. For any  $r > 0$ , put  $\eta = (2(4 + 3\Lambda)r)/R_0$  and  $\psi = (\phi_Q)_\eta$ . Then it is easy to see that for any  $Q \in \partial D$  and  $r > 0$ ,

$$\Delta_Q(Q, r, 3r) = \eta \Delta^\psi.$$

We can show that  $\Delta^\psi \subset B(0, \frac{1}{2}R_0)$  by the same argument in the proof of Lemma 5.10. On the other hand, it is easy to see that  $\Delta^\psi$  is a  $\kappa_1$ -fat open set with  $\kappa_1 = \kappa_1(\Lambda, R_0)$  for every Lipschitz function  $\psi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$  with Lipschitz constant  $\Lambda$ . Therefore by Proposition 3.4, there exists positive constant  $r_2$  such that for every  $Q \in \partial D$  and  $r \in (0, r_2]$ , we have

$$G_{\Delta_Q(Q,r,3r)}^Y(x, y) \leq 2 G_{\Delta_Q(Q,r,3r)}(x, y), \quad x, y \in \Delta_Q(Q, r, 3r).
 \tag{5.20}$$

The next theorem is a boundary Harnack principle for bounded convex domains and it is the main result of this section. Maybe a word of caution is in order here. The boundary Harnack principle here is a little different from the ones proved in [4] and [29] in the sense that in the boundary Harnack principle below we require our harmonic functions to vanish on the whole complement of the bounded convex domain. However, this will not affect our application later since we are mainly interested in the case when the harmonic functions are given by the Green functions of the convex domain. Recall that  $D$  is a bounded convex domain with Lipschitz characteristics  $(R_0, \Lambda)$  and  $\kappa$ -fat characteristics  $(R_0, \kappa_0)$ .

**Theorem 5.12** *There exist constants  $c > 1$  and  $r_3 > 0$ , depending on  $d, \alpha$  and  $D$  such that for any  $Q \in \partial D, r < r_3$  and any nonnegative functions  $u, v$  which are regular harmonic with respect to  $Y$  in  $D \cap B(Q, 6(3 + 2\Lambda)r)$  and vanish in  $D^c$ , we have*

$$\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)} \quad \text{for any } x, y \in D \cap B\left(Q, \frac{r}{1 + \Lambda}\right).
 \tag{5.21}$$

*Proof* Without loss of generality, we may assume  $Q = 0$ . Let  $r_3 := (r_0 \wedge r_2 \wedge R_0)/(6(3 + 2\Lambda))$  and fix a  $r < r_3$ . For notational convenience, we denote  $\phi(\tilde{x}) := \phi_0(\tilde{x}), \delta(x) := \delta_0(x), F_1 = F_{1,0}^r, F_2 := F_{2,0}^r, \Delta(a, b) := \Delta_0(0, a, b)$  and  $\Delta := \Delta(r, 3r)$ . Note that if  $y \in B(0, \frac{r}{1+\Lambda}) \cap D$ , then

$$y_d - \phi(\tilde{y}) \leq y_d + \Lambda|\tilde{y}| < \frac{r}{1 + \Lambda} + \Lambda \frac{r}{1 + \Lambda} = r.$$

So  $B(0, \frac{r}{1+\Lambda}) \cap D \subset \Delta(r, r)$ . Thus it is enough to consider  $x, y \in \Delta(r, r)$ . By Lemma 5.10,  $\Delta(2r, 4r) \subset B(0, 2(3+2\Lambda)r)$ . Thus by Lemma 5.9, there exists a positive constant  $c_1 = c_1(D, d, \alpha)$

$$\sup_{\Delta(2r, 4r)} u \leq c_1 u(A) \tag{5.22}$$

where  $B(A, \kappa_0 r) \subset D \cap B(0, \frac{4}{3}(3+2\Lambda)r)$ . Let  $\Delta_1 := \{z \in D \setminus \Delta : \text{dist}(z, \Delta) < 1\}$ . By (5.20) and Theorem 4.1, we have

$$K_{\Delta}^Y(x, z) \leq 2 K_{\Delta}(x, z), \quad z \in \Delta_1. \tag{5.23}$$

Thus,

$$\begin{aligned} \mathbb{E}_x [u(Y_{\tau_{\Delta}}) : Y_{\tau_{\Delta}} \in D \setminus \Delta(2r, 4r)] &= \int_{\Delta_1 \setminus \Delta(2r, 4r)} K_{\Delta}^Y(x, z) u(z) dz \\ &\leq 2 \mathbb{E}_x [u(X_{\tau_{\Delta}}) : X_{\tau_{\Delta}} \in \Delta_1 \setminus \Delta(2r, 4r)]. \end{aligned}$$

By Lemma 5.11, we have

$$\begin{aligned} \mathbb{E}_x [u(X_{\tau_{\Delta}}) : X_{\tau_{\Delta}} \in \Delta_1 \setminus \Delta(0, 2r, 4r)] &\leq c_2 r^{\alpha} \mathbb{P}_x(F_1) \int_{\Delta_1 \setminus \Delta(0, 2r, 4r)} (\text{dist}(z, \Delta))^{-d-\alpha} u(z) dz \\ &\leq c_3 r^{\alpha} \mathbb{P}_x(F_1) \int_{\Delta_1 \setminus \Delta(0, 2r, 4r)} |z|^{-d-\alpha} u(z) dz \end{aligned}$$

for some constants  $c_i = c_i(d, \alpha, D), i = 2, 3$ . In the last inequality above, we have used the fact that for  $z \in \Delta_1 \setminus \Delta(2r, 4r)$ ,

$$|z| \leq \text{dist}(z, \Delta) + \text{diam}(\Delta) \leq \text{dist}(z, \Delta) + 2(4+3\Lambda)r \leq 3(3+2\Lambda) \text{dist}(z, \Delta).$$

Therefore, by (5.22) and (5.23), for every  $x \in \Delta(r, r)$ ,

$$\begin{aligned} u(x) &= \mathbb{E}_x [u(Y_{\tau_{\Delta}}) : Y_{\tau_{\Delta}} \in \Delta(2r, 4r)] + \mathbb{E}_x [u(Y_{\tau_{\Delta}}) : Y_{\tau_{\Delta}} \in D \setminus \Delta(2r, 4r)] \\ &\leq c_1 u(A) \mathbb{P}_x (Y_{\tau_{\Delta}} \in \Delta(2r, 4r)) + 2 \mathbb{E}_x [u(X_{\tau_{\Delta}}) : X_{\tau_{\Delta}} \in \Delta_1 \setminus \Delta(2r, 4r)] \\ &\leq 2 c_1 u(A) \mathbb{P}_x (X_{\tau_{\Delta}} \in \Delta(2r, 4r)) + 2 c_3 r^{\alpha} \mathbb{P}_x(F_1) \int_{\Delta_1 \setminus \Delta(2r, 4r)} |z|^{-d-\alpha} u(z) dz \\ &= 2 c_1 u(A) \mathbb{P}_x(F_1) + 2 c_3 r^{\alpha} \mathbb{P}_x(F_1) \int_{\Delta_1 \setminus \Delta(2r, 4r)} |z|^{-d-\alpha} u(z) dz. \end{aligned} \tag{5.24}$$

On the other hand, by Lemma 5.8,

$$u(A) \geq c_4 r^{\alpha} \int_{A(0, \frac{9}{4}(3+2\Lambda)r, 1+3(3+2\Lambda)r)} |z|^{-d-\alpha} u(z) dz \tag{5.25}$$

for some constant  $c_4 = c_4(D, d, \alpha)$ . Moreover, since  $u$  is harmonic with respect to  $Y$  in  $B(A, \frac{1}{2}\kappa_0 r)$  and  $B(0, \frac{9}{4}(3 + 2\Lambda)r) \subset B(A, 1 - \frac{1}{2}\kappa_0 r)$ , by (4.7)

$$\begin{aligned} u(A) &\geq \mathbb{E}_A \left[ u \left( Y_{\tau_{B(A, \frac{1}{2}\kappa_0 r)}} \right) : Y_{\tau_{B(A, \frac{1}{2}\kappa_0 r)}} \in B \left( 0, \frac{9}{4}(3 + 2\Lambda)r \right) \setminus \Delta(2r, 4r) \right] \\ &= \int_{B(0, \frac{9}{4}(3+2\Lambda)r) \setminus \Delta(2r, 4r)} K_{B(A, \frac{1}{2}\kappa_0 r)}^Y(A, z) u(z) dz \\ &\geq \int_{B(0, \frac{9}{4}(3+2\Lambda)r) \setminus \Delta(2r, 4r)} K_{B(A, \frac{1}{2}\kappa_0 r)}(A, z) u(z) dz. \end{aligned}$$

So by (4.9),

$$\begin{aligned} u(A) &\geq c_5 \int_{B(0, \frac{9}{4}(3+2\Lambda)r) \setminus \Delta(2r, 4r)} \frac{\left(\frac{1}{2}\kappa_0 r\right)^\alpha}{\left(|z - A|^2 - \left(\frac{1}{2}\kappa_0 r\right)^2\right)^{\frac{\alpha}{2}}} \frac{1}{|z - A|^d} u(z) dz \\ &\geq c_6 r^\alpha \int_{B(0, \frac{9}{4}(3+2\Lambda)r) \setminus \Delta(2r, 4r)} |z|^{-d-\alpha} u(z) dz \end{aligned} \tag{5.26}$$

for some constant  $c_i = c_i(D, d, \alpha), i = 5, 6$ . Since  $\Delta_1 \subset B(0, 1 + 3(3 + 2\Lambda)r)$ , by combining (5.25) and (5.26) we get

$$u(A) \geq c_7 r^\alpha \int_{B(0, 1+3(3+2\Lambda)r) \setminus \Delta(2r, 4r)} |z|^{-d-\alpha} u(z) dz \geq c_7 r^\alpha \int_{\Delta_1 \setminus \Delta(2r, 4r)} |y|^{-d-\alpha} u(z) dz \tag{5.27}$$

for some constant  $c_7 = c_7(D, d, \alpha)$ . Putting (5.24) and (5.27) together, we have

$$u(x) \leq c_8 u(A) \mathbb{P}_x(F_1) = c_8 \frac{u(A)}{v(A)} v(A) \mathbb{P}_x(F_1) \tag{5.28}$$

for some constant  $c_8 = c_8(D, d, \alpha)$ . By Lemma 6 in [7], we have

$$\begin{aligned} \mathbb{P}_x(F_1) &\leq c_9 \mathbb{P}_x(F_2) = c_9 \mathbb{P}_x(X_{\tau_\Delta} \in \Delta(2r, 3r)) = c_9 \int_{\Delta(2r, 3r)} K_\Delta(x, z) dz \\ &= c_9 \int_{\Delta(2r, 3r)} A(d, -\alpha) \int_\Delta \frac{G_\Delta(x, y)}{|y - z|^{d+\alpha}} dy dz \\ &\leq c_9 \int_{\Delta(2r, 3r)} A(d, -\alpha) \int_\Delta \frac{G_\Delta^Y(x, y)}{|y - z|^{d+\alpha}} dy dz \end{aligned} \tag{5.29}$$

for some constant  $c_9 = c_9(D, d, \alpha)$ . We have used (3.1) in the last inequality above. For every  $z \in \Delta(2r, 3r)$  and  $y \in \Delta, |y - z| \leq |y| + |z| < 4(3 + 2\Lambda)r < r_0 < 1$ . So (5.29) is, in fact, equal to

$$c_9 \int_{\Delta(2r, 3r)} K_\Delta^Y(x, z) dz = c_9 \mathbb{P}_x(Y_{\tau_\Delta} \in \Delta(2r, 3r)).$$

By the Harnack inequality (Theorem 4.9),

$$v(A) \mathbb{P}_x(Y_{\tau_\Delta} \in \Delta(2r, 3r)) \leq c_{10} \mathbb{E}_x[v(Y_{\tau_\Delta}) : Y_{\tau_\Delta} \in \Delta(2r, 3r)] \leq c_{10} v(x) \tag{5.30}$$

for some constant  $c_{10} = c_{10}(D, d, \alpha)$ . From (5.28) and (5.30), we conclude

$$\frac{u(x)}{v(x)} \leq c_{11} \frac{u(A)}{v(A)}, \quad x \in \Delta(r, r) \tag{5.31}$$

for some constant  $c_{11} = c_{11}(D, d, \alpha)$ . The above argument also implies that

$$\frac{v(y)}{u(y)} \leq c_{11} \frac{v(A)}{u(A)}, \quad y \in \Delta(r, r).$$

Therefore

$$\frac{u(x)}{v(x)} \leq c_{11} \frac{u(A)}{v(A)} \leq c_{11}^2 \frac{u(y)}{v(y)}, \quad x, y \in \Delta(r, r).$$

□

In the remainder of this section, we fix  $r_3 > 0$  from Theorem 5.12. The following result is analogous to Lemma 5 of [4]. We recall from Definition 3.1 that for each  $z \in \partial D$  and  $r \in (0, R_0)$ ,  $A_r(z)$  is a point in  $D \cap B(z, r)$  satisfying  $B(A_r(z), \kappa_0 r) \subset D \cap B(z, r)$ .

**Lemma 5.13** *There exist positive constants  $C = C(D, d, \alpha)$  and  $\gamma = \gamma(d, \alpha) < \alpha$  such that for any  $Q \in \partial D$  and  $r \in (0, r_3)$ , and nonnegative function  $u$  which is harmonic with respect to  $Y$  in  $D \cap B(Q, r)$  we have*

$$u(A_s(Q)) \geq C(s/r)^\gamma u(A_r(Q)), \quad s \in (0, r).$$

*Proof* Without loss of generality, we may assume  $Q = 0$ . Fix  $r < r_3$  and let

$$r_k := \left(\frac{2}{\kappa_0}\right)^{-k} r, \quad A_k := A_{r_k}(0) \quad \text{and} \quad B_k := B(A_k, r_{k+1}), \quad k = 0, 1, \dots$$

Note that the  $B_k$ 's are disjoint. So by the harmonicity of  $u$ , we have

$$u(A_k) \geq \sum_{l=0}^{k-1} \mathbb{E}_{A_k} \left[ u(Y_{\tau_{B_k}}) : Y_{\tau_{B_k}} \in B_l \right] = \sum_{l=0}^{k-1} \int_{B_l} K_{B_k}^Y(A_k, z) u(z) dz.$$

Since  $r < r_3$ , (4.7) and Corollary 4.10 imply that

$$\int_{B_l} K_{B_k}^Y(A_k, z) u(z) dz \geq c_1 u(A_l) \int_{B_l} K_{B_k}(A_k, z) dz$$

for some constant  $c_1 = c_1(d, \alpha)$ . Using the explicit formula of  $K_{B_k}$ , one can easily check that

$$\int_{B_l} K_{B_k}(A_k, z) dz \geq c_2 \left(\frac{\kappa_0}{2}\right)^{-(k-l)\alpha}, \quad z \in B_l,$$

for some constant  $c_2 = c_2(d, \alpha)$ . Therefore,

$$\left(\frac{2}{\kappa_0}\right)^{k\alpha} u(A_k) \geq c_3 \sum_{l=0}^{k-1} \left(\frac{2}{\kappa_0}\right)^{l\alpha} u(A_l)$$

for some constant  $c_3 = c_3(d, \alpha)$ . The remainder of the proof is same as in the proof of Lemma 5 in [4] and so we omit it.  $\square$

The next lemma is analogous to Lemma 14 of [4].

**Lemma 5.14** *Suppose  $M := 6(3 + 2\Lambda)(1 + \Lambda)$ . Then there exist positive constants  $c_1 = c_1(D, d, \alpha)$  and  $c_2 = c_2(D, d, \alpha) < 1$  such that for any  $Q \in \partial D, r < \frac{r_3}{1+\Lambda}$  and nonnegative function  $u$  which is regular harmonic with respect to  $Y$  in  $D \cap B(Q, Mr)$  and vanishes in  $D^c$ ,*

$$\mathbb{E}_x \left[ u(Y_{\tau_{D \cap B_k}}) : Y_{\tau_{D \cap B_k}} \in A(Q, r, 1 + M^{-k}r) \right] \leq c_1 c_2^k u(x), \quad x \in D \cap B_k,$$

where  $B_k := B(Q, M^{-k}r), k = 0, 1, \dots$

*Proof* Without loss of generality, we may assume  $Q = 0$ . Fix  $r < \frac{r_3}{1+\Lambda}$  and a nonnegative function  $u$  which is harmonic with respect to  $Y$  in  $D \cap B(0, Mr)$  and vanishes in  $\mathbb{R}^d \setminus D$ .

Let  $r_k := M^{-k}r, B_k := B(0, r_k)$  and

$$u_k(x) := \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B_k}}) : Y_{\tau_{D \cap B_k}} \in A(0, r, 1 + r_k) \right], \quad x \in D \cap B_k.$$

Note that

$$\begin{aligned} u_{k+1}(x) &= \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B_{k+1}}}) : Y_{\tau_{D \cap B_{k+1}}} \in A(0, r, 1 + r_{k+1}) \right] \\ &= \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B_{k+1}}}) : \tau_{D \cap B_{k+1}} = \tau_{D \cap B_k}, Y_{\tau_{D \cap B_{k+1}}} \in A(0, r, 1 + r_{k+1}) \right] \\ &= \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B_k}}) : \tau_{D \cap B_{k+1}} = \tau_{D \cap B_k}, Y_{\tau_{D \cap B_k}} \in A(0, r, 1 + r_{k+1}) \right] \\ &\leq \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B_k}}) : Y_{\tau_{D \cap B_k}} \in A(0, r, 1 + r_{k+1}) \right] \\ &\leq \mathbb{E}_x \left[ u(Y_{\tau_{D \cap B_k}}) : Y_{\tau_{D \cap B_k}} \in A(0, r, 1 + r_k) \right] \end{aligned}$$

Thus

$$u_{k+1}(x) \leq u_k(x). \tag{5.32}$$

Let  $A_k := A_{r_k}(0)$ . We have

$$\begin{aligned} u_k(A_k) &= \mathbb{E}_{A_k} \left[ u(Y_{\tau_{D \cap B_k}}) : Y_{\tau_{D \cap B_k}} \in A(0, r, 1 + r_k) \right] \\ &\leq \mathbb{E}_{A_k} \left[ u(Y_{\tau_{B_k}}) : Y_{\tau_{B_k}} \in A(0, r, 1 + r_k) \right] \\ &\leq \int_{A(0, r, 1-r_k)} K_{B_k}^Y(A_k, z) u(z) dz + \int_{A(0, 1-r_k, 1+r_k)} K_{B_k}^Y(A_k, z) u(z) dz. \end{aligned}$$

For  $z \in A(0, r, 1 - r_k)$ , by (4.8) and (4.9) we get

$$K_{B_k}^Y(A_k, z) \leq 2 K_{B_k}(A_k, z) \leq c_1 \frac{M^{-k\alpha} r^\alpha}{|z|^{d+\alpha}}$$

for some constant  $c_1 = c_1(d, \alpha)$ . For  $z \in A(0, 1 - r_k, 1 + r_k)$ , we use Lemma 4.4 and get

$$K_{B_k}^Y(A_k, z) \leq c_2 M^{-k\alpha} r^\alpha \leq c_3 \frac{M^{-k\alpha} r^\alpha}{|z|^{d+\alpha}}$$

for some constant  $c_i = c_i(d, \alpha), i = 2, 3$ . Therefore

$$u_k(A_k) \leq c_4 M^{-k\alpha} r^\alpha \int_{A(0,r,1+r_k)} u(z) \frac{dz}{|z|^{d+\alpha}} \tag{5.33}$$

for some constant  $c_4 = c_4(d, \alpha)$ . From Lemma 5.8, we have

$$u(A_0) \geq c_5 r^\alpha \int_{A(0,r,1+r)} u(z) \frac{dz}{|z|^{d+\alpha}} \tag{5.34}$$

for some constant  $c_5 = c_5(D, d, \alpha)$ . (5.33) and (5.34) imply that  $u_k(A_k) \leq c_6 M^{-k\alpha} u(A_0)$  for some constant  $c_6 = c_6(D, d, \alpha)$ . On the other hand, using Lemma 5.13, we get  $u(A_0) \leq c_7 M^{k\gamma} u(A_k)$  for some constant  $c_7 = c_7(D, d, \alpha)$ . Thus,  $u_k(A_k) \leq c_6 c_7 M^{-k(\alpha-\gamma)} u(A_k)$ . By (5.32) and (5.31), we have

$$\frac{u_k(x)}{u(x)} \leq \frac{u_{k-1}(x)}{u(x)} \leq c_8 \frac{u_{k-1}(A_{k-1})}{u(A_{k-1})} \leq c_6 c_7 c_8 M^{-k(\alpha-\gamma)}$$

for some constant  $c_8 = c_8(D, d, \alpha)$ . □

Now the next theorem follows from Lemma 5.13, Theorem 5.12 and Lemma 5.14 (instead of using Lemma 5, Lemma 13 and Lemma 14 in [4], respectively) in very much the same way as in the case of symmetric stable process proved in Lemma 16 of [4]. We omit the details.

**Theorem 5.15** *There exist positive constants  $r_4, M_1, C$  and  $v$  depending on  $D$  and  $\alpha$  such that for any  $Q \in \partial D, r < r_4$  and nonnegative functions  $u, v$  which are regular harmonic with respect to  $Y$  in  $D \cap B(Q, M_1 r)$ , vanish in  $\mathbb{R}^d \setminus D$ , and satisfy  $u(A_r(Q)) = v(A_r(Q)) > 0$ , we have*

$$\left| \frac{u(x)}{v(x)} - \frac{u(y)}{v(y)} \right| \leq C \left( \frac{|x - y|}{r} \right)^v, \quad x, y \in D \cap B(Q, r).$$

*In particular, the limit  $\lim_{D \ni x \rightarrow w} u(x)/v(x)$  exists for every  $w \in \partial D \cap B(Q, r)$ .*

Using the results above and repeating the arguments in the proof of Theorem 4.1 of [29] we can get the following result identifying the Martin boundary of any bounded convex domain. For the definition and basic results on Martin boundary, one can see [23] and [29].

**Theorem 5.16** *Suppose that  $D$  is a bounded convex domain in  $\mathbb{R}^d$ . Then both the Martin boundary and the minimal Martin boundary of  $D$  with respect to  $Y$  coincide with the Euclidean boundary of  $D$ .*

*Proof* We omit the details. □

### 6 Counterexample

In this section, we present an example of a bounded non-convex domain on which the boundary Harnack principle for  $Y$  fails.

Consider the domain in  $R^d$

$$D := (-100, 100)^d \setminus \left( (-100, 50]^{d-1} \times [-1/2, 0] \right).$$

Of course there are nothing special about the numbers 100 and 50 above, they are just two big numbers.

Suppose the boundary Harnack principle (not necessarily scale invariant) is true for  $D$  at the origin. i.e., there exist constants  $R_1 > 0$  and  $M_1 > 1$  such that for any  $r < R_1$  and any nonnegative functions  $u, v$  which are regular harmonic with respect to  $Y$  in  $D \cap B(0, M_1 r)$  and vanish in  $D^c$ , we have

$$\frac{u(x)}{v(x)} \leq c \frac{u(y)}{v(y)} \quad \text{for any } x, y \in D \cap B(0, r), \tag{6.1}$$

where  $c = c(D, r) > 0$  is independent of harmonic functions  $u$  and  $v$ . Choose an  $r_1 < R_1$  with  $M_1 r_1 < 1/2$  and let  $A := (\tilde{0}, \frac{1}{2}r_1)$ . We define a function  $v$  by

$$v(x) := \mathbb{P}_x \left( Y_{\tau_{D \cap B(0, M_1 r_1)}} \in \{y \in D; y_d > 0\} \right).$$

By definition  $v$  is regular harmonic in  $D \cap B(0, M_1 r_1)$  with respect to  $Y$  and vanishes in  $D^c$ . Applying  $v$  above to (6.1), we have a Carleson type estimate at 0, i.e., there exists constant  $c_1 = c_1(D, r_1) > 0$  such that for any nonnegative function  $u$  which is regular harmonic with respect to  $Y$  in  $D \cap B(0, M_1 r_1)$  and vanishes in  $D^c$  we have

$$u(A) \geq c_1 u(x), \quad x \in D \cap B(0, r_1). \tag{6.2}$$

We will construct a bounded positive function  $u$  which is regular in  $D \cap B(0, M_1 r)$  with respect to  $Y$  and vanishes in  $D^c$  for which (6.2) fails.

For  $n \geq 1$ , we put

$$\begin{aligned} C_n &:= \left\{ (\tilde{x}, x_d) \in D; \quad |\tilde{x}| \leq \frac{r_1}{8}, \quad x_d \leq -1 + 2^{-n}r_1^2 \right\} \\ D_n &:= \left\{ (\tilde{y}, y_d) \in D; \quad y_d > 0, \quad |x - y| < 1 \quad \text{for some } x \in C_n \right\}. \end{aligned}$$

Note that  $D_n \supset D_{n+1} \supset \dots$  and  $\bigcap_1^\infty D_n = \emptyset$ . Moreover, it is easy to see that

$$D_n \subset B(0, r_1) \cap D, \quad \text{for } n \geq 3. \tag{6.3}$$

In fact, for any  $y \in D_n$ , we have  $y_d \in (0, 2^{-n}r_1^2)$  and  $|y - x| < 1$  for some  $x \in C_n$ , thus  $y_d - x_d > -x_d \geq 1 - 2^{-n}r_1^2$  and

$$|\tilde{y} - \tilde{x}|^2 + |y_d - x_d|^2 = |(\tilde{y}, y_d) - (\tilde{x}, x_d)|^2 < 1.$$

Hence

$$|\tilde{y}| \leq |\tilde{x}| + |\tilde{y} - \tilde{x}| \leq \frac{r_1}{8} + \sqrt{1 - |y_d - x_d|^2} < \frac{r_1}{8} + \sqrt{2^{-n+1}r_1^2}.$$

Since  $r_1 < 1$ , we get for  $n \geq 3$

$$|\tilde{y}|^2 + y_d^2 < 2^{-n}r_1^2 + \left( \frac{r_1}{8} + 2^{(-n+1)/2}r_1 \right)^2 < r_1^2.$$

For any  $n$ , let  $T_{D_n}$  be the first hitting time of  $D_n$  by the process  $Y$ . Note that since  $\bigcap_1^\infty D_n = \emptyset$ ,

$$\mathbb{P}_A(\tau_{D \cap B(0, M_1 r_1)} > T_{D_n}) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Choose  $n \geq 3$  large so that

$$\mathbb{P}_A(\tau_{D \cap B(0, M_1 r_1)} > T_{D_n}) < \frac{c_1}{2} \tag{6.4}$$

and define

$$u(x) := \mathbb{P}_x \left( Y_{\tau_{D \cap B(0, M_1 r_1)}} \in C_n \right).$$

$u$  is a nonnegative bounded function which is regular harmonic in  $D \cap B(0, M_1 r_1)$  with respect to  $Y$  and vanishes in  $D^c$ . It also vanishes continuously on  $\partial D \cap B(0, M_1 r_1)$ . Note that by Theorem 4.1,

$$\mathbb{P}_A \left( Y_{\tau_{D \cap B(0, M_1 r_1)}} \in C_n, \tau_{D \cap B(0, M_1 r_1)} \leq T_{D_n} \right) = \mathbb{P}_A \left( Y_{\tau_{D \cap B(0, M_1 r_1) \setminus D_n}} \in C_n \right) = 0.$$

Thus by the strong Markov property,

$$\begin{aligned} u(A) &= \mathbb{P}_A \left( Y_{\tau_{D \cap B(0, M_1 r_1)}} \in C_n, \tau_{D \cap B(0, M_1 r_1)} > T_{D_n} \right) \\ &= \mathbb{E}_A \left[ \mathbb{P}^{Y_{T_{D_n}}} \left( Y_{\tau_{D \cap B(0, M_1 r_1)}} \in C_n \right); \tau_{D \cap B(0, M_1 r_1)} > T_{D_n} \right] \\ &\leq \mathbb{P}_A \left( \tau_{D \cap B(0, M_1 r_1)} > T_{D_n} \right) \left( \sup_{x \in D_n} u(x) \right) < \frac{c_1}{2} \left( \sup_{x \in D \cap B(0, r_1)} u(x) \right). \end{aligned}$$

In the last inequality above, we have used (6.3) and (6.4). But by (6.2),  $u(A) \geq c_1 \sup_{x \in D \cap B(0, r_1)} u(x)$ , which gives a contradiction. Thus the boundary Harnack principle is not true for  $D$  at the origin.

By smoothing off the corners of  $D$ , we can easily construct a smooth bounded non-convex domain on which the boundary Harnack principle fails for the truncated stable process  $Y$ .

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