Sharp Heat Kernel Estimates for Relativistic Stable Processes in Open Sets

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Abstract

In this paper, we establish sharp two-sided estimates for the transition densities of relativistic stable processes (or equivalently, for the heat kernels of the operators $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$) in $C^{1,1}$ open sets. The estimates are uniform in $m\in(0,M]$ for each fixed $M>0$. Letting $m\downarrow 0$, the estimates given in this paper recover the Dirichlet heat kernel estimates for $-(-\Delta)^{\alpha/2}$ in $C^{1,1}$-open sets obtained in [9]. Sharp two-sided estimates are also obtained for Green functions of relativistic stable processes in half-space-like $C^{1,1}$ open sets and bounded $C^{1,1}$ open sets.

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1 Introduction

Throughout this paper we assume that $d\geq 1$ and $\alpha\in(0,2)$. For any $m\geq 0$, a relativistic α-stable process X^m on $\mathbb{R}^d$ with weight m is a Lévy process with characteristic function given by

$$\mathbb{E}\left[\exp\left(i\xi\cdot(X^m_t-X^m_0)\right)\right]=\exp\left(-t\left(\left(|\xi|^2+m^{2/\alpha}\right)^{\alpha/2}-m\right)\right),\qquad \xi\in\mathbb{R}^d. \tag{1.1}$$

X^0, which will be denoted by X, is simply a (rotationally) symmetric α-stable process on $\mathbb{R}^d$. The infinitesimal generator of X^m is $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$. When $\alpha=1$, the infinitesimal generator reduces to the free relativistic Hamiltonian $m-\sqrt{-\Delta+m^2}$. Here the kinetic energy of a relativistic particle is $\sqrt{-\Delta+m^2}-m$, instead of $-\Delta$ for a nonrelativistic particle. The reason to subtract the constant m in the free Hamiltonian is to ensure that its spectrum is $[0,\infty)$ (see [6]). There exists a huge literature on the properties of relativistic Hamiltonians (see, for example, [6, 20, 24, 29, 30]).

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Various fine properties of relativistic α-stable processes have been studied recently in [12, 15, 23, 25, 26, 28, 31].

The objective of this paper is to establish sharp two-sided estimates on the transition density $p^m_D(t,x,y)$ of the subprocess of X^m in any $C^{1,1}$ open set $D \subset \mathbb{R}^d$. $p^m_D(t,x,y)$ is also the heat kernel of the restriction of $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$ in D with zero exterior condition. A precise definition of $C^{1,1}$ open set will be given in Section 2. Along the way, we also obtain sharp two-sided estimates on the transition density $p^m(t,x,y)$ of X^m in bounded time intervals; see Theorem 3.6 below.

The main result of this paper is Theorem 1.1 below. The open set D below is not necessarily bounded or connected. In this paper, we use ":=" as a way of definition. For $a,b \in \mathbb{R}$, $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$.

Theorem 1.1 *Let $M>0$ be a constant, D a $C^{1,1}$ open set in $\mathbb{R}^d$ and $\delta_D(x)$ the distance between x and D^c.*

(i) *For any $T>0$, there exist $C_1=C_1(\alpha,M,D,T)>1$ and $C_2=C_2(\alpha,M,D,T)>1$ such that for any $m\in(0,M]$ and $(t,x,y)\in(0,T]\times D\times D$,*

$$\frac{1}{C_1}\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge\frac{t\phi(C_2m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\right)\le p^m_D(t,x,y)$$
$$\le C_1\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha}\wedge\frac{t\phi(m^{1/\alpha}|x-y|/C_2)}{|x-y|^{d+\alpha}}\right), \qquad (1.2)$$

where $\phi(r)=e^{-r}(1+r^{(d+\alpha-1)/2})$.

(ii) *Suppose in addition that D is bounded. For any $T>0$, there exist $C_3=C_3(\alpha,M,D,T)>0$ and $C_4=C_4(\alpha,M,D,T)>0$ such that for any $m\in(0,M]$ and $(t,x,y)\in[T,\infty)\times D\times D$,*

$$C_3\,e^{-t\,\lambda_1^{\alpha,m,D}}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2}\;\le\;p^m_D(t,x,y)\;\le\;C_4\,e^{-t\,\lambda_1^{\alpha,m,D}}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2},$$

where $\lambda_1^{\alpha,m,D}>0$ is the smallest eigenvalue of the restriction of $(m^{2/\alpha}-\Delta)^{\alpha/2}-m$ in D with zero exterior condition.

Remark 1.2 (i) Note that the estimates in Theorem 1.1 are uniform in $m\in(0,M]$. When $m\downarrow 0$, $m-(m^{2/\alpha}-\Delta)^{\alpha/2}$ converges to the fractional Laplacian $\Delta^{\alpha/2}:=-(-\Delta)^{\alpha/2}$ in the distributional sense and it is easy to check that X^m converges weakly to X in the Skorokhod space $\mathbb{D}([0,\infty),\mathbb{R}^d)$. It follows from [16, Theorm 1.1] together with the uniform Hölder continuity result of [11, Theorem 4.14] that $p^m_D(t,x,y)$ converges pointwisely to $p_D(t,x,y)$, the transition density function of the subprocess X^D of X in D, and that, if D is bounded, $\lim_{m\downarrow 0}\lambda_1^{\alpha,m,D}=\lambda_1^{\alpha,D}$, the smallest eigenvalue of $(-\Delta)^{\alpha/2}$ in D with zero exterior condition. So letting $m\downarrow 0$ in Theorem 1.1 recovers the sharp two-sided estimates of $p_D(t,x,y)$ for $C^{1,1}$ open set D, which were obtained recently in [9].

(ii) When D is bounded, the functions $\phi(m^{1/\alpha}|x-y|/C_2)$ and $\phi(C_2m^{1/\alpha}|x-y|)$ are bounded between two positive constants independent of $m\in(0,M]$. Thus it follows from Theorem 1.1(i) above and Theorem 1.1(i) of [9] that, for each $T>0$, the heat kernel $p^m_D(t,x,y)$ is uniformly comparable to the heat kernel $p_D(t,x,y)$ on $(0,T]\times D\times D$ when D is a bounded $C^{1,1}$ open set. However when D is unbounded, these two are not comparable.

(iii) In fact, the upper bound estimates in both Theorem 1.1 and Theorem 1.3 below hold for any open set D satisfying (a weak version of) the *uniform exterior ball condition* in place of the $C^{1,1}$ condition, while the lower bound estimates in both Theorem 1.1 and Theorem 1.3 below hold for any open set D satisfying the *uniform interior ball condition* in place of the $C^{1,1}$ condition. (See the paragraph before Theorem 3.8 and the beginning of Section 4 for the definitions of uniform exterior and interior ball conditions respectively.)

Although two-sided estimates on transition densities of jump processes in $\mathbb{R}^d$ have been studied by several authors in the last several years, (see [7, 8, 11, 12] and the references therein), due to the complication near the boundary, sharp two-sided estimates on transition densities of jump processes in open sets have only been studied very recently in [9, 10] by the authors and in [17] by Chen and Tokle for symmetric stable processes and censored stable processes. See [4] for some recent development of heat kernel estimates for symmetric stable processes in general open sets.

This paper is a natural continuation of our previous work [9, 10]. We point out that, although this paper adopts the main strategy from [9], there are many new difficulties and differences between obtaining estimates on transition densities of relativistic stable processes in open sets and those of symmetric stable processes and censored stable processes in open sets studied [9, 10]. For example, unlike symmetric stable processes and censored stable processes, relativistic stable processes do not have the stable-scaling property, which is one of the main ingredients used in the approaches of [9, 10]. As in [9, 10], the Lévy system of X^m, which describes how the process jumps (see (2.5)), is the basic tool used throughout our argument as X^m moves by "pure jumps". But the Lévy density of X^m does not have a simple form and has exponential decay at infinity as opposed to the polynomial decay of the Lévy density of symmetric stable processes. Moreover, in this paper we aim at obtaining sharp estimates that are uniform in $m \in (0, M]$; that is, the comparing constants in Theorem 1.1 are independent of $m \in (0, M]$. This requires very careful and detailed estimates throughout our proofs. Furthermore, unlike symmetric stable processes considered in [9], sharp lower bound on transition densities of relativistic stable processes in $\mathbb{R}^d$ was not available. Thus we need to first establish sharp two-sided estimates on the transition densities of relativistic stable processes in $\mathbb{R}^d$ in bounded time intervals, which is done in Theorem 3.6. These estimates are sharper than the ones established earlier in [12] for more general jump processes with exponentially decaying jump kernels.

We overcome the above mentioned difficulties by using estimates on transition densities of symmetric stable processes in bounded open sets, Meyer's construction (see [1, Remarks 3.4 and 3.5]), the uniform estimates on exit times in Theorem 2.6, the uniform parabolic Harnack inequality in Theorem 2.7, and the uniform comparison between Green functions of X^m and X in small balls (see Theorem 2.5). In particular, this uniform comparability between Green functions is used in several places to get the boundary decay rate of $p^m_D(t, x, y)$. Our approach uses a combination of probabilistic and analytic techniques, but it is mainly probabilistic.

When D is a bounded $C^{1,1}$ open set, integrating the estimates on $p^m_D(t, x, y)$ from Theorem 1.1 over t yields sharp two-sided sharp estimates on the Green function $G^m_D(x, y) := \int_0^\infty p^m_D(t, x, y)dt$.

To state this result, we define a function V_D^α on $D \times D$ by

$$V_D^\alpha(x,y) := \begin{cases} \left(1 \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^\alpha}\right)|x-y|^{\alpha-d} & \text{when } d > \alpha, \\ \log\left(1 + \dfrac{\delta_D(x)^{1/2}\delta_D(y)^{1/2}}{|x-y|}\right) & \text{when } d = 1 = \alpha, \\ \big(\delta_D(x)\delta_D(y)\big)^{(\alpha-1)/2} \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|} & \text{when } d = 1 < \alpha. \end{cases}$$

Theorem 1.3 *Let $M > 0$ be a constant and D a bounded $C^{1,1}$-open set in $\mathbb{R}^d$. Then there exist positive constants $C_5 < C_6$ depending only on D, α, M such that for every $m \in (0, M]$ and $(x,y) \in D \times D$,*

$$C_5 V_D^\alpha(x,y) \le G_D^m(x,y) \le C_6 V_D^\alpha(x,y).$$

The proof of Theorem 1.3 is the same as that of [9, Corollary 1.2]. Theorem 1.3 extends the Green function estimates obtained in [15, 26, 31] in the sense that the case $d = 1$ is allowed. Theorem 1.3 improves the Green function estimates obtained in [15, 22, 31] in the sense that our estimates are uniform in $m \in (0, M]$.

Following [17], we say an open set D in $\mathbb{R}^d$ is half-space-like if there is an orthonormal coordinate system $y = (y_1, \cdots, y_d)$ for $\mathbb{R}^d$ so that $H_a \subset D \subset H_b$ for some real numbers $a > b$. Here for $a \in \mathbb{R}$, $H_a := \big\{(y_1, \cdots, y_d) \in \mathbb{R}^d : \ y_d > a\big\}$. Although we do not yet have large time heat kernel estimates when D is unbounded, by using the short time heat kernel estimates in Theorem 1.1(i), the two-sided Green function estimates on the upper half space from [23] and a comparison idea from [17], we are able to obtain sharp two-sided estimates on the Green function $G_D^m(x,y)$ when D is a half-space-like $C^{1,1}$ open set. To state our result, we define a function $\widetilde{V}_D^{\alpha,m}$ on $D \times D$ as follows: when $d \ge 3$,

$$\widetilde{V}_D^{\alpha,m}(x,y) := \begin{cases} m^{(2-\alpha)/\alpha}\left(1 \wedge \frac{\left(\delta_D(x)+m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2}\right)\left(\delta_D(y)+m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2}\right)}{|x-y|^2}\right)|x-y|^{2-d} & \\ \hfill \text{when } |x-y| > 3m^{-1/\alpha}, & \\ \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-d} \qquad \text{when } |x-y| \le 3m^{-1/\alpha}; & \end{cases}$$

when $d = 2$,

$$\widetilde{V}_D^{\alpha,m}(x,y) := \begin{cases} m^{(2-\alpha)/\alpha}\ln\left(1 + \frac{\left(\delta_D(x)+m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2}\right)\left(\delta_D(y)+m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2}\right)}{|x-y|^2}\right) & \\ \hfill \text{when } |x-y| > 3m^{-1/\alpha}, & \\ \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-2} + m^{(2-\alpha)/\alpha}\ln(1 + m^{1/\alpha}(\delta_D(x) \wedge \delta_D(y))) & \\ \hfill \text{when } |x-y| \le 3m^{-1/\alpha}; & \end{cases}$$

when $d = 1 < \alpha$,

$$\widetilde{V}^{\alpha,m}_D(x,y) := \begin{cases} \dfrac{e^{-m^{1/\alpha}|x-y|}}{|x-y|^{1-(\alpha/2)}}(m^{-1/\alpha} \wedge \delta_D(x) \wedge \delta_D(y))^{\alpha/2} + m^{(2-\alpha)/\alpha}(\delta_D(x) \wedge \delta_D(y)) \\ \quad + m^{(2-\alpha)/(2\alpha)}\,(\delta_D(x) \wedge \delta_D(y))^{\alpha/2} \qquad \text{when } |x-y| > 3m^{-1/\alpha}, \\ \big(\delta_D(x)\delta_D(y)\big)^{(\alpha-1)/2} \wedge \dfrac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|} + m^{(2-\alpha)/\alpha}\,\delta_D(x)^{1/2}\delta_D(y)^{1/2} \\ \qquad \text{when } |x-y| \le 3m^{-1/\alpha}; \end{cases}$$

when $d = 1 = \alpha$,

$$\widetilde{V}^{\alpha,m}_D(x,y) := \begin{cases} \dfrac{e^{-m|x-y|}}{|x-y|^{1/2}}(m^{-1} \wedge \delta_D(x) \wedge \delta_D(y))^{1/2} + m(\delta_D(x) \wedge \delta_D(y)) \\ \quad + m^{1/2}\,(\delta_D(x) \wedge \delta_D(y)^{1/2} \qquad \text{when } |x-y| > 3m^{-1/\alpha}, \\ \log\left(1 + \dfrac{\delta_D(x)^{1/2}\delta_D(y)^{1/2}}{|x-y|}\right) + m^{1/2}\,\delta_D(x)^{1/2}\delta_D(y)^{1/2} \\ \qquad \text{when } |x-y| \le 3m^{-1/\alpha}; \end{cases}$$

and when $d = 1 > \alpha$,

$$\widetilde{V}^{\alpha,m}_D(x,y) := \begin{cases} \dfrac{m^{-1/2}e^{-m^{1/\alpha}|x-y|}}{|x-y|^{1-(\alpha/2)}}\left(1 \wedge \dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} + m^{(2-\alpha)/\alpha}(\delta_D(x) \wedge \delta_D(y)) \\ \quad + m^{(2-\alpha)/(2\alpha)}(\delta_D(x) \wedge \delta_D(y))^{\alpha/2} \qquad \text{when } |x-y| > 3m^{-1/\alpha}, \\ |x-y|^{\alpha-1}\left(1 \wedge \dfrac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2} + m^{(2-\alpha)/\alpha}(\delta_D(x) \wedge \delta_D(y)) \\ \qquad \text{when } |x-y| \le 3m^{-1/\alpha}. \end{cases}$$

Theorem 1.4 *Let $M > 0$ be a constant and D a half-space-like $C^{1,1}$ open set in $\mathbb{R}^d$. Then there exist $C_7 < C_8$ depending only on D, α and M such that for any $m \in (0, M]$ and $(x,y) \in D \times D$,*

$$C_7 \widetilde{V}^{\alpha,m}_D(x,y) \le G^m_D(x,y) \le C_8 \widetilde{V}^{\alpha,m}_D(x,y).$$

Any half space satisfies the assumption of the theorem above. When D is the half space $H := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : x_d > 0\}$, the two-sided estimates on G^1_H were essentially obtained in [23, Theorems 2.13 and 3.2] for $d = 1$ and in [23, Theorem 5.3] for $d \ge 2$. However there are some errors in the statement of [23, Theorem 2.13] for the case of $\alpha < d = 1$ and in the statement of [23, Theorem 5.3] for the case of $|x - y| \le 3$; see the proofs of Theorem 5.1 and 5.2 below for details, which corrects both errors in [23].

The rest of the paper is organized as follows. In Section 2 we recall some basic facts about X^m and prove some preliminary uniform results on X^m, like uniform estimates on the Green function G^m_B for small balls and annuli, uniform parabolic Harnack inequality, uniform Harnack principle and uniform boundary Harnack principle. The upper bound in Theorem 1.1(i) is proved in Section 3, while the lower bound in Theorem 1.1(i) is proved in Section 4. Theorem 1.1(ii) is also proved in Section 4. Theorem 1.4 is proved in the last section.

In the remainder of this paper, we assume that $m > 0$. We will use capital letters $C_1, C_2, \ldots$ to denote constants in the statements of results, and their labeling will be fixed. The lower case constants $c_1, c_2, \ldots$ will denote generic constants used in proofs, whose exact values are not important and can change from one appearance to another. The labeling of the lower case constants starts anew in every proof. The dependence of the constant c on the dimension d will not be mentioned explicitly. We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote its Lebesgue measure.

2 Relativistic stable processes and preliminary uniform estimates

The Lévy measure of the relativistic α-stable process X^m, defined in (1.1), has a density

$$J^m(x) = j^m(|x|) := \frac{\alpha}{2\Gamma(1-\frac{\alpha}{2})} \int_0^\infty (4\pi u)^{-d/2} e^{-\frac{|x|^2}{4u}} e^{-m^{2/\alpha}u} u^{-(1+\frac{\alpha}{2})} du, \tag{2.1}$$

which is continuous and radially decreasing on $\mathbb{R}^d \setminus \{0\}$ (see [31, Lemma 2]). Here and in the rest of this paper Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$ for every $\lambda > 0$. Put $J^m(x, y) := j^m(|x-y|)$. Let $\mathcal{A}(d, -\alpha) := \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})\Gamma(1-\frac{\alpha}{2})^{-1}$. Using change of variables twice, first with $u = |x|^2 v$ then with $v = 1/s$, we get

$$J^m(x, y) = \mathcal{A}(d, -\alpha)|x-y|^{-d-\alpha}\psi(m^{1/\alpha}|x-y|) \tag{2.2}$$

where

$$\psi(r) := 2^{-(d+\alpha)}\, \Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}}\, ds, \tag{2.3}$$

which is a decreasing smooth function of r^2 satisfying $\psi(r) \le 1$ and

$$c_1^{-1} e^{-r} r^{(d+\alpha-1)/2} \le \psi(r) \le c_1 e^{-r} r^{(d+\alpha-1)/2} \qquad \text{on } [1, \infty) \tag{2.4}$$

for some $c_1 > 1$ (see [15, pp. 276-277] for details). We denote the Lévy density of X by

$$J(x, y) := J^0(x, y) = \mathcal{A}(d, -\alpha)|x-y|^{-(d+\alpha)}.$$

The Lévy density gives rise to a Lévy system for X^m, which describes the jumps of the process X^m: for any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$, $x \in \mathbb{R}^d$ and stopping time T (with respect to the filtration of X^m),

$$\mathbb{E}_x\left[\sum_{s\le T} f(s, X^m_{s-}, X^m_s)\mathbf{1}_{\{X^m_{s-}\neq X^m_x\}}\right] = \mathbb{E}_x\left[\int_0^T \left(\int_{\mathbb{R}^d} f(s, X^m_s, y)J^m(X^m_s, y)dy\right) ds\right]. \tag{2.5}$$

(See, for example, [11, Proof of Lemma 4.7] and [12, Appendix A]).

For $r \in (0, \infty)$, we define

$$\xi(r) := \begin{cases} r^2 & \text{when } d+\alpha > 2, \\ r^{1+\alpha} & \text{when } d = 1 > \alpha, \\ r^2 \ln(\frac{1}{r}) & \text{when } d = 1 = \alpha. \end{cases} \tag{2.6}$$

Lemma 2.1 *There exists $C_9 = C_9(d,\alpha) > 0$ such that for all $r \in (0,1)$,*

$$1 - \psi(r) \le C_9 \xi(r).$$

Proof. Note that

$$\int_0^{r^2} s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}} (1 - e^{-\frac{r^2}{s}})\, ds \le \int_0^{r^2} s^{\frac{d+\alpha}{2}-1} ds \le c_1 r^{d+\alpha}, \tag{2.7}$$

and, by the mean-value theorem,

$$\int_{r^2}^{\infty} s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}} (1 - e^{-\frac{r^2}{s}})\, ds \le r^2 \int_{r^2}^{\infty} s^{\frac{d+\alpha}{2}-2} e^{-\frac{s}{4}}\, ds \le c_2 \xi(r). \tag{2.8}$$

Since

$$1 - \psi(r) \;=\; 2^{-(d+\alpha)} \Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \left(\int_0^{r^2} + \int_{r^2}^{\infty}\right) s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}} (1 - e^{-\frac{r^2}{s}})\, ds,$$

we arrive at the conclusion of this lemma by combining (2.7)–(2.8). □

The next two inequalities, which can be seen easily from (2.4), will be used several times in this paper. For any $a > 0$ and $M > 0$, there exist positive constants C_{10} and C_{11} depending on a and M such that for any $m \in (0, M]$,

$$j^m(r) \le C_{10} j^m(2r), \qquad \forall r \in (0,a) \tag{2.9}$$

and

$$j^m(r) \le C_{11} j^m(r+a), \qquad \forall r > a. \tag{2.10}$$

We will use $p^m(t,x,y) = p^m(t,x-y)$ to denote the transition density of X^m and use $p(t,x,y)$ to denote the transition density of X. It is well known that (cf. [11])

$$p(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \qquad \text{on } (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.11}$$

Here for two non-negative functions f, g, $f \asymp g$ means that there is a positive constant $c_0 > 1$ so that $c_0^{-1} f \le g \le c_0 f$ on their common domain of definitions. It is also known that

$$p^1(t,x) \;=\; e^t \int_0^{\infty} (4\pi u)^{-d/2} e^{-\frac{|x|^2}{4u}} e^{-u} \theta_\alpha(t,u) du, \tag{2.12}$$

where $\theta_\alpha(t,u)$ is the transition density of an $\frac{\alpha}{2}$-stable subordinator with the Laplace transform $e^{t\lambda^{\alpha/2}}$. It follows from [34, (2.5.17)–(2.5.18)] and [2, Theorem 2.1] that

$$\theta_\alpha(t,u) \le c t u^{-1-\alpha/2} \qquad \text{for every } t > 0, u > 0,$$

thus by (2.1) and (2.12), there exists $L = L(\alpha) > 0$ such that

$$p^1(t,x,y) \;\le\; Lt\, e^t J^1(x,y) \quad \text{for all } (t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d. \tag{2.13}$$

From (1.1), one can easily see that X^m has the following scaling property:

$$p^m(t,x,y) \;=\; m^{d/\alpha} p^1(mt, m^{1/\alpha}x, m^{1/\alpha}y), \tag{2.14}$$

i.e,

$$\{m^{-1/\alpha}(X^1_{mt} - X^1_0), t \geq 0\} \text{ has the same distribution as that of } \{X^m_t - X^m_0, t \geq 0\}. \tag{2.15}$$

Thus it follows from (2.13) and (2.14) that

$$p^m(t,x,y) \;\leq\; L\, t\, e^{mt} J^m(x,y) \qquad (t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \tag{2.16}$$

and it follows from [31, Lemma 3] that there exists $M_2 = M_2(\alpha) > 0$ such that

$$p^m(t,x,y) \;\leq\; M_2(m^{d/\alpha - d/2} t^{-d/2} + t^{-d/\alpha}). \tag{2.17}$$

For any open set D, we use τ^m_D to denote the first exit time from D for X^m, i.e., $\tau^m_D = \inf\{t > 0 : X^m_t \notin D\}$ and let τ_D be the first exit time from D for X. We define $X^{m,D}$ by $X^{m,D}_t(\omega) = X^m_t(\omega)$ if $t < \tau^m_D(\omega)$ and $X^{m,D}_t(\omega) = \partial$ if $t \geq \tau^m_D(\omega)$. We define X^D similarly. $X^{m,D}$ is called the subprocess of X^m killed upon exiting D (or, the killed relativistic stable process in D with weight m), and X^D is called the killed symmetric α-stable process in D.

It is known (see [12]) that $X^{m,D}$ has a continuous transition density $p^m_D(t,x,y)$ with respect to the Lebesgue measure. We will use $G^m_D(x,y) := \int_0^\infty p^m_D(t,x,y)dt$ to denote the Green function of $X^{m,D}$. We use $p_D(t,x,y)$ and $G_D(x,y)$ to denote the transition density and the Green function of X^D respectively.

$p^m_D(t,x,y)$ also has the following scaling property:

$$p^m_D(t,x,y) \;=\; m^{d/\alpha} p^1_{m^{1/\alpha}D}(mt, m^{1/\alpha}x, m^{1/\alpha}y). \tag{2.18}$$

Thus the Green function $G^m_D(x,y)$ of $X^{m,D}$ satisfies

$$G^m_D(x,y) = m^{(d-\alpha)/\alpha} G^1_{m^{1/\alpha}D}(m^{1/\alpha}x, m^{1/\alpha}y) \qquad \text{for every } x,y \in D. \tag{2.19}$$

Let

$$J_m(x,y) := J(x,y) - J^m(x,y) = \mathcal{A}(d,\, -\alpha)|x-y|^{-d-\alpha}(1 - \psi(m^{1/\alpha}|x-y|)).$$

Then

$$\int_{\mathbb{R}^d} J_m(x,y)dy = m \qquad \text{for all } x \in \mathbb{R}^d. \tag{2.20}$$

(See [31, Lemma 2].) Thus X^m can be constructed from X by reducing jumps via Meyer's construction; see [1, Remarks 3.4 and 3.5]. By (3.18) of [1, Lemma 3.6], we have

$$p^m_D(t,x,y) \;\leq\; e^{mt} p_D(t,x,y), \qquad \text{for every } (t,x,y) \in (0,\infty) \times D \times D. \tag{2.21}$$

Lemma 2.2 *Suppose that $B = (0,2) \subset \mathbb{R}$ and $\alpha > 1$. Define*

$$f(x,y) = \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}, \qquad \forall x,y \in B. \tag{2.22}$$

(i) *If* $f(x,w) \ge 4$, *there exists* $C_{12} = C_{12}(\alpha) > 0$ *such that*

$$\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} \le C_{12}\delta_B(y)^{(\alpha-1)/2}\,\delta_B(z)^{(\alpha-1)/2} \le C_{12}, \quad y,z \in B.$$

(ii) *There exists* $C_{13} = C_{13}(\alpha) > 0$ *such that*

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le C_{13} \qquad \text{for every } x,y,z \in B.$$

Proof. Note that (see [18, p. 187])

$$|x-y| \le \delta_B(x) \wedge \delta_B(y) \qquad \text{if } f(x,y) \ge 4, \tag{2.23}$$

$$\delta_B(x) \wedge \delta_B(y) \ge \frac{1}{2}\left(\delta_B(x) \vee \delta_B(y)\right) \qquad \text{if } f(x,y) \ge 4, \tag{2.24}$$

$$3|x-y| \ge \delta_B(x) \vee \delta_B(y) \qquad \text{if } f(x,y) \le 4. \tag{2.25}$$

We know from [9, Corollary 1.2] that

$$G_B(x,y) \asymp \big(\delta_B(x)\delta_B(y)\big)^{(\alpha-1)/2} \wedge \frac{\delta_B(x)^{\alpha/2}\delta_B(y)^{\alpha/2}}{|x-y|}. \tag{2.26}$$

So when $f(x,w) \ge 4$, we have by (2.26) that

$$\begin{aligned}\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} \quad &\le \quad \frac{c_1}{(\delta_B(x)\delta_B(w))^{(\alpha-1)/2}}\left(\big(\delta_B(x)\delta_B(y)\big)^{(\alpha-1)/2} \wedge \frac{\delta_B(x)^{\alpha/2}\delta_B(y)^{\alpha/2}}{|x-y|}\right)\\ &\quad \times\left(\big(\delta_B(z)\delta_B(w)\big)^{(\alpha-1)/2} \wedge \frac{\delta_B(z)^{\alpha/2}\delta_B(w)^{\alpha/2}}{|z-w|}\right).\end{aligned} \tag{2.27}$$

If $f(x,w) \ge 4$, $f(x,y) \ge 4$ and $f(z,w) \ge 4$, we have

$$\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} \le c_2\big(\delta_B(y)\delta_B(z)\big)^{(\alpha-1)/2}.$$

If $f(x,w) \ge 4$, $f(x,y) \le 4$ and $f(z,w) \ge 4$, we have by (2.27) that

$$\begin{aligned}\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} \quad &\le \quad c_3\,\frac{\delta_B(x)^{1/2}\delta_B(y)^{\alpha/2}\delta_B(z)^{(\alpha-1)/2}}{|x-y|}\\ &= \quad c_3\,(f(x,y))^{1/2}\,\delta_B(y)^{(\alpha-1)/2}\,\delta_B(z)^{(\alpha-1)/2}\\ &\le \quad 2\,c_3\,\delta_B(y)^{(\alpha-1)/2}\,\delta_B(z)^{(\alpha-1)/2}.\end{aligned}$$

Similarly, if $f(x,w) \ge 4$, $f(x,y) \ge 4$ and $f(z,w) \le 4$,

$$\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} \le c_4\delta_B(z)^{(\alpha-1)/2}\delta_B(y)^{(\alpha-1)/2}.$$

If $f(x,w) \ge 4$, $f(x,y) \le 4$ and $f(z,w) \le 4$, then

$$\begin{aligned}\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} \quad &\le \quad c_5\frac{\delta_B(x)^{1/2}}{|x-y|}\frac{\delta_B(w)^{1/2}}{|z-w|}\delta_B(y)^{\alpha/2}\delta_B(z)^{\alpha/2}\\ &= \quad c_5(f(x,y))^{1/2}(f(z,w))^{1/2}\delta_B(y)^{(\alpha-1)/2}\,\delta_B(z)^{(\alpha-1)/2}\\ &\le \quad 4c_5\delta_B(y)^{(\alpha-1)/2}\,\delta_B(z)^{(\alpha-1)/2}.\end{aligned}$$

The proof of (i) is now finished.

Now we prove (ii). It follows from (i) we have

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le c_6 \quad \text{if } f(x,z) \ge 4. \tag{2.28}$$

By symmetry we may assume $f(x,y) \ge f(y,z)$ and consider the following cases separately: If $f(x,z) \le 4$ and $f(x,y) \ge f(y,z) \ge 4$, then by (2.24) $\delta_B(x)\delta_B(z) \ge \frac{1}{2}(\delta_B(x)\delta_B(y) \vee \delta_B(y)\delta_B(z))$. This together with $|x-z| \le 2(|x-y| \vee |y-z|)$ implies that $G_B(x,z) \ge c_7(G_B(x,y) \wedge G_B(y,z))$ and so

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le c_7^{-1}(G_B(x,y) \vee G_B(y,z)).$$

If $f(x,z) \le 4$ and $f(x,y) \ge 4 > f(y,z)$, by (2.23)–(2.26)

$$\begin{aligned}
\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} &\le c_8 G_B(x,y)\frac{\delta_B(y)^{\alpha/2}}{\delta_B(x)^{\alpha/2}}\frac{|x-z|}{|y-z|} \\
&\le c_8 G_B(x,y)\left(\frac{|x-y|}{\delta_B(x)}+1\right)^{\alpha/2}\left(\frac{|y-z|+|x-y|}{|y-z|}\right) \\
&\le c_9 G_B(x,y)\left(1+\frac{|x-y|}{|y-z|}\right) \\
&\le c_9 G_B(x,y)\left(1+\frac{\delta_D(y)}{|y-z|}\right) \le c_9 G_B(x,y)\left(1+3\frac{|y-z|}{|y-z|}\right).
\end{aligned}$$

If $f(x,z) \le 4$, $f(x,y) \le 4$ and $f(y,z) \le 4$, then by (2.25) and (2.26)

$$\begin{aligned}
\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} &\le c_{10}\frac{\delta_B(y)^{\alpha}|x-z|}{|x-y||y-z|} \le c_{10}\left(\frac{\delta_B(y)^{\alpha}}{|x-y|}+\frac{\delta_B(y)^{\alpha}}{|y-z|}\right) \\
&\le c_{10}3^{\alpha/2}(|x-y|^{\alpha-1}+|y-z|^{\alpha-1}).
\end{aligned}$$

Therefore, by combining with (2.28), we have

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le c_{11}, \qquad \text{for all } x,y,z \in B.$$

□

Lemma 2.3 *Suppose that* $B=(0,2) \subset \mathbb{R}$ *and* $\alpha = 1$. *Let* f *be as in* (2.22) *and define* $F(x,y) := \log\left(1+f(x,y)^{1/2}\right)$.

(i) *If* $f(x,w) \ge 4$, *there exists* $C_{14} > 0$ *such that*

$$\frac{G_B(x,y)G_B(z,w)}{G_B(x,w)} \le C_{14}F(x,y)F(z,w), \quad y,z \in B.$$

(ii) *There exists* $C_{15} > 0$ *such that*

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le C_{15}(1+F(x,y)+F(y,z)) \quad x,y,z \in B.$$

Proof. (i) is an immediate consequence of [9, Corollary 1.2]. Using [9, Corollary 1.2], (ii) can be proved by following the argument of the proof of [18, Theorem 6.24]. We omit the details. □

For $r \in (0, 1]$, we define

$$\sigma(r) = \begin{cases} r^{2-\alpha-d} & \text{when } d+\alpha > 2, \\ 1 & \text{when } d = 1 > \alpha, \\ \ln(1/r) & \text{when } d = 1 = \alpha. \end{cases}$$

The following result will be used to prove Theorem 2.5.

Lemma 2.4 *(i) If B is a ball of radius 1 in $\mathbb{R}^d$, then,*

$$\sup_{x,y\in B, x\neq y} \int_{B\times B} \frac{G_B(x,w)\sigma(|w-z|)G_B(z,w)}{G_B(x,y)} dwdz < \infty.$$

(ii) If $d \geq 2$ and U is an annulus of inner radius 1 and outer radius $3/2$ in $\mathbb{R}^d$, then

$$\sup_{x,y\in U, x\neq y} \int_{U\times U} \frac{G_U(x,w)\sigma(|w-z|)G_U(z,w)}{G_U(x,y)} dwdz < \infty.$$

Proof. We only present the proof of (i). The proof of (ii) is similar to the proof of (i) for the case $d > \alpha$. We prove (i) by dealing with two separate cases.

Case 1: $d > \alpha$. In this case, by repeating the argument in [14, Example 2], we know that there exists $c_1 = c_1(d, \alpha) > 0$ such that

$$\begin{aligned} \frac{G_B(x,w)\sigma(|w-z|)G_B(z,w)}{G_B(x,y)} \leq c_4 \Bigg(& \frac{1}{|z-y|^{d-\alpha/2}|w-z|^{d+\alpha-\beta}} + \frac{1}{|x-w|^{d-\alpha/2}|w-z|^{d+\alpha-\beta}} \\ & \frac{1}{|z-y|^{d-\alpha}|w-z|^{d+\alpha-\beta}} + \frac{1}{|x-w|^{d-\alpha}|w-z|^{d+\alpha-\beta}} \\ & \frac{1}{|x-w|^{d-\alpha/2}|z-y|^{d-\alpha/2}|w-z|^{3\alpha/2-\beta}} + \frac{1}{|x-w|^{d-\alpha/2}|z-y|^{d-\alpha}|w-z|^{2\alpha-\beta}} \Bigg), \end{aligned}$$

where $\beta = 2$ when $d \geq 2$ and $\beta = 1 + \alpha$ when $d = 1 > \alpha$. The conclusion now follows immediately.

Case 2: $d = 1 \leq \alpha$. In this case, it follows from the first part of the proof of Proposition 3.17 in [22] that

$$\sup_{x,y\in B, x\neq y, f(x,y)\leq 4} \int_{B\times B} \frac{G_B(x,w)\sigma(|w-z|)G_B(z,y)}{G_B(x,y)} dwdz < \infty,$$

where the f is the function defined in (2.22). The inequality

$$\sup_{x,y\in B, x\neq y, f(x,y)\geq 4} \int_{B\times B} \frac{G_B(x,w)\sigma(|w-z|)G_B(z,y)}{G_B(x,y)} dwdz < \infty$$

follows easily from Lemmas 2.2–2.3. □

The following result will be used later in this paper. Note that this result does not follow from the main result in [22], since the constants in the following results are uniform in $m \in (0, \infty)$ and $r \in (0, R_0 m^{-1/\alpha}]$.

Theorem 2.5 *There exist positive constants* R_0 *and* $C_{16} > 1$ *depending only on* d *and* α *such that for any* $m \in (0, \infty)$*, any ball* B *of radius* $r \le R_0 m^{-1/\alpha}$*,*

$$C_{16}^{-1} G_B(x, y) \le G_B^m(x, y) \le C_{16} G_B(x, y), \quad x, y \in B.$$

Moreover, in the case $d \ge 2$*, there exists a constant* $C_{17} = C_{17}(d, \alpha) > 1$ *such that for any* $m \in (0, \infty)$*,* $r \in (0, R_0 m^{-1/\alpha}]$ *and any annulus* U *of inner radius* r *and outer radius* $3r/2$*,*

$$C_{17}^{-1} G_U(x, y) \le G_U^m(x, y) \le C_{17} G_U(x, y), \quad x, y \in U.$$

Proof. We only present the proof for balls, the case of annuli is similar. By [9], $G_B(x, y) \asymp V_B^\alpha(x, y)$. Hence by (2.19), we only need to prove the theorem for $m = 1$. In this proof we will use B_r to denote the ball $B(0, r)$.

Put

$$F(x, y) := \frac{J^1(x, y)}{J(x, y)} - 1 = \psi(|x - y|) - 1, \qquad x, y \in \mathbb{R}^d.$$

Then it follows from (2.1)–(2.3) that there exists $c_1 = c_1(d, \alpha) > 0$ such that for any $m \in (0, \infty)$ and $r \in (0, \infty)$, $\inf_{x,y \in B_r} F(x, y) \ge c_1 - 1$. It follows from Lemma 2.1 that there exists $c_2 = c_2(d, \alpha) > 0$ such that for any $r \in (0, \infty)$ and $x, y \in B_1$,

$$|F(rx, ry)| + |\ln(1 + F(rx, ry))| + (e^{4|\ln(1+F(rx,ry))|} - 1) \le c_2 \xi(r|x - y|). \tag{2.29}$$

Put

$$q_{B_r}(x) := \int_{B_r^c} J_1(x, y) dy = \mathcal{A}(d, -\alpha) \int_{B_r^c} |x - y|^{-d-\alpha} (1 - \psi(|x - y|)) dy, \qquad x \in B_r.$$

Then it follows from [15, Section 3] that

$$G_{B_r}^1(x, y) = G_{B_r}(x, y) \mathbb{E}_x^y [K^{B_r}(\tau_{B_r})] \qquad \text{for every } x, y \in B_r$$

where

$$K^{B_r}(t) := \exp\big(\sum_{0 < s \le t} \ln(1 + F(X_{s-}^{B_r}, X_s^{B_r})) - \int_0^t \int_{B_r} F(X_s^{B_r}, y) J(X_s^{B_r}, y) dy ds + \int_0^t q_{B_r}(X_s^{B_r}) ds \big).$$

Using the scaling property of G_{B_r}, we get

$$\begin{aligned} & \sup_{x,y \in B_r, x \ne y} \int_{B_r \times B_r} \frac{G_{B_r}(x, w)(e^{4|\ln(1+F(w,z))|} - 1) G_{B_r}(z, w)}{G_{B_r}(x, y)|w - z|^{d+\alpha}} dw dz \\ = & \sup_{x,y \in B_1, x \ne y} \int_{B_1 \times B_1} \frac{G_{B_1}(x, w)(e^{4|\ln(1+F(rw,rz))|} - 1) G_B(z, w)}{G_B(x, y)|w - z|^{d+\alpha}} dw dz, \end{aligned} \tag{2.30}$$

$$\begin{aligned} & \sup_{x,y \in B_r, x \ne y} \int_{B_r \times B_r} \frac{G_{B_r}(x, w)|F(w, z))| G_{B_r}(z, w)}{G_{B_r}(x, y)|w - z|^{d+\alpha}} dw dz \\ = & \sup_{x,y \in B_1, x \ne y} \int_{B_1 \times B_1} \frac{G_{B_1}(x, w)|F(rw, rz))| G_{B_1}(rz, rw)}{G_{B_1}(x, y)|w - z|^{d+\alpha}} dw dz \end{aligned} \tag{2.31}$$

and

$$\sup_{x,y\in B_r, x\neq y}\int_{B_r}\frac{G_{B_r}(x,w)G_{B_r}(w,y)}{G_{B_r}(x,y)}q_{B_r}(w)dw = r^\alpha\cdot\sup_{x,y\in B_1, x\neq y}\int_{B_1}\frac{G_{B_1}(x,w)G_{B_1}(w,y)}{G_{B_1}(x,y)}q_{B_r}(rw)dw. \tag{2.32}$$

Using (2.29)–(2.31) and Lemma 2.4, we have for $r\in(0,1]$

$$\sup_{x,y\in B_r, x\neq y}\int_{B_r\times B_r}\frac{G_{B_r}(x,w)(e^{4|\ln(1+F(w,z))|}-1)G_{B_r}(z,w)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}dwdz\le c_3 r$$

and

$$\sup_{x,y\in B_r, x\neq y}\int_{B_r\times B_r}\frac{G_{B_r}(x,w)|F(w,z))|G_{B_r}(z,w)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}dwdz\le c_3 r.$$

By applying (2.20), the 3G inequality (Lemma 2.2(ii) and Lemma 2.3(ii) for $d=1$) and (2.32), we also have

$$\sup_{x,y\in B_r, x\neq y}\int_{B_r}\frac{G_{B_r}(x,w)G_{B_r}(w,y)}{G_{B_r}(x,y)}q_{B_r}(w)dw\le c_3 r^\alpha.$$

Now choose $R_0>0$ small enough so that for $r\le R_0$

$$\sup_{x,y\in B_r, x\neq y}\int_{B_r\times B_r}\frac{G_{B_r}(x,w)(e^{4|\ln(1+F(w,z))|}-1)G_{B_r}(z,w)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}dwdz\le\frac{1}{2},$$

$$\sup_{x,y\in B_r, x\neq y}\int_{B_r\times B_r}\frac{G_{B_r}(x,w)|F(w,z))|G_{B_r}(z,w)}{G_{B_r}(x,y)|w-z|^{d+\alpha}}dwdz\le\frac{1}{8}$$

and

$$\sup_{x,y\in B_r, x\neq y}\int_{B_r}\frac{G_{B_r}(x,w)G_{B_r}(w,y)}{G_{B_r}(x,y)}dw\le\frac{1}{8}.$$

Using the three displays above, we can follow the argument in [14, Proposition 2.3] to conclude that for all $r\le R_0$,

$$\sup_{x,y\in B_r, x\neq y}\mathbb{E}_x^y[K^{B_r}(\tau_{B_r})]\le 2^{3/4}.$$

Now the upper bound on $G^1_{B_r}$ follows immediately. The lower bound on $G^1_{B_r}$ is an easy consequence of Jensen's inequality, see [14, Remark 2] for details. □

In the remainder of this paper, R_0 will always stand for the constant in Theorem 2.5.

Later in this paper, we will also need the following exit time estimate and parabolic Harnack inequality that are uniform in $m\in(0,M]$. These results are extensions of Proposition 4.9 and Theorem 4.12 of [12], respectively.

Theorem 2.6 *For any $M>0$, $R>0$, $A>0$ and $0<B<1$, there exists $\gamma=\gamma(A,B,M,R)\in(0,1/2)$ such that for every $m\in(0,M]$, $r\in(0,R]$ and $x\in\mathbb{R}^d$,*

$$\mathbb{P}_x\left(\tau^m_{B(x,Ar)}<\gamma r^\alpha\right)\le B.$$

Proof. Let Y^m be a symmetric pure jump process on $\mathbb{R}^d$ with jump kernel given by

$$J_0^m(x,y) = \begin{cases} j^m(|x-y|) & \text{if } |x-y| \le 1, \\ j^m(1)|x-y|^{-(d+\alpha)} & \text{if } |x-y| > 1. \end{cases}$$

Note that $J_0^m(x,y) \ge J^m(x,y)$. In view of (2.1)-(2.4) and (2.20), there are constants $c_i = c_i(M,d,\alpha) > 0, i = 1,2$, such that

$$\frac{c_2}{|x-y|^{d+\alpha}} \le J_0^m(x,y) \le \frac{c_1}{|x-y|^{d+\alpha}} \qquad \text{for every } m \in (0,M] \text{ and } x,y \in \mathbb{R}^d \tag{2.33}$$

and

$$\sup_{x\in\mathbb{R}^d} \mathcal{J}_0^m(x) \le M \qquad \text{for every } m \in (0,M] \text{ and } x,y \in \mathbb{R}^d, \tag{2.34}$$

where $\mathcal{J}_0^m(x) := \int_{\mathbb{R}^d} (J_0^m(x,y) - J^m(x,y))\, dy$. In view of (2.33), it follows from [11, Proposition 4.1] that for each $M > 0$, $R > 0$, $A > 0$ and $0 < B < 1$, there is $\gamma = \gamma(A,B,M,R) \in (0,1)$ such that for every $m \in (0,M]$, $r \in (0,R]$ and $x \in \mathbb{R}^d$,

$$\mathbb{P}_x\left(\tau^{Y^m}_{B(x,\,Ar)} < \gamma\, r^\alpha\right) \le B/2,$$

where $\tau^{Y^m}_{B(x,\,Ar)}$ is the first time the process Y^m exits the set $B(x,\,Ar)$. On the other hand, in view of (2.34), X^m can be obtained from Y^m by adding new jumps according to the jump kernel $J_0^m(x,y) - J^m(x,y)$ through Meyer's construction (see [1, Remarks 3.4]). Hence we have for every $m \in (0,M]$, $r \in (0,R]$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} & \mathbb{P}_x\left(\tau^m_{B(x,\,Ar)} < \gamma\, r^\alpha\right) \\ \le\ & \mathbb{P}_x\left(\tau^{Y^m}_{B(x,\,Ar)} < \gamma\, r^\alpha \text{ and there is no new jumps added to } Y^m \text{ by time } \gamma r^\alpha\right) \\ & + \mathbb{P}_x\left(\text{there is at least one new jump added to } Y^m \text{ by time } \gamma r^\alpha\right) \\ \le\ & B/2 + \left(1 - e^{-\gamma r^\alpha \|\mathcal{J}_0^m\|_\infty}\right) \ \le\ B/2 + \left(1 - e^{-\gamma R^\alpha M}\right) < B, \end{aligned}$$

where the last inequality is achieved by decreasing the value of γ if necessary. □

We now introduce the space-time process $Z_s^m := (V_s, X_s^m)$, where $V_s = V_0 + s$. The filtration generated by Z^m satisfying the usual condition will be denoted as $\{\widetilde{\mathcal{F}}_s;\ s \ge 0\}$. The law of the space-time process $s \mapsto Z_s^m$ starting from (t,x) will be denoted as $\mathbb{P}^{(t,x)}$.

We say that a non-negative Borel function $h(t,x)$ on $[0,\infty)\times\mathbb{R}^d$ is *parabolic* with respect to the process X^m in a relatively open subset D of $[0,\infty)\times\mathbb{R}^d$ if for every relatively compact open subset D_1 of D, $h(t,x) = \mathbb{E}^{(t,x)}\left[h\big(Z^m(\tau^{Z^m}_{D_1})\big)\right]$ for every $(t,x) \in D_1$, where $\tau^{Z^m}_{D_1} = \inf\{s > 0 :\ Z_s^m \notin D_1\}$. For each $r,t > 0$, we define $Q(t,x,r) := [t, t+\gamma\, r^\alpha] \times B(x,r)$.

Theorem 2.7 *For any $R > 0$, and $M > 0$, there exists $C_{18} > 0$ such that for every $m \in (0,M]$, $0 < \delta \le 1$, $z \in \mathbb{R}^d$, $r \in (0,R]$ and every non-negative function h on $[0,\infty)\times\mathbb{R}^d$ that is bounded and parabolic on $[0,\gamma\,(2r)^\alpha] \times B(z,2r)$ with respect to X^m,*

$$\sup_{(t,y)\in Q(\delta r^\alpha, z,\, r)} h(t,y) \le C_{18} \inf_{y\in B(z,r)} h(0,y).$$

Proof. Since ψ is decreasing, we have for any $|y| \geq 2r$,

$$\frac{1}{r^d}\int_{B(0,r)} \frac{\psi(m^{1/\alpha}|z-y|)dz}{|z-y|^{d+\alpha}} \geq \frac{\psi(m^{1/\alpha}|y|)}{r^d|y|^\alpha}\int_{\{|w|\leq r/|y|, |w-y/|y||\leq 1\}} \frac{dw}{|w-\frac{y}{|y|}|^d} \geq c_0\frac{\psi(m^{1/\alpha}|y|)}{|y|^{d+\alpha}}.$$

Thus there is a constant $c>0$ so that for every $m>0$,

$$J^m(x,y) \leq \frac{c}{r^d}\int_{B(x,r)} J^m(z,y)dz \qquad \text{for every } r \leq \frac{|x-y|}{2}.$$

The conclusion of the theorem now follows from [8, Theorem 4.5]. $\Box$

Recall that an open set D in $\mathbb{R}^d$ (when $d \geq 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $R>0$ and a constant $\Lambda_0>0$ such that for every $z \in \partial D$, there is a $C^{1,1}$-function $\phi=\phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0)=\nabla\phi(0)=0$, $\|\nabla\phi\|_\infty \leq \Lambda_0$, $|\nabla\phi(x)-\nabla\phi(z)| \leq \Lambda_0|x-z|$, and an orthonormal coordinate system $y=(y_1,\cdots,y_{d-1},y_d) := (\widetilde{y},\, y_d)$ such that $B(z,R)\cap D = B(z,R)\cap\{y : y_d > \phi(\widetilde{y})\}$. By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be expressed as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. Note that a $C^{1,1}$ open set can be unbounded and disconnected. It is well known that a $C^{1,1}$ open set D satisfies both the *uniform interior ball condition* and the *uniform exterior ball condition*: there exists $r_0 < R$ such that for every $x \in D$ with $\delta_{\partial D}(x) < r_0$ and $y \in \mathbb{R}^d \setminus \overline{D}$ with $\delta_{\partial D}(y) < r_0$, there are $z_x, z_y \in \partial D$ so that $|x-z_x| = \delta_{\partial D}(x)$, $|y-z_y| = \delta_{\partial D}(y)$ and that $B(x_0,r_0)\subset D$ and $B(y_0,r_0)\subset \mathbb{R}^d\setminus\overline{D}$ for $x_0 = z_x + r_0(x-z_x)/|x-z_x|$ and $y_0 = z_y + r_0(y-z_y)/|y-z_y|$. For simplicity, in this paper we call the pair (r_0,Λ_0) the characteristics of the $C^{1,1}$ open set D. Without loss of generality, we will assume that $r_0 \leq R_0$, the constant in Theorem 2.5.

A real-valued function u defined on $\mathbb{R}^d$ is said to be harmonic in an open set $D\subset\mathbb{R}^d$ with respect to X^m if for every open set B whose closure is a compact subset of D,

$$\mathbb{E}_x\left[\left|u(X^m_{\tau^m_B})\right|\right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x\left[u(X^m_{\tau^m_B})\right] \qquad \text{for every } x \in B. \tag{2.35}$$

A real-valued function u defined on $\mathbb{R}^d$ is said to be regular harmonic in an open set $D\subset\mathbb{R}^d$ with respect to X^m if

$$\mathbb{E}_x\left[\left|u(X^m_{\tau^m_D})\right|\right] < \infty \quad \text{and} \quad u(x) = \mathbb{E}_x\left[u(X^m_{\tau^m_D})\right] \qquad \text{for every } x \in D.$$

Clearly, a regular harmonic function in D is harmonic in D.

The following uniform Harnack principle is a consequence of Theorem 2.7 and the scaling property (2.15).

Theorem 2.8 (Uniform Harnack principle) *There exists a constant $C_{19} = C_{19}(\alpha, d) > 0$ such that for any $m \in (0,\infty)$ and $r \in (0, 4m^{-1/\alpha}]$, $x_0 \in \mathbb{R}^d$ and any function u which is nonnegative in $\mathbb{R}^d$ and harmonic in $B(x_0,r)$ with respect to X^m we have*

$$u(x) \leq C_{19}u(y) \qquad \textit{for all } x,y \in B(x_0, r/2).$$

The following uniform boundary Harnack principle will be needed in the proof of Theorem 1.4. Note that this result is not a consequence of the boundary Harnack principle in [27], since the constant C below is uniform in $m\in(0,\infty)$ and $r \in (0, R_0 m^{-1/\alpha}]$.

Theorem 2.9 (Uniform boundary Harnack principle) *Suppose $M \in (0,\infty)$ and that D is an open set in $\mathbb{R}^d$, $z \in \partial D$, $r \in (0, R_0 M^{-1/\alpha})$ and that $B(A,\kappa r) \subset D \cap B(z,r)$. There exists $C_{20} = C_{20}(d,\alpha,\kappa,M) > 0$ such that for every $m \in (0,M]$, and any functions $u, v \geq 0$ on $\mathbb{R}^d$, positive regular harmonic for X^m in $D \cap B(z,2r)$ and vanishing on $D^c \cap B(z,2r)$, we have*

$$C_{20}^{-1}\frac{u(A)}{v(A)} \leq \frac{u(x)}{v(x)} \leq C_{20}\frac{u(A)}{v(A)}, \qquad x \in D \cap B(z,r).$$

Proof. By using (2.1)–(2.2) and Theorem 2.5, we can easily get uniform estimates on the Poisson kernel

$$K^m_{B(x_0,r)}(x,z) := \int_{B(x_0,r)} G^m_{B(x_0,r)}(x,y) J^m(y,z)dy$$

of $B(x_0,r)$ with respect to X^m for $r \in (0, R_0 M^{-1/\alpha}]$. In particular, for $r < |z - x_0| < 2R_0 M^{-1/\alpha}$, $K^m_{B(x_0,r)}(x,z)$ is comparable to $K_{B(x_0,r)}(x,z)$, the Poisson kernel of $B(x_0,r)$ with respect to X for $r \in (0, R_0 M^{-1/\alpha}]$. Then using the uniform estimates on $K^m_{B(x_0,r)}(x,z)$ and Theorem 2.5 we can easily see that [32, Lemma 3.3] can be proved in the same way. Using the uniform estimates on the Poisson kernel of $B(x_0,r)$, (2.1)–(2.2) and Theorem 2.5 we can adapt the argument in [3, 27, 32] to get our uniform boundary Harnack principle. We omit the details. □

3 Upper bound estimate

The goal of this section is to establish the sharp upper bound for $p^m_D(t,x,y)$. Before establishing such upper bound, we first give some preliminary lower bounds on $p^m_D(t,x,y)$ and sharp two-sided estimates for $p^m(t,x,y)$, which will be used later.

Lemma 3.1 *For any positive constants M, T, b and a, there exists $C_{21} = C_{21}(a,b,M,\alpha,T) > 0$ such that for any $m \in (0,M]$, $z \in \mathbb{R}^d$ and $\lambda \in (0,T]$,*

$$\inf_{\substack{y\in\mathbb{R}^d \\ |y-z|\leq b\lambda^{1/\alpha}}} \mathbb{P}_y\left(\tau^m_{B(z,2b\lambda^{1/\alpha})} > a\lambda\right) \geq C_{21}.$$

Proof. By Theorem 2.6, there exists $\varepsilon = \varepsilon(b,\alpha,M,T) > 0$ such that for any $m \in (0,M]$ and $\lambda \in (0,T]$,

$$\inf_{y\in\mathbb{R}^d} \mathbb{P}_y(\tau^m_{B(y,b\lambda^{1/\alpha}/2)} > \varepsilon\lambda) \geq \frac{1}{2}.$$

We may assume that $\varepsilon > a$. Applying Theorem 2.7 at most $1 + [(a-\varepsilon)(4/b)^\alpha]$ times, we get that there exists $c_1 = c_1(\alpha,M,a,T) > 0$ such that for all $m \in (0,M]$,

$$c_1\, p^m_{B(y,b\lambda^{1/\alpha})}(\varepsilon\lambda,y,w) \leq p^m_{B(y,b\lambda^{1/\alpha})}(a\lambda,y,w) \qquad \text{for } w \in B(y,b\lambda^{1/\alpha}/2).$$

Thus for any $m \in (0,M]$,

$$\begin{aligned}
\mathbb{P}_y\left(\tau^m_{B(y,b\lambda^{1/\alpha})} > a\lambda\right) &= \int_{B(y,b\lambda^{1/\alpha})} p^m_{B(y,b\lambda^{1/\alpha})}(a\lambda,y,w)dw \\
&\geq \int_{B(y,b\lambda^{1/\alpha}/2)} p^m_{B(y,b\lambda^{1/\alpha})}(a\lambda,y,w)dw \\
&\geq c_1 \int_{B(y,\varepsilon\lambda^{1/\alpha}/2)} p^m_{B(y,\varepsilon\lambda^{1/\alpha}/2)}(a\lambda,y,w)dw \geq c_1/2.
\end{aligned}$$

This proves the lemma. ☐

For the next four results, D is an arbitrary nonempty open set and we use the convention that $\delta_D(\cdot) \equiv \infty$ when $D = \mathbb{R}^d$.

Proposition 3.2 *Let M and T be positive constants. Suppose that $(t,x,y) \in (0,T] \times D \times D$ with $\delta_D(x) \geq t^{1/\alpha} \geq 2|x-y|\psi(m^{1/\alpha}|x-y|)^{-1/(d+\alpha)}$. Then there exists a positive constant $C_{22} = C_{22}(M,\alpha,T)$ such that for any $m \in (0,M]$,*

$$p^m_D(t,x,y) \geq C_{22}\, t^{-d/\alpha}. \tag{3.1}$$

Proof. Let $t \leq T$ and $x,y \in D$ with $\delta_D(x) \geq t^{1/\alpha} \geq 2|x-y|\psi(m^{1/\alpha}|x-y|)^{-1/(d+\alpha)}$. By the uniform parabolic Harnack inequality (Theorem 2.7), there exists $c_1 = c_1(\alpha, M, T) > 0$ such that for any $m \in (0,M]$,

$$p^m_D(t/2,x,w) \leq c_1\, p^m_D(t,x,y) \quad \text{for every } w \in B(x, 2t^{1/\alpha}/3).$$

This together with Lemma 3.1 yields that for any $m \in (0,M]$,

$$\begin{aligned} p^m_D(t,x,y) &\geq \frac{1}{c_1\,|B(x,t^{1/\alpha}/2)|}\int_{B(x,t^{1/\alpha}/2)} p^m_D(t/2,x,w)dw \\ &\geq c_2 t^{-d/\alpha}\int_{B(x,t^{1/\alpha}/2)} p^m_{B(x,t^{1/\alpha}/2)}(t/2,x,w)dw \\ &= c_2 t^{-d/\alpha}\,\mathbb{P}_x\left(\tau^m_{B(x,t^{1/\alpha}/2)} > t/2\right) \geq c_3\, t^{-d/\alpha}, \end{aligned}$$

where $c_i = c_i(T,\alpha,M) > 0$ for $i = 2,3$. ☐

Lemma 3.3 *Let M and T be positive constants. Suppose that $(t,x,y) \in (0,T] \times D \times D$ with $\min\{\delta_D(x),\, \delta_D(y)\} \geq t^{1/\alpha}$ and $|x-y|^\alpha \geq 2^{-\alpha} t\psi(m^{1/\alpha}|x-y|)^{\alpha/(d+\alpha)}$. Then there exists a constant $C_{23} = C_{23}(\alpha,T,M) > 0$ such that for all $m \in (0,M]$,*

$$\mathbb{P}_x\left(X^{m,D}_t \in B\big(y,\, 2^{-1}t^{1/\alpha}\big)\right) \geq C_{23}\, t^{d/\alpha+1} J^m(x,y).$$

Proof. By Lemma 3.1, starting at $z \in B(y,\, 4^{-1}t^{1/\alpha})$, with probability at least $c_1 = c_1(\alpha,M,T) > 0$, for any $m \in (0,M]$, the process X^m does not move more than $6^{-1}t^{1/\alpha}$ by time t. Thus, it is sufficient to show that there exists a constant $c_2 = c_2(\alpha,M,T) > 0$ such that for any $m \in (0,M]$, $t \in (0,T]$ and (x,y) with $|x-y|^\alpha \geq 2^{-\alpha}t\psi(m^{1/\alpha}|x-y|)^{\alpha/(d+\alpha)}$,

$$\mathbb{P}_x\left(X^{m,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \geq c_2\, t^{d/\alpha+1} J^m(x,y). \tag{3.2}$$

Let $B_x := B(x,\, 6^{-1}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})$, $B_y := B(y,\, 6^{-1}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})$ and $\tau^m_x := \tau^m_{B_x}$. It follows from Lemma 3.1, there exists $c_3 = c_3(\alpha,M,T) > 0$ such that for all $m \in (0,M]$,

$$\mathbb{E}_x\left[t \wedge \tau^m_x\right] \geq t\,\mathbb{P}_x\left(\tau^m_x \geq t\right) \geq c_3\, t \qquad \text{for } t > 0. \tag{3.3}$$

By the Lévy system in (2.5),

$$\begin{aligned}
& \mathbb{P}_x\left(X^{m,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \\
\geq\;& \mathbb{P}_x(X^m_{t\wedge\tau^m_x} \in B(y,\, 4^{-1}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha}) \text{ and } t\wedge\tau^m_x \text{ is a jumping time }) \\
\geq\;& \mathbb{E}_x\left[\int_0^{t\wedge\tau^m_x}\int_{B_y} J^m(X^m_s,u)\,duds\right], \qquad (3.4)
\end{aligned}$$

where in the first inequality we used the fact that $\psi(r) \leq 1$ for all $r \geq 0$.

We consider two cases separately.

(i) Suppose $|x-y| \leq T^{1/\alpha}$. Note that $|x-y| \geq 2^{-1}t^{1/\alpha}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}$. Moreover, if $s < \tau^m_x$ and $u \in B_y$,

$$|X^m_s - u| \leq |x-y| + |x - X^m_s| + |y-u| \leq 2|x-y|.$$

Thus from (3.4), for any $m \in (0, M]$,

$$\begin{aligned}
& \mathbb{P}_x\left(X^{m,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \\
\geq\;& \mathbb{E}_x\left[t\wedge\tau^m_x\right]\int_{B_y} j^m(2|x-y|)\,du \;\geq\; c_4\, t\,|B_y|\, j^m(2|x-y|) \;\geq\; c_5\, t^{d/\alpha+1} j^m(2|x-y|)
\end{aligned}$$

for some positive constants $c_i = c_i(\alpha, M, T)$, $i = 4, 5$. Here in the second inequality above, we used (3.3). Therefore using (2.9), we see that the assertion of the lemma is valid for $|x-y| \leq T^{1/\alpha}$.

(ii) Suppose $|x-y| > T^{1/\alpha}$. In this case, if $s < \tau^m_x$ and $u \in B_y$,

$$\begin{aligned}
& |X^m_s - u| \;\leq\; |x-y| + |x - X^m_s| + |u-y| \\
\leq\;& |x-y| + 3^{-1}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha} \leq |x-y| + 3^{-1}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}T^{1/\alpha}.
\end{aligned}$$

Thus from (3.4), for any $m \in (0, M]$,

$$\begin{aligned}
& \mathbb{P}_x\left(X^{m,D} \text{ hits the ball } B(y,\, 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \\
\geq\;& \mathbb{E}_x\left[t\wedge\tau^m_x\right]\int_{B_y} j^m\left(|x-y| + 3^{-1}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}T^{1/\alpha}\right) du \\
\geq\;& c_6\, t\,|B_y|\, j^m\left(|x-y| + 3^{-1}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}T^{1/\alpha}\right) \\
\geq\;& c_7\, t^{d/\alpha+1} j^m\left(|x-y| + 3^{-1}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}T^{1/\alpha}\right)
\end{aligned}$$

for some positive constants $c_i = c_i(\alpha, M, T)$, $i = 6, 7$. Here in the second inequality, (3.3) is used. Since $|x-y| > T^{1/\alpha} \geq 3^{-1}\psi(m^{1/\alpha}T^{1/\alpha})^{1/(d+\alpha)}T^{1/\alpha}$, using (2.10), we see that the assertion of the lemma is valid for $|x-y| > T^{1/\alpha}$. □

Proposition 3.4 *Let M and T be positive constants. Suppose that $(t,x,y) \in (0,T] \times D \times D$ with $\min\{\delta_D(x),\, \delta_D(y)\} \geq (t/2)^{1/\alpha}$ and $|x-y|^\alpha \geq 2^{-\alpha-1}\, t\psi(m^{1/\alpha}|x-y|)^{\alpha/(d+\alpha)}$. Then there exists a constant $C_{24} = C_{24}(\alpha, M, T) > 0$ such that for all $m \in (0, M]$,*

$$p^m_D(t,x,y) \;\geq\; C_{24}\, tJ^m(x,y).$$

Proof. By the semigroup property, Proposition 3.2 and Lemma 3.3, there exist positive constants $c_1 = c_1(\alpha, T, M)$ and $c_2 = c_2(\alpha, T, M)$ such that for all $m \in (0, M]$,

$$\begin{aligned} p^m_D(t,x,y) &= \int_D p^m_D(t/2,x,z)p^m_D(t/2,z,y)dz \\ &\geq \int_{B(y,\,2^{-1}(t/2)^{1/\alpha})} p^m_D(t/2,x,z)p^m_D(t/2,z,y)dz \\ &\geq c_1 t^{-d/\alpha}\mathbb{P}_x\left(X^{m,D}_{t/2} \in B(y, 2^{-1}(t/2)^{1/\alpha})\right) \geq c_2\, tJ^m(x,y). \end{aligned}$$

□

Combining Propositions 3.2 and 3.4, we have the following preliminary lower bound for $p^m_D(t,x,y)$.

Proposition 3.5 *Let M and T be positive constants. Suppose that $(t,x,y) \in (0,T] \times D \times D$ with $\min\{\delta_D(x),\, \delta_D(y)\} \geq t^{1/\alpha}$. Then there exists a constant $C_{25} = C_{25}(\alpha, M, T) > 0$ such that for all $m \in (0, M]$,*

$$p^m_D(t,x,y) \geq C_{25}\,(t^{-d/\alpha} \wedge tJ^m(x,y)).$$

Combining (2.16)-(2.17) with Proposition 3.5 we have the following sharp two-sided estimates for $p^m(t,x,y)$.

Theorem 3.6 *Let M and T be positive constants. Then there exists a constant $C_{26} = C_{26}(\alpha, M, T) > 1$ such that for all $m \in (0, M]$, $t \in (0,T]$ and $x, y \in \mathbb{R}^d$,*

$$C_{26}^{-1}\left(t^{-d/\alpha} \wedge tJ^m(x,y)\right) \leq p^m(t,x,y) \leq C_{26}\left(t^{-d/\alpha} \wedge tJ^m(x,y)\right).$$

Lemma 3.7 *Let $M > 0$ be a constant and $E = \{x \in \mathbb{R}^d : |x| > r_0\}$. For every $T > 0$, there is a constant $C_{27} = C_{27}(r_0, \alpha, M, T) > 0$ such that for any $m \in (0, M]$,*

$$p^m_E(t,x,y) \leq C_{27}\sqrt{t}\,\delta_E(x)^{\alpha/2} j^m(|x-y|/16) \qquad \text{for } r_0 < |x| < 5r_0/4,\ |y| \geq 2r_0 \text{ and } t \leq T.$$

Proof. Define $U := \left\{z \in \mathbb{R}^d : r_0 < |z| < 3r_0/2\right\}$. It is well-known (see, e.g., [33]) that $X^m_{\tau_U} \notin \partial U$. For $r_0 < |x| < 5r_0/4$, $|y| \geq 2r_0$ and $t \in (0,T]$, it follows from the strong Markov property and (2.5) that

$$\begin{aligned} & p^m_E(t,x,y) = \mathbb{E}_x\left[p^m_E(t-\tau^m_U, X^m_{\tau^m_U}, y) : \tau^m_U < t\right] \\ = & \int_0^t \left(\int_U p^m_U(s,x,z)\left(\int_{\{w:|w|>3r_0/2\}} J^m(z,w)\; p^m_E(t-s,w,y)dw\right)dz\right)ds \\ = & \int_0^t \left(\int_U p^m_U(s,x,z)\left(\int_{\{w:\,(3r_0/4)+(|y|/2)\geq|w|>3r_0/2\}} J^m(z,w)p^m_E(t-s,w,y)dw\right)dz\right)ds \\ & + \int_0^t \left(\int_U p^m_U(s,x,z)\left(\int_{\{w:\,|w|>(3r_0/4)+(|y|/2)\}} J^m(z,w)p^m_E(t-s,w,y)dw\right)dz\right)ds. \\ =: & \; I + II. \end{aligned}$$

Since $p^m_E \le p^m$, we have

$$I \le \int_0^t \left(\int_U p^m_U(s,x,z) \left(\int_{\{w:\,(3r_0/4)+(|y|/2)\ge|w|>3r_0/2\}} J^m(z,w)\, p^m(t-s,w,y)dw \right) dz \right) ds.$$

Since $|w-y| \ge |y|-|w| \ge \frac{1}{2}\left(|y|-\frac{3r_0}{2}\right) \ge \frac{|y|}{8} \ge \frac{|x-y|}{16}$ for $|w| \le (3r_0/4)+(|y|/2)$, by (2.16) we have for $|w| \le (3r_0/4)+(|y|/2)$ and $0<s<t<T$,

$$p^m(t-s,w,y) \le p^m(t-s,x/16,y/16) \le L\,t\,e^{mT}\,J^m(x/16,y/16).$$

Therefore

$$\begin{aligned} I &\le L\,t\,e^{mT}\,J^m(x/16,y/16)\int_0^t \left(\int_U p^m_U(s,x,z) \left(\int_{\{w:\,3|x-y|/4\ge|w|>3r_0/2\}} J^m(z,w)dw \right) dz \right) ds \\ &= L\,t\,e^{mT}\,J^m(x/16,y/16)\,\mathbb{P}_x\left(3r_0/2 < |X^m_{\tau^m_U}| \le 3|x-y|/4;\ \tau^m_U \le t\right) \\ &\le L\,T\,e^{MT}\,j^m(|x-y|/16)\,\mathbb{P}_x\left(|X^m_{\tau^m_U}| > 3r_0/2\right). \end{aligned}$$

By Theorem 2.5,

$$\begin{aligned} \mathbb{P}_x\left(|X^m_{\tau^m_U}| > 3r_0/2\right) &= \int_{\{|z|>3r_0/2\}}\int_U G^m_U(x,y)J^m(y,z)dydz \\ &\le C_{16}\int_{\{|z|>3r_0/2\}}\int_U G_U(x,y)J(y,z)dydz \\ &= C_{16}\mathbb{P}_x\left(|X_{\tau_U}| > 3r_0/2\right) \\ &\le c_1\delta^\alpha_U(x) = c_1\delta^\alpha_E(x) \end{aligned}$$

for some positive constant $c_1 = c_1(M,r_0,\alpha)$. Thus we have

$$I \le c_2\,t\,e^{MT}\,\delta_E(x)^{\alpha/2} j^m(|x-y|/(16)), \quad m \in (0,M] \tag{3.5}$$

for some positive constant $c_2 = c_2(r_0,\alpha,M)$.

On the other hand, for $z \in U$ and $w \in \mathbb{R}^d$ with $|w| > (3r_0/4)+(|y|/2)$, we have

$$|z-w| \ge |w|-|z| \ge \frac{1}{2}\left(|y|-\frac{3r_0}{2}\right) \ge \frac{|y|}{8} \ge \frac{|x-y|}{16}.$$

Thus by the symmetry of $p^m_E(t-s,w,y)$ in (w,y), we have that there exists $c_3 = c_3(M,r_0,\alpha) > 0$ such that for any $m \in (0,M]$,

$$\begin{aligned} II &\le \int_0^t \left(\int_U p^m_U(s,x,z) \left(\int_{\{w:\,|w|>(3r_0/4)+(|y|/2)\}} J^m(x/16,y/16)p^m_E(t-s,y,w)dw \right) dz \right) ds \\ &\le c_3 j^m(|x-y|/16)\int_0^t \left(\int_U p^m_U(s,x,z)dz \right) ds. \end{aligned}$$

By (2.21), there exists $c_4 = c_4(\alpha,T) > 0$ such that

$$p^m_U(s,x,z) \le e^{ms}p_U(s,x,z) \le c_4 e^{ms}\frac{\delta_U(x)^{\alpha/2}}{\sqrt{s}}\left(s^{-d/\alpha} \wedge \frac{s}{|x-z|^{d+\alpha}}\right), \quad s \le T.$$

The last inequality above comes from [9, Theorem 1.1]. Thus

$$
\begin{aligned}
&\int_0^t \left(\int_U p_U^m(s,x,z)dz\right) ds \\
\le\;& c_4 e^{mT}\delta_U(x)^{\alpha/2}\left(\int_0^t\int_{\{|z|\le s^{1/\alpha}\}} s^{-d/\alpha-1/2}dzds+\int_0^t\int_{\{|z|>s^{1/\alpha}\}}\frac{\sqrt{s}}{|z|^{d+\alpha}}dzds\right)\\
\le\;& c_5\delta_E(x)^{\alpha/2}\sqrt{t}.
\end{aligned}
$$

This together with our estimate on I above completes the proof the lemma. □

In the remainder of this section we assume that D is an open set satisfying the following (weak version of) uniform exterior ball condition with radius $r_0 > 0$: for every $z \in \partial D$ and $r \in (0, r_0)$, there is a ball B^z of radius r such that $B^z \subset \mathbb{R}^d \setminus \overline{D}$ and $\partial B^z \cap \partial D = \{z\}$.

Theorem 3.8 *Let $M > 0$ be a constant and D an open set satisfying the uniform exterior ball condition with radius $r_0 \le R_0$. For every $T > 0$, there exists a positive constant $C_{28} = C_{28}(T, r_0, \alpha, M)$ such that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D^m(t,x,y) \le C_{28}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge tj^m(|x-y|/16)\right).$$

Proof. In view of (2.16)-(2.17), it suffices to prove the theorem for $x \in D$ with $\delta_D(x) < r_0/4$. By (2.21) and [9, Theorem 1.1], there exists $c_1 = c_1(\alpha, T, D) > 0$ such that on $0, T] \times D \times D$

$$p_D^m(t,x,y) \;\le\; e^{mt}p_D(t,x,y) \;\le\; c_1e^{MT}\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{3.6}$$

For $x, y \in D$, let $z \in \partial D$ so that $|x - z| = \delta_D(x)$. Let $B_z \subset D^c$ be the ball with radius r_0 so that $\partial B_z \cap \partial D = \{z\}$. When $\delta_D(x) < r_0/4$ and $|x - y| \ge 5r_0$, we have $\delta_{B_z^c}(y) > 2r_0$ and so by Lemma 3.7, there is a constant $c_2 = c_2(r_0, T, M, \alpha) > 0$ such that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,

$$p_D^m(t,x,y) \;\le\; p^m_{(\overline{B}_z)^c}(t,x,y) \;\le\; c_2\,\delta_{(\overline{B}_z)^c}(x)^{\alpha/2}\sqrt{t}j^m(|x-y|/16) \;=\; c_2\,\delta_D(x)^{\alpha/2}\sqrt{t}j^m(|x-y|/16). \tag{3.7}$$

Since there exist constants c_3 and c_4 depending only on M, α and r_0 such that

$$\frac{c_3}{|x-y|^{d+\alpha}} \le j^m(|x-y|/16) \le \frac{c_4}{|x-y|^{d+\alpha}} \qquad \text{for } m \in (0,M] \text{ and } |x-y| < 5r_0,$$

combining (3.6)-(3.7), we arrive at the conclusion of the theorem. □

Theorem 3.9 *Let M and T be positive constants. Suppose that D is an open set satisfying the uniform exterior ball condition with radius $r_0 \le R_0$. Then there exists a constant $C_{29} = C_{29}(T, r_0, M, \alpha) > 0$ such that for all $m \in (0, M]$, $t \in (0, T]$ and $x, y \in D$,*

$$p_D^m(t,x,y) \le C_{29}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\left(t^{-d/\alpha} \wedge \frac{t\phi(m^{1/\alpha}|x-y|/(16))}{|x-y|^{d+\alpha}}\right). \tag{3.8}$$

Proof. Fix $T > 0$ and $M > 0$. By Theorem 3.8, symmetry and the semigroup property, we get that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,

$$\begin{aligned} p^m_D(t,x,y) &= \int_D p^m_D(t/2,x,z)p^m_D(t/2,z,y)dz \\ &\le c_1^2 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \\ &\quad \cdot \int_D \Big((t/2)^{-d/\alpha} \wedge (t/2)J^m(x/16,z/16)\Big)\Big((t/2)^{-d/\alpha} \wedge (t/2)J^m(z/16,y/16)\Big)dz. \end{aligned}$$

By a change of variable,

$$\begin{aligned} &\int_D \Big((t/2)^{-d/\alpha} \wedge (t/2)J^m(x/16,z/16)\Big)\Big((t/2)^{-d/\alpha} \wedge (t/2)J^m(z/16,y/16)\Big)dz \\ \le\ & (16)^d \int_{\mathbb{R}^d} \Big((t/2)^{-d/\alpha} \wedge (t/2)J^m(x/16,w)\Big)\Big((t/2)^{-d/\alpha} \wedge (t/2)J^m(w,y/16)\Big)dw. \end{aligned}$$

Thus by Theorem 3.6 and the semigroup property, there exists a positive constant $c_2 = c_2(\alpha, M, T)$ such that for any $m \in (0, M]$ and $(t, x, y) \in (0, T] \times D \times D$,

$$\begin{aligned} p^m_D(t,x,y) &\le c_2 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \int_{\mathbb{R}^d} p^m(t/2,x/16,z)p^m(t/2,z,y/16)dz \\ &= c_2 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) p^m(t,x/16,y/16) \\ &\le c_3 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \Big(t^{-d/\alpha} \wedge tJ^m(x/16,y/16)\Big). \end{aligned}$$

This completes the proof of the theorem. □

4 Lower bound estimate

Throughout this section, the open set D is assumed to satisfy the uniform interior ball condition with radius $r_0 > 0$ in the following sense: For every $x \in D$ with $\delta_D(x) < r_0$, there is $z_x \in \partial D$ so that $|x - z_x| = \delta_D(x)$ and $B(x_0, r_0) \subset D$ for $x_0 := z_x + r_0(x - z_x)/|x - z_x|$.

The goal of this section is to prove the following lower bound for $p^m_D(t, x, y)$.

Theorem 4.1 *For any $M > 0$ and $T > 0$ there exists positive constant $C_{30} = C_{30}(\alpha, T, M, r_0)$ such that for all $m \in (0, M]$, $(t, x, y) \in (0, T] \times D \times D$,*

$$p^m_D(t,x,y) \ge C_{30} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\Big(t^{-d/\alpha} \wedge tj^m(|x-y|)\Big).$$

We will first establish Theorem 4.1 for small T, that is, we will first assume that

$$t \le T_0 := \left(\frac{r_0}{16}\right)^\alpha. \tag{4.1}$$

Lemma 4.2 *Let $M>0$ be a constant. Suppose that $(t,x)\in(0,T_0]\times D$ with $\delta_D(x)\le 3t^{1/\alpha}<r_0/4$ and $\kappa\in(0,1)$. Let $z_x\in\partial D$ be such that $|z_x-x|=\delta_D(x)$ and let $\mathbf{n}(z_x)=(x-z_x)/|z_x-x|$. Put $x_1=z_x+3t^{1/\alpha}\mathbf{n}(z_x)$ and $B=B(x_1,3t^{1/\alpha})$. Suppose that x_0 is a point on the line segment connecting z_x and $z_x+6t^{1/\alpha}\mathbf{n}(z_x)$ such that $B(x_0,2\kappa t^{1/\alpha})\subset B\setminus\{x\}$. Then for any $a>0$, there exists a constant $C_{31}=C_{31}(M,\kappa,\alpha,r_0,a)>0$ such that for all $m\in(0,M]$,*

$$\mathbb{P}_x\left(X^m_{at}\in B(x_0,\kappa t^{1/\alpha})\right)\ \ge\ C_{31}\,t^{-1/2}\delta_D(x)^{\alpha/2}. \tag{4.2}$$

Proof. Let $0<\kappa_1\le\kappa$ and assume first that $2^{-4}\kappa_1t^{1/\alpha}<\delta_D(x)\le 3t^{1/\alpha}$. Repeating the proof of Lemma 3.3, we get that, in this case, there exists a constant $c_1=c_1(\alpha,\kappa_1,M,r_0,a)>0$ such that for all $m\in(0,M]$ and $t\le T_0$,

$$\mathbb{P}_x\left(X^m_{at}\in B(x_0,\kappa_1t^{1/\alpha})\right)\ \ge\ c_1t^{d/\alpha+1}J^m(x,x_0).$$

Using the fact that $|x-x_0|\in[2\kappa t^{1/\alpha},6t^{1/\alpha}]$, we have

$$\mathbb{P}_x\left(X^m_{at}\in B(x_0,\kappa_1t^{1/\alpha})\right)\ \ge\ c_2>0\qquad\text{for every } m\in(0,M], \tag{4.3}$$

where $c_2=c_2(\alpha,\kappa_1,M,r_0,a)$. By taking $\kappa_1=\kappa$, this shows that (4.2) holds for all $a>0$ in the case when $2^{-4}\kappa_1t^{1/\alpha}<\delta_D(x)\le3t^{1/\alpha}$.

So it suffices to consider the case that $\delta_D(x)\le2^{-4}\kappa t^{1/\alpha}$. We now show that there is some $a_0>1$ so that (4.2) holds for every $a\ge a_0$ and $\delta_D(x)\le2^{-4}\kappa t^{1/\alpha}$. For simplicity, we assume without loss of generality that $x_0=0$ and let $\widehat{B}:=B(0,\kappa t^{1/\alpha})$. Let $x_2=z_x+4^{-1}\kappa\mathbf{n}(z_x)t^{1/\alpha}$ and $B_2:=B(x_2,4^{-1}\kappa t^{1/\alpha})$. Observe that since $B(0,2\kappa t^{1/\alpha})\subset B\setminus\{x\}$,

$$\kappa/2t^{1/\alpha}\le|y-z|\le6t^{1/\alpha}\qquad\text{for } y\in B_2 \text{ and } z\in B(0,\kappa t^{1/\alpha}). \tag{4.4}$$

By the strong Markov property of X^m at the first exit time $\tau^m_{B_2}$ from B_2 and Lemma 3.1, there exists $c_3=c_3(a,\kappa,\alpha,M,T)>0$ such that for all $m\in(0,M]$,

$$\begin{aligned}
&\mathbb{P}_x\left(X^m_{at}\in B(0,\kappa t^{1/\alpha})\right)\\
\ge\ &\mathbb{P}_x\left(\tau^m_{B_2}<at,\ X^m_{\tau^m_{B_2}}\in B(0,2^{-1}\kappa t^{1/\alpha})\text{ and }|X^m_s-X^m_{\tau_{B_2}}|<\kappa/2\text{ for }s\in[\tau^m_{B_2},\tau^m_{B_2}+at^{1/\alpha}]\right)\\
\ge\ &c_3\,\mathbb{P}_x\left(\tau^m_{B_2}<at\text{ and }X^m_{\tau^m_{B_2}}\in B(0,2^{-1}\kappa t^{1/\alpha})\right).
\end{aligned} \tag{4.5}$$

It follows from Theorem 2.5 and the explicit formula for the Poisson kernel of balls with respect to X that there exist $c_4=c_4(\alpha,M)>0$ and $c_5=c_5(\alpha,M,\kappa,r_0)>0$ such that for all $m\in(0,M]$,

$$\mathbb{P}_x\left(X^m_{\tau^m_{B_2}}\in B(0,2^{-1}\kappa t^{1/\alpha})\right)\ge\mathbb{P}_x\left(X_{\tau_{B_2}}\in B(0,2^{-1}\kappa t^{1/\alpha})\right)\ge c_5\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}. \tag{4.6}$$

Applying Theorem 2.5 and the estimates for G_B (see, for instance, [13, (1.4)]), we get that there exist $c_6=c_6(\alpha,M)>0$ and $c_7=c_7(\alpha,M,\kappa,r_0)>0$ such that for all $m\in(0,M]$,

$$\mathbb{P}_x(\tau^m_{B_2}\ge at)\ \le\ (at)^{-1}\,\mathbb{E}_x[\tau^m_{B_2}]\le c_6(at)^{-1}\,\mathbb{E}_x[\tau_{B_2}]\ \le\ a^{-1}c_7\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}.$$

Define $a_0 = 2c_7/(c_5)$. We have by (4.5)–(4.6) and the display above that for $a \ge a_0$ and $m \in (0, M]$,

$$\mathbb{P}_x(X^m_{at} \in \widehat{B}) \ge c_2 \left(\mathbb{P}_x(X^m_{\tau^m_{B_2}} \in B(0, 2^{-1}\kappa t^{1/\alpha})) - \mathbb{P}_x\left(\tau^m_{B_2} \ge at\right)\right) \ge c_2\,(c_5/2)\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}. \quad (4.7)$$

(4.3) and (4.7) show that (4.2) holds for every $a \ge a_0$ and for every $x \in D$ with $\delta_D(x) \le 3t^{1/\alpha}$.

Now we deal with the case $0 < a < a_0$ and $\delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$. If $\delta_D(x) \le 3(at/a_0)^{1/\alpha}$, we have from (4.2) for the case of $a = a_0$ that there exist $c_8 = c_8(\kappa, \alpha, M) > 0$ and $c_9 = c_9(\kappa, \alpha, M, a) > 0$ such that for all $m \in (0, M]$,

$$\begin{aligned}\mathbb{P}_x\left(X^m_{at} \in B(x_0, \kappa t^{1/\alpha})\right) &\ge \mathbb{P}_x\left(X^m_{a_0(at/a_0)} \in B(x_0, \kappa(at/a_0)^{1/\alpha})\right)\\ &\ge c_8\left(\frac{\delta_D(x)}{(at/a_0)^{1/\alpha}}\right)^{\alpha/2} = c_9\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha/2}.\end{aligned}$$

If $3(at/a_0)^{1/\alpha} < \delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$ (in this case $\kappa > 3 \cdot 2^4(a/a_0)^{1/\alpha}$), we get (4.2) from (4.3) by taking $\kappa_1 = (a/a_0)^{1/\alpha}$. The proof of the lemma is now complete. □

Proposition 4.3 *Let $M > 0$ be a constant. Suppose that $(t, x, y) \in (0, T_0] \times D \times D$ with $|x - y| \le t^{1/\alpha}$ and $\delta_D(x) \le 2t^{1/\alpha}$. Then there exists a constant $C_{32} = C_{32}(\alpha, M, r_0) > 0$ such that for all $m \in (0, M]$,*

$$p^m_D(t, x, y) \ge C_{32}\, t^{-d/\alpha-1}\, \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}. \quad (4.8)$$

Proof. Note that under the assumptions of the Proposition, we have $\delta_D(y) \le |x - y| + \delta_D(x) \le 3t^{1/\alpha} < r_0/5$. So there are points z_x, $z_y \in \partial D$ and x_0, $y_0 \in D$ such that $\delta_D(x) = |x - z_x|$, $\delta_D(y) = |y - z_y|$, $\partial B(x_0, 4t^{1/\alpha}) \cap \partial D = \{z_x\}$ and $\partial B(y_0, 4t^{1/\alpha}) \cap \partial D = \{z_y\}$. Observe that

$$\delta_D(x_0) = \delta_D(y_0) = 4t^{1/\alpha} \quad \text{and} \quad |x - x_0|, |y - y_0| \in [t^{1/\alpha}, 4t^{1/\alpha}).$$

By the semigroup property, with $B := B(x_0, 4^{-1}t^{1/\alpha})$ and $\widetilde{B} := B(y_0, 4^{-1}t^{1/\alpha})$

$$\begin{aligned}p^m_D(t, x, y) &= \int_D p^m_D(t/3, x, z)\int_D p^m_D(t/3, z, w)p^m_D(t/3, w, y)dwdz\\ &\ge \int_B p^m_D(t/3, x, z)\int_{\widetilde{B}} p^m_D(t/3, z, w)p^m_D(t/3, w, y)dwdz\\ &\ge \inf_{(z,w)\in B\times\widetilde{B}} p^m_D(t/3, z, w)\int_B p^m_D(t/3, x, z)dz\int_{\widetilde{B}} p^m_D(t/3, w, y)dw.\end{aligned}$$

Since for $z \in B$ and $w \in \widetilde{B}$, $\delta_D(z) \ge \delta_D(x_0) - |x_0 - z| \ge t^{1/\alpha}$, $\delta_D(w) \ge \delta_D(y_0) - |y_0 - w| \ge t^{1/\alpha}$, $|z - w| \le |z - x_0| + |x_0 - x| + |x - y| + |y - y_0| + |y_0 - w| < 10t^{1/\alpha}$, and $|z - w| < \frac{10}{\psi(m^{1/\alpha}10T^{1/\alpha})}\psi(m^{1/\alpha}|z - w|)t^{1/\alpha}$, by combining Proposition 3.2 and Proposition 3.4, we have that there exists $c_1 = c_1(\alpha, r_0, M) > 0$ such that for all $m \in (0, M]$,

$$\inf_{(z,w)\in B\times\widetilde{B}} p^m_D(t/3, z, w) \ge c_1 t^{-d/\alpha}.$$

Since $\delta_D(x) \le 2t^{1/\alpha} < r_0/8$ and $\delta_D(y) \le 3t^{1/\alpha}$, we have by Lemma 4.2 that there is a positive constant $c_2 = c_2(\alpha, M, r_0)$ such that for all $m \in (0, M]$,

$$p_D^m(t,x,y) \ \ge\ c_2\, t^{-d/\alpha-1}\, \delta_D(x)^{\alpha/2}\, \delta_D(y)^{\alpha/2}.$$

□

Proposition 4.4 *Let $M > 0$ be a constant. Suppose that $(t, x, y) \in (0, T_0] \times D \times D$ with $\delta_D(x) \le (t/2)^{1/\alpha} \le \delta_D(y)$ and $|x-y|^\alpha \ge t\psi(m^{1/\alpha}|x-y|)^{\alpha/(d+\alpha)}$. Then there exists a constant $C_{33} = C_{33}(\alpha, M, r_0) > 0$ such that for all $m \in (0, M]$,*

$$p_D^m(t,x,y) \ \ge\ C_{33}\, t^{1/2}\, \delta_D(x)^{\alpha/2} J^m(x,y). \tag{4.9}$$

Proof. Since $\delta_D(x) \le (t/2)^{1/\alpha} \le r_0/16$, there are $z_x \in \partial D$ and $z_0 \in D$ such that $\delta_D(x) = |x - z_x|$ and $\partial B(z_0, 2t^{1/\alpha}) \cap \partial D = \{z_x\}$. Choose x_0 in $B(z_0, 2\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})$ and $\kappa = \kappa(\alpha) \in (0,1)$ such that

$$\begin{aligned} & B\left(x_0, 2\kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha}\right) \\ \subset\ & B\left(z_0, (2-2^{-2/\alpha})\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha}\right) \cap B\left(x, (1-2^{-1-2/\alpha})\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha}\right). \end{aligned}$$

Such a ball $B(x_0, 2\kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})$ always exists because $2 < (2-2^{-1}) + (1-2^{-2}) < (2-2^{-2/\alpha}) + (1-2^{-1-2/\alpha})$.

We consider two cases separately.

(i) Suppose $|x-y| \le T_0^{1/\alpha}$. Note that $|x-y| \ge t^{1/\alpha}\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}$. Thus for every $z \in B(x_0, \kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})$, we have $\delta_D(z) \ge (t/4)^{1/\alpha}$ and

$$\begin{aligned} &|y-z| \ge |y-x| - |z-x| \ge 2^{-1}(t/4)^{1/\alpha}\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)} \\ &\ge 2^{-1}(t/4)^{1/\alpha}\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}\psi(m^{1/\alpha}|y-z|)^{1/(d+\alpha)}. \end{aligned}$$

On the other hand, for every $z \in B(x_0, \kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})$,

$$|z-y| \ \le\ |z-x| + |x-y| \ \le\ (1-2^{-1-2/\alpha})\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha} + |x-y| \ <\ 2|x-y|.$$

Thus by the semigroup property, Proposition 3.2 and Proposition 3.4, there exist positive constants $c_i = c_i(\alpha, M, r_0)$, $i = 1, 2, 3$, such that for all $m \in (0, M]$,

$$\begin{aligned} p_D^m(t,x,y) &= \int_D p_D^m(t/2,x,z)p_D^m(t/2,z,y)dz \\ &\ge \int_{B(x_0,\kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})} p_D^m(t/2,x,z)p_D^m(t/2,z,y)dz \\ &\ge c_1 t \int_{B(x_0,\kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})} p_D^m(t/2,x,z)J^m(z,y)dz \\ &\ge c_2 t j^m(2|x-y|) \int_{B(x_0,\kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})} p_D^m(t/2,x,z)dz \\ &= c_3 t j^m(2|x-y|)\, \mathbb{P}_x\left(X_{t/2}^{m,D} \in B(x_0, \kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})\right). \end{aligned}$$

Applying Lemma 4.2 and (2.9), we arrive at the conclusion of the Proposition for $|x-y| \le T_0^{1/\alpha}$.

(ii) Suppose $|x-y| > T_0^{1/\alpha}$. For every $z \in B(x_0, \kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})$, we have

$$|x-y| \,\le\, (1-2^{-1-2/\alpha})\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}T_0^{1/\alpha} + |y-z| \,\le\, T_0^{1/\alpha} + |y-z|.$$

Thus for every $z \in B(x_0, \kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})$, we have $\delta_D(z) \ge (t/4)^{1/\alpha}$ and

$$\begin{aligned}
&|y-z| \ge |y-x| - |z-x| \ge T_0^{1/\alpha} - (1-2^{-1-2/\alpha})\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha} \\
&\ge 2^{-1}(t/4)^{1/\alpha}\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)} \ge 2^{-1}(t/4)^{1/\alpha}\psi(m^{1/\alpha}|x-y|^{1/\alpha})^{1/(d+\alpha)} \\
&\ge 2^{-1}(t/4)^{1/\alpha}\psi\left(m^{1/\alpha}T_0^{1/\alpha} + m^{1/\alpha}|y-z|\right)^{1/(d+\alpha)} \ge c_4 2^{-1}(t/4)^{1/\alpha}\psi\left(m^{1/\alpha}|y-z|\right)^{1/(d+\alpha)}
\end{aligned}$$

for some $c_4 = c_4(\alpha, r_0, M) > 0$, where in the last inequality we used (2.4). On the other hand, for every $z \in B(x_0, \kappa\psi(m^{1/\alpha}|x-y|)^{1/(d+\alpha)}t^{1/\alpha})$,

$$\begin{aligned}
|z-y| \;&\le\; |z-x| + |x-y| \,\le\, (1-2^{-1-2/\alpha})\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha} + |x-y| \\
&\le\; T_0^{1/\alpha} + |x-y|.
\end{aligned}$$

Thus by the semigroup property and Proposition 3.5, there exist positive constants $c_i = c_i(\alpha, M, r_0)$, $i = 5, 6, 7$, such that for all $m \in (0, M]$,

$$\begin{aligned}
p_D^m(t,x,y) \;&\ge\; c_5 t \int_{B(x_0,\kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})} p_D^m(t/2,x,z)J^m(z,y)dz \\
&\ge\; c_6 t j^m(|x-y| + T_0^{1/\alpha}) \int_{B(x_0,\kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})} p_D^m(t/2,x,z)dz \\
&=\; c_7 t j^m(|x-y| + T_0^{1/\alpha})\, \mathbb{P}_x\left(X_{t/2}^{m,D} \in B(x_0, \kappa\psi(m^{1/\alpha}T_0^{1/\alpha})^{1/(d+\alpha)}t^{1/\alpha})\right).
\end{aligned}$$

Applying Lemma 4.2 and (2.10), we arrive at the conclusion of the proposition for $|x-y| > T_0^{1/\alpha}$. □

Proposition 4.5 *Let $M > 0$ be a constant. Suppose that $(t, x, y) \in (0, T_0] \times D \times D$ with*

$$\max\{\delta_D(x),\, \delta_D(y)\} \le (t/2)^{1/\alpha} \le |x-y|\psi(m^{1/\alpha}|x-y|)^{-1/(d+\alpha)}.$$

Then there exists a constant $C_{34} = C_{34}(\alpha, M, r_0) > 0$ such that for all $m \in (0, M]$,

$$p_D^m(t,x,y) \,\ge\, C_{34}\, \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}J^m(x,y). \tag{4.10}$$

Proof. As in the first paragraph of the proof of Proposition 4.3, choose $z_x \in \partial D$ and $x_0 \in D$ so that $|x - z_x| = \delta_D(x)$ and $\partial B(x_0, 3t^{1/\alpha}) \cap \partial D = \{z_x\}$. Let $\kappa := 1 - 2^{-1/\alpha}$. Note that for every $z \in B(x_0, \kappa t^{1/\alpha})$, we have

$$\delta_D(z) \ge 2(t/2)^{1/\alpha} \quad \text{and} \quad |y-z| \ge \delta_D(z) - \delta_D(y) \ge (t/2)^{1/\alpha} \ge (t/2)^{1/\alpha}\psi(m^{1/\alpha}|y-z|)^{\alpha/(d+\alpha)}.$$

Thus, by the semigroup property and Proposition 4.4, there exists positive constant $c_1 = c_1(\alpha, M, D)$ such that for all $m \in (0, M]$,

$$\begin{aligned} p_D^m(t,x,y) &\geq \int_{B(x_0,\kappa t^{1/\alpha})} p_D^m(t/2,x,z) p_D^m(t/2,z,y) dz \\ &\geq c_1 \int_{B(x_0,\kappa t^{1/\alpha})} p_D^m(t/2,x,z) t^{1/2}\, \delta_D(y)^{\alpha/2} J^m(z,y) dz. \end{aligned} \tag{4.11}$$

If $|x-y| \leq T_0^{1/\alpha}$, for every $z \in B(x_0, \kappa t^{1/\alpha})$

$$\begin{aligned} |z-y| &\leq |x-y| + |x_0 - x| + |x_0 - z| \;\leq\; |x-y| + 4t^{1/\alpha} \\ &\leq |x-y| + \frac{2^{2+1/\alpha}}{\psi(m^{1/\alpha} T_0^{1/\alpha})^{1/(d+\alpha)}} (t/2)^{1/\alpha} \psi(m^{1/\alpha}|x-y|)^{1/(d+\alpha)} \\ &\leq (1 + \frac{2^{2+1/\alpha}}{\psi(m^{1/\alpha} T_0^{1/\alpha})^{1/(d+\alpha)}}) |x-y|. \end{aligned}$$

If $|x-y| > T_0^{1/\alpha}$, for every $z \in B(x_0, \kappa t^{1/\alpha})$

$$|z-y| \;\leq\; |x-y| + |x_0 - x| + |x_0 - z| \;\leq\; |x-y| + 4t^{1/\alpha} \;\leq\; |x-y| + 4T_0^{1/\alpha}.$$

Thus by (2.9), (2.10) and Lemma 4.2, we get from (4.11) that there exist positive constants $c_i = c_i(\alpha, M, D), i = 2, 3$, such that for all $m \in (0, M]$,

$$\begin{aligned} p_D^m(t,x,y) &\geq c_2\, t^{1/2}\, \delta_D(y)^{\alpha/2} J^m(x,y) \int_{B(x_0,\kappa t^{1/\alpha})} p_D^m(t/2,x,z) dz \\ &= c_2\, t^{1/2}\, \delta_D(y)^{\alpha/2} J^m(x,y)\, \mathbb{P}_x \left(X_{t/2}^{m,D} \in B(x_0, \kappa t^{1/\alpha}) \right) \\ &= c_3\, \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} J^m(x,y). \end{aligned}$$

□

Proof of Theorem 4.1. We first assume that $t \leq T_0$. Combining Propositions 3.2 and 3.4, we get the conclusion of Theorem 4.1 in the case

$$\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha} \geq |x-y| \psi(m^{1/\alpha}|x-y|)^{-1/(d+\alpha)}.$$

Using symmetry and Proposition 4.3 we the conclusion of Theorem 4.1 in the case

$$\max\{\delta_D(x), \delta_D(y), |x-y|\psi(m^{1/\alpha}|x-y|)^{-1/(d+\alpha)}\} \leq t^{1/\alpha}.$$

Now we consider the case $|x-y|\psi(m^{1/\alpha}|x-y|)^{-1/(d+\alpha)} \geq t^{1/\alpha}$. Using symmetry and combining Propositions 4.4 and 4.5, we get the conclusion of Theorem 4.1 in the case $\min\{\delta_D(x), \delta_D(y)\} \leq (t/2)^{1/\alpha}$ and $|x-y|\psi(m^{1/\alpha}|x-y|)^{-1/(d+\alpha)} \geq t^{1/\alpha}$. Proposition 3.4 covers the remaining case $\min\{\delta_D(x), \delta_D(y)\} \geq (t/2)^{1/\alpha}$ and $|x-y|\psi(m^{1/\alpha}|x-y|)^{-1/(d+\alpha)} \geq t^{1/\alpha}$. We have thus arrived at the conclusion of Theorem 4.1 with $c_2 = 1$ for $t \leq T_0$.

Assume $T = 2T_0$. Recall that $T_0 = (r_0/16)^\alpha$. For $(t,x,y) \in (T_0, 2T_0] \times D \times D$, let $x_0, y_0 \in D$ be such that $\max\{|x-x_0|, |y-y_0|\} < r_0$ and $\min\{\delta_D(x_0), \delta_D(y_0)\} \geq r_0/2$. Note that, if $|x-y| \geq 4r_0$, then $|x-y|-2r_0 \le |x_0-y_0| \le |x-y|+2r_0$, so by (2.10), $c_0^{-1}J^m(x_0,y_0) \le J^m(x,y) \le c_0 J^m(x_0,y_0)$ for some constant $c=c(M)>1$. Thus by considering the cases $|x-y|\ge 4r_0$ and $|x-y|<4r_0$, we have

$$(t/2)^{-d/\alpha} \wedge \frac{t}{2}J^m(x_0,y_0) \ge c_1\left(t^{-d/\alpha} \wedge tJ^m(x,y)\right). \tag{4.12}$$

Similarly, there is a positive constant c_2 such that

$$\begin{aligned}
(t/3)^{-d/\alpha} \wedge (t/3)J^m(x,z) &\ge c_2\left((t/(12))^{-d/\alpha} \wedge \frac{t}{12}J^m(x_0,z)\right), \quad z\in D,\\
(t/3)^{-d/\alpha} \wedge \frac{t}{3}J^m(w,y) &\ge c_2\left((t/(12))^{-d/\alpha} \wedge \frac{t}{12}J^m(w,y_0)\right), \quad w\in D.
\end{aligned} \tag{4.13}$$

By (4.12) and the lower bound estimate in Theorem 4.1 for p_D^m on $(0,T_0]\times D\times D$, we have

$$\begin{aligned}
p_D^m(t,x,y) &= \int_{D\times D} p_D^m(t/3,x,z)p_D^m(t/3,z,w)p_D^m(t/3,w,y)dzdw\\
&\ge c_3\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t/3}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t/3}}\right)\int_{D\times D}\left((t/3)^{-d/\alpha}\wedge(t/3)J^m(x,z)\right)\left(1\wedge\frac{\delta_D(z)^{\alpha/2}}{\sqrt{t/3}}\right)\\
&\qquad\cdot p_D^m(t/3,z,w)\left((t/3)^{-d/\alpha}\wedge\frac{t}{3}J^m(w,y)\right)\left(1\wedge\frac{\delta_D(w)^{\alpha/2}}{\sqrt{t/3}}\right)dzdw\\
&\ge c_4\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)\int_{D\times D}\left((t/(12))^{-d/\alpha}\wedge\frac{t}{12}J^m(x_0,z)\right)\\
&\qquad\cdot p_D^m(t/3,z,w)\left((t/(12))^{-d/\alpha}\wedge\frac{t}{12}J^m(w,y_0)\right)dzdw
\end{aligned}$$

for some positive constants $c_i, i=3,4$. Let $D_1:=\{z\in D:\delta_D(z)>r_0/4\}$. Clearly, $x_0,y_0\in D_1$ and

$$\min\{\delta_{D_1}(x_0),\delta_{D_1}(y_0)\}\ge r_0/4=4(T_0)^{1/\alpha}\ge 4(t/2)^{1/\alpha}. \tag{4.14}$$

By Theorem 3.6 and (4.13), we have

$$\begin{aligned}
&\int_{D\times D}\left((t/(12))^{-d/\alpha}\wedge\frac{t}{12}J^m(x_0,z)\right)p_D^m(t/3,z,w)\left((t/(12))^{-d/\alpha}\wedge\frac{t}{12}J^m(w,y_0)\right)dzdw\\
&\ge c_5\int_{D_1\times D_1}p^m(t/(12),x_0,z)p_{D_1}^m(t/3,z,w)p^m(t/(12),w,y_0)dzdw\\
&\ge c_5\int_{D_1\times D_1}p_{D_1}^m(t/(12),x_0,z)p_{D_1}^m(t/3,z,w)p_{D_1}^m(t/(12),w,y_0)dzdw\\
&= c_5\,p_{D_1}^m(t/2,x_0,y_0)\ge c_6\left((t/2)^{-d/\alpha}\wedge\frac{t}{2}J^m(x_0,y_0)\right)\ge c_7\left(t^{-d/\alpha}\wedge tJ^m(x,y)\right)
\end{aligned}$$

for some positive constants $c_i, i=5,\cdots,7$. Here Proposition 3.5 is used in the third inequality in view of (4.14). By repeating the argument above, we have proved Theorem 4.1. □

Proof of Theorem 1.1. Theorem 1.1(i) is a combination of Theorems 3.9 and 4.1, so we only need to prove Theorem 1.1(ii).

Since D is bounded, the functions $\phi(m^{1/\alpha}|x-y|/C_2)$ and $\phi(C_2 m^{1/\alpha}|x-y|)$ are bounded between two positive constants independent of $m \in (0, M]$. Thus it follows from Theorem 1.1 (i) that there exist positive constants $c_i = c_i(\alpha, M, D) > 0$, $i = 1, 2$, such that for all $m \in (0, M]$,

$$c_1 p_D(t, x, y) \le p^m_D(t, x, y) \le c_2 p_D(t, x, y) \qquad \text{on } (0, T_1] \times D \times D, \tag{4.15}$$

where $T_1 = \text{diam}(D)^\alpha$. Let $\{\lambda^{\alpha,m,D}_k, k = 1, 2 \dots\}$ be the eigenvalues of $((m^{2/\alpha} - \Delta)^{\alpha/2} - m)|_D$, arranged in increasing order and repeated according to multiplicity, and let $\{\phi^{m,\alpha,D}_k, k = 1, 2, \dots\}$ be the corresponding eigenfunctions normalized to have unit L^2-norm on D. It is well known that $\lambda^{\alpha,m,D}_1$ is simple and $\phi^{m,\alpha,D}_1$ can be chosen to be non-negative. It follows from [16] that the function $m \mapsto \lambda^{\alpha,m,D}_1$ is continuous on $[0, M]$, and thus it is bounded between two positive numbers on the interval $[0, M]$. Since

$$\phi^{m,\alpha,D}_1(x) = e^{\lambda^{\alpha,m,D}_1 T_1} \int_D p^m_D(T_1, x, y)\, \phi^{m,\alpha,D}_1(y) dy,$$

it follows from (4.15) that there exist $c_i = c_i(\alpha, M, D) > 0$, $i = 3, 4$, such that

$$\begin{aligned} \phi^{m,\alpha,D}_1(x) &\le c_3 T_1^{-d/\alpha-1} \delta_D(x)^{\alpha/2} \int_D \delta_D(y)^{\alpha/2} \phi^{m,\alpha,D}_1(y) dy, && (4.16)\\ \phi^{m,\alpha,D}_1(x) &\ge c_4 T_1^{-d/\alpha-1} \delta_D(x)^{\alpha/2} \int_D \delta_D(y)^{\alpha/2} \phi^{m,\alpha,D}_1(y) dy. && (4.17)\end{aligned}$$

Using the Sobolev embedding theorem and the simplicity of $\lambda^{\alpha,m,D}_1$, we conclude from Example 5.1 of [16] that the first eigenfunction $\phi^{\alpha,m,D}_1$ is continuous in $L^2(D; m)$ in $m \in [0, M]$. Consequently,

$$m \mapsto \int_D \delta_D(y)^{\alpha/2} \phi^{m,\alpha,D}_1(y) dy$$

is a continuous positive function of $m \in [0, M]$ and so it is bounded between two positive constants over the interval $[0, M]$. Thus by (4.16)–(4.17), there exist positive constants $c_i = c_i(\alpha, M, D) > 0$, $i = 5, 6$, such that for every $m \in (0, M]$ and $x \in D$,

$$c_5 \delta^{\alpha/2}_D(x) \le \phi^{m,\alpha,D}_1(x) \le c_6 \delta^{\alpha/2}_D(x). \tag{4.18}$$

It follows from the paragraph after Theorem 2.1.1 in [19] that there exists a decreasing function $g(t)$ of t

$$p_D(t, x, y) \le g(t) \phi^{0,\alpha,D}_1(x)\, \phi^{0,\alpha,D}_1(y) \qquad (t, x, y) \in (0, \infty) \times D \times D.$$

Thus by (4.15)–(4.18), we have for all $m \in (0, M]$,

$$p^m_D(t, x, y) \le c_7 g(t) \phi^{m,\alpha,D}_1(x)\, \phi^{m,\alpha,D}_1(y) \qquad (t, x, y) \in (0, 1] \times D \times D$$

for some positive constant c_7. Put

$$\widetilde{p}^m_D(t, x, y) := e^{\lambda^{\alpha,m,D}_1 t} \frac{p^m_D(t, x, y)}{\phi^{m,\alpha,D}_1(x)\, \phi^{m,\alpha,D}_1(y)} \qquad (t, x, y) \in (0, \infty) \times D \times D.$$

Then it follows from the arguments in [19, Theorems 4.2.5 and 2.1.4] that there exists a decreasing function $\tilde{g}(t)$ of t such that

$$|\widetilde{p}^m_D(t, x, y) - 1| \le \tilde{g}^2(t/3) \sum_{k=1}^\infty e^{-(\lambda^{\alpha,m,D}_k - \lambda^{\alpha,m,D}_1)t} \le \tilde{g}^2(t/3) \tilde{g}^4(t/(24)) e^{-(\lambda^{\alpha,m,D}_2 - \lambda^{\alpha,m,D}_1)t}.$$

Since the function $m \mapsto \lambda_2^{\alpha,m,D} - \lambda_1^{\alpha,m,D}$ is continuous on $[0, M]$ (see [16]) and positive, there exists $c_8 = c_8(M) > 0$ such that for all $m \in (0, M]$, $\lambda_2^{\alpha,m,D} - \lambda_1^{\alpha,m,D} > c_8$. Consequently we have

$$|\widetilde{p}_D^m(t,x,y) - 1| \le \tilde{g}^2(t/3)\tilde{g}^4(t/(24))e^{-c_5 t}.$$

Thus there exists $T_2 = T_2(M) > 0$ such that for all $m \in (0, M]$, $|\widetilde{p}_D^m(t,x,y) - 1| \le \frac{1}{2}$ for $t \ge T_2$, that is, for all $m \in (0, M]$ and $t \ge T_2$,

$$\frac{1}{2}e^{-\lambda_1^{\alpha,m,D}t}\phi_1^{m,\alpha,D}(x)\,\phi_1^{m,\alpha,D}(y) \le p_D^m(t,x,y) \le \frac{3}{2}e^{-\lambda_1^{\alpha,m,D}t}\phi_1^{m,\alpha,D}(x)\,\phi_1^{m,\alpha,D}(y).$$

Therefore, by (4.18) , we have for all $m \in (0, M]$ and $t \ge T_2$,

$$\frac{1}{2}c_4^2 e^{-\lambda_1^{\alpha,m,D}t}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2} \;\le p_D^m(t,x,y) \le \frac{3}{2}c_3^2 e^{-\lambda_1^{\alpha,m,D}t}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2}.$$

If $T < T_2$, then by Theorem 1.1(i), there is a constant $c_9 \ge 1$ such that for all $m \in (0, M]$

$$c_9^{-1}\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2} \;\le\; p_D(t,x,y) \;\le\; c_9\,\delta_D(x)^{\alpha/2}\,\delta_D(y)^{\alpha/2} \qquad \text{for } t \in [T, T_2) \text{ and } x, y \in D.$$

This establishes Theorem 1.1(ii). □

5 Green function estimates

In this section, we present the proof of Theorem 1.4.

Theorem 5.1 *Let $H := \{(x_1, \cdots, x_d) \in \mathbb{R}^d : \ x_d > 0\}$ be the upper half space in $\mathbb{R}^d$, $d \ge 2$. There exists $C_{35} > 1$ such that for all $m > 0$,*

$$C_{35}^{-1}\widetilde{V}_H^{\alpha,m}(x,y) \le G_H^m(x,y) \le C_{35}\widetilde{V}_H^{\alpha,m}(x,y), \qquad x, y \in H,$$

where $\widetilde{V}_H^{\alpha,m}$ is defined preceding Theorem 1.4.

Proof. Since by (2.19),

$$G_H^m(x,y) = m^{(d-\alpha)/\alpha}\,G_H^1(m^{1/\alpha}x,\, m^{1/\alpha}y) \qquad \text{for } x, y \in H, \tag{5.1}$$

it suffices to consider $m = 1$. When $m = 1$, this theorem is essentially established as Theorem 5.3 in [23]. However there is an error in the statement of [23, Theorem 5.3] for the case of $|x - y| \le 3$, where the terms $\left(\frac{x_d \wedge y_d}{|x-y|}\right)^{\alpha/2}$ should be replaced by $\left(\frac{x_d y_d}{|x-y|^2}\right)^{\alpha/2}$. The error in [23, Theorem 5.3] stems from [5, Theorem 3.2], where the same error occurred in the estimate of the 1-resolvent of X^1 in upper half space. [5, Theorem 3.2] is a corollary of [5, Lemma 3.1]. A typo occurred in the display preceding the statement of [5, Theorem 3.2], which resulted in all these errors. That display should be

$$\left(\sqrt{\frac{4\delta(x)\delta(y)}{|x-y|^2} + 1} - 1\right)|x - y| \approx \frac{\delta(x)\delta(y)}{|x-y|} \quad \text{for } \frac{\delta(x)\delta(y)}{|x-y|^2} \le 1.$$

Another typo occurred in [5, (21) and (22)], where the term $\delta(x)\wedge\delta(y)\wedge 1$ should be $\dfrac{\delta(x)\delta(y)}{|x-y|}\wedge 1$. With these corrections, the desired Green function estimates can then be established as in [23, Theorem 5.3]. $\square$

In the next theorem, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 depending only on α so that $c_1 g(x) \le f(x) \le c_2 g(x)$ in the common domain of definition for f and g.

Theorem 5.2 *Let $H=(0,\infty)\subset\mathbb{R}$. Then*

(i) *For $\alpha\ge 1$ and $x,y>0$,*

$$G^m_H(x,y)\asymp\begin{cases}\dfrac{e^{-m^{1/\alpha}|x-y|}}{|x-y|^{1-(\alpha/2)}}(m^{-1/\alpha}\wedge x\wedge y)^{\alpha/2}+m^{(2-\alpha)/\alpha}(x\wedge y)+m^{(2-\alpha)/(2\alpha)}(x\wedge y)^{\alpha/2} \\ \qquad\qquad\qquad\qquad \text{when } |x-y|\ge m^{-1/\alpha}\wedge x\wedge y, \\ (m^{-1/\alpha}\wedge x\wedge y)^{\alpha-1}+m^{(2-\alpha)/\alpha}(x\wedge y)+m^{(2-\alpha)/(2\alpha)}(x\wedge y)^{\alpha/2} \\ \qquad\qquad\qquad\qquad \text{when } \alpha>1 \text{ and } |x-y|<m^{-1/\alpha}\wedge x\wedge y, \\ \ln\left(2\,\dfrac{m^{-1}\wedge x\wedge y}{|x-y|}\right)+m(x\wedge y)+m^{1/2}(x\wedge y)^{1/2} \\ \qquad\qquad\qquad\qquad \text{when } \alpha=1 \text{ and } |x-y|<m^{-1/\alpha}\wedge x\wedge y.\end{cases}$$

(ii) *For $0<\alpha<1$ and $x,y>0$,*

$$G^m_H(x,y)\asymp\begin{cases}\dfrac{m^{-1/2}e^{-m^{1/\alpha}|x-y|}}{|x-y|^{1-(\alpha/2)}}\left(1\wedge\dfrac{xy}{|x-y|^2}\right)^{\alpha/2}+m^{(2-\alpha)/\alpha}(x\wedge y)+m^{(2-\alpha)/(2\alpha)}(x\wedge y)^{\alpha/2} \\ \qquad\qquad\qquad\qquad \text{when } |x-y|\ge m^{-1/\alpha}, \\ |x-y|^{\alpha-1}\left(1\wedge\dfrac{xy}{|x-y|^2}\right)^{\alpha/2}+m^{(2-\alpha)/\alpha}(x\wedge y)+m^{(2-\alpha)/(2\alpha)}(x\wedge y)^{\alpha/2} \\ \qquad\qquad\qquad\qquad \text{when } |x-y|<m^{-1/\alpha}.\end{cases}$$

Proof. In view of (5.1), it suffices to establish the above estimates for $m=1$. When $m=1$, the proof of this theorem is essentially given in [23, Theorems 2.13 and 3.2]. However there is an error in the statement of [23, Theorem 2.13] for the case of $\alpha<d=1$, which is the same error as described in the proof of Theorem 5.1 above. With this correction, we can get the desired estimates. $\square$

Proof of Theorem 1.4. We will just give the proof for the case $d\ge 2$. The argument can be adapted to give a proof for the case $d=1$. To save space, we leave the details to interested readers.

In the rest of this proof, we assume $d\ge 2$. Without loss of generality, we assume $M=1$ and that $H_{1/4}\subset D\subset H_{-1/4}$, where $H_a:=\{y=(\widehat{y},y_d)\in\mathbb{R}^d:\ y_d>a\}$. In this proof, the notation $f\asymp g$ means that there are positive constants c_1 and c_2 depending only on D and α so that $c_1 g(x)\le f(x)\le c_2 g(x)$ in the common domain of definition for f and g.

If D is a general $C^{1,1}$ open set by (4.3), (4.4), (4.6), (4.7) in [9] and our Theorem 1.1(i), we have for $|x-y| \le 3m^{-1/\alpha}$,

$$\int_0^1 p_D^m(t,x,y)dt \asymp \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha}}\right). \tag{5.2}$$

Now we assume that D is a half-space-like $C^{1,1}$ open set in $\mathbb{R}^d$. Observe that if $\delta_D(x) \ge 1$, then

$$\frac{1}{2}\,\delta_D(x) \le \delta_D(x) - \frac{1}{2} \le \delta_{H_{1/4}}(x) \le \delta_D(x) \le \delta_{H_{-1/4}}(x) \le \delta_D(x) + \frac{1}{2} \le \frac{3}{2}\,\delta_D(x). \tag{5.3}$$

(i) First assume that $|x-y| > 3m^{-1/\alpha}$.

If $1 \le \delta_D(x) \le \delta_D(y)$, by (5.3) and Theorem 5.1, we know that there exists $c_3 > 1$ such that for all $m > 0$ and $d \ge 2$,

$$c_3^{-1}\widetilde{V}_D^{\alpha,m}(x,y) \le G_{H_{1/4}}^m(x,y) \ \le\ G_D^m(x,y) \ \le\ G_{H_{-1/4}}^m(x,y) \le c_3\widetilde{V}_D^{\alpha,m}(x,y).$$

In the remainder of this proof, for each $x \in D$, we define

$$x_0 := (\widehat{x}, x_d + 1). \tag{5.4}$$

Since $|x-y| > 3m^{-1/\alpha}$, we have $|x_0 - y| > 2m^{-1/\alpha}$.

Now we consider the case $\delta_D(x) < 1$. Choose a $Q_x \in D$ such that $\delta_D(x) = |x - Q_x|$. Note that $x_0 \in B(Q_x, 2)$. It is easy to see that one can choose a constant $c_4 = c_4(D)$ and a bounded $C^{1,1}$ open set U, whose $C^{1,1}$-characteristics depends on D but is independent of x, such that $B(Q_x, \frac{10}{4}) \cap D \subset U \subset B(Q_x, \frac{11}{4}) \cap D$ and $(U \cap \{\delta_U(z) > c_4\}) \setminus B(Q_x, \frac{9}{4})$ is nonempty. Note that $\delta_U(x) = \delta_D(x)$. Choose an $x_1 \in (U \cap \{\delta_U(z) > c_4\}) \setminus B(Q_x, \frac{9}{4})$. By Theorem 2.9 (uniform boundary Harnack principle), Theorem 2.8 (uniform Harnack principle) and Theorem 1.3,

$$G_D^m(x,y) \asymp \frac{G_U^m(x,x_1)}{G_U^m(x_0,x_1)} G_D^m(x_0,y) \asymp \delta_U(x)^{\alpha/2} G_D^m(x_0,y) = \delta_D(x)^{\alpha/2} G_D^m(x_0,y). \tag{5.5}$$

If $\delta_D(x) < 1 \le \delta_D(y)$, then

$$\delta_D(y) \le \delta_D(x) + |x-y| \le 1 + |x-y| \le \frac{4}{3}|x-y|.$$

By (5.5), Theorem 5.1 and (5.3), we have for $d \ge 3$

$$\begin{aligned}
&G_D^m(x,y) \asymp \delta_D(x)^{\alpha/2} G_D^m(x_0,y) \le \delta_D(x)^{\alpha/2} G_{H_{-1/4}}^m(x_0,y)\\
\asymp\ & \delta_D(x)^{\alpha/2} m^{(2-\alpha)/\alpha} \left(1 \wedge \frac{(1+m^{-(2-\alpha)/(2\alpha)})(\delta_{H_{-1/4}}(y) + m^{-(2-\alpha)/(2\alpha)}\delta_{H_{-1/4}}(y)^{\alpha/2})}{|x_0-y|^2}\right) \frac{1}{|x_0-y|^{d-2}}\\
\asymp\ & m^{(2-\alpha)/\alpha} \left(\delta_D(x)^{\alpha/2} \wedge \frac{(\delta_D(x)^{\alpha/2} + m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2})(\delta_D(y) + m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2})}{|x-y|^2}\right)\\
& \cdot \frac{1}{|x-y|^{d-2}}\\
\le\ & m^{(2-\alpha)/\alpha} \left(1 \wedge \frac{(\delta_D(x) + m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2})(\delta_D(y) + m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2})}{|x-y|^2}\right) \frac{1}{|x-y|^{d-2}}\\
=\ & \widetilde{V}^{\alpha,m}(x,y),
\end{aligned}$$

and

$$\begin{aligned}
& G^m_D(x,y) \asymp \delta_D(x)^{\alpha/2} G^m_D(x_0,y) \geq \delta_D(x)^{\alpha/2} G^m_{H_{1/4}}(x_0,y) \\
\asymp\ & \delta_D(x)^{\alpha/2} m^{(2-\alpha)/\alpha} \frac{\left(1+m^{-(2-\alpha)/(2\alpha)}\right)\left(\delta_{H_{1/4}}(y)+m^{-(2-\alpha)/(2\alpha)}\delta_{H_{1/4}}(y)^{\alpha/2}\right)}{|x_0-y|^2} \frac{1}{|x_0-y|^{d-2}} \\
\asymp\ & m^{(2-\alpha)/\alpha} \frac{\left(\delta_D(x)+m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2}\right)\left(\delta_D(y)+m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2}\right)}{|x-y|^2} \frac{1}{|x-y|^{d-2}} \\
\geq\ & \widetilde{V}^{\alpha,m}(x,y).
\end{aligned}$$

Similarly, when $\delta_D(x) < 1 \leq \delta_D(y)$ and $d=2$, we get

$$G^m_D(x,y) \asymp m^{(2-\alpha)/\alpha} \ln\left(1+\frac{(\delta_D(x)+m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2})(\delta_D(y)+m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2})}{|x-y|^2}\right).$$

Now we suppose that $\delta_D(x) \leq \delta_D(y) < 1$. In this case we have $|x_0-y_0| = |x-y| > 3m^{-1/\alpha}$ Using (5.5), Theorem 2.8 (uniform Harnack principle), Theorem 2.8 (uniform Harnack principle) and Theorem 1.3, similar to the argument before (5.5) with y instead of x, we get

$$G^m_D(x,y) \asymp \delta_D(x)^{\alpha/2} G^m_D(x_0,y) \asymp \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G^m_D(x_0,y_0). \tag{5.6}$$

Thus by (5.3), (5.6) and Theorem 5.1, we have for $d \geq 3$,

$$\begin{aligned}
& G^m_D(x,y) \asymp \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G^m_D(x_0,y_0) \leq \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G^m_{H_{-1/4}}(x_0,y_0) \\
\asymp\ & \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} m^{(2-\alpha)/\alpha} \frac{\left(1+m^{-(2-\alpha)/(2\alpha)}\right)\left(1+m^{-(2-\alpha)/(2\alpha)}\right)}{|x_0-y_0|^2} \frac{1}{|x_0-y_0|^{d-2}} \\
\asymp\ & m^{(2-\alpha)/\alpha} \frac{\left(\delta_D(x)+m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2}\right)\left(\delta_D(y)+m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2}\right)}{|x-y|^2} \frac{1}{|x-y|^{d-2}} \\
\asymp\ & \widetilde{V}^{\alpha,m}(x,y),
\end{aligned}$$

and

$$\begin{aligned}
& G^m_D(x,y) \asymp \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G^m_D(x_0,y_0) \geq \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} G^m_{H_{1/4}}(x_0,y_0) \\
\asymp\ & \delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2} m^{(2-\alpha)/\alpha} \frac{\left(1+m^{-(2-\alpha)/(2\alpha)}\right)\left(1+m^{-(2-\alpha)/(2\alpha)}\right)}{|x_0-y_0|^2} \frac{1}{|x_0-y_0|^{d-2}} \\
\asymp\ & m^{(2-\alpha)/\alpha} \frac{\left(\delta_D(x)+m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2}\right)\left(\delta_D(y)+m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2}\right)}{|x-y|^2} \frac{1}{|x-y|^{d-2}} \\
\asymp\ & \widetilde{V}^{\alpha,m}(x,y).
\end{aligned}$$

Similarly, when $\delta_D(x) \leq \delta_D(y) < 1$, $|x-y| > 3m^{-1/\alpha}$ and $d=2$, we get

$$G^m_D(x,y) \asymp m^{(2-\alpha)/\alpha} \ln\left(1+\frac{(\delta_D(x)+m^{-(2-\alpha)/(2\alpha)}\delta_D(x)^{\alpha/2})(\delta_D(y)+m^{-(2-\alpha)/(2\alpha)}\delta_D(y)^{\alpha/2})}{|x-y|^2}\right).$$

(ii) Now assume that $|x-y| \leq 3m^{-1/\alpha}$.

When $t > 1$, we have by Theorem 1.1 that

$$\begin{aligned}
p^m_D(t,x,y) &= \int_{D\times D} p^m_D(1/2,x,z)p^m_D(t-1,z,w)p^m_D(1/2,w,y)dz \\
&\asymp (1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}\int_{D\times D}(1\wedge\delta_D(z))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|x-z|^{d+\alpha}}\right) \\
&\qquad \cdot p^m_D(t-1,z,w)(1\wedge\delta_D(w))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{d+\alpha}}\right)dzdw.
\end{aligned}$$

Therefore it follows from Theorem 5.1

$$\begin{aligned}
&\int_1^\infty p^m_D(t,x,y)dt \\
&\le c_5\,(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}\int_{D\times D}(1\wedge\delta_D(z))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|x-z|^{d+\alpha}}\right) \\
&\qquad \cdot G^m_D(z,w)(1\wedge\delta_D(w))^{\alpha/2}\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{d+\alpha}}\right)dzdw \\
&\le c_5\,(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}\int_{D\times D}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|z-x|^{d+\alpha}}\right) \\
&\qquad \cdot G^m_{H_1}(z,w)\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{d+\alpha}}\right)dzdw. \qquad (5.7)
\end{aligned}$$

Observe that the following $3G$ inequality is valid,

$$\frac{|x-w|^{d-\alpha}}{|x-z|^{d-\alpha}|z-w|^{d-\alpha}} \le c_6\left(\frac{1}{|x-z|^{d-\alpha}}+\frac{1}{|z-w|^{d-\alpha}}\right). \qquad (5.8)$$

When $d\ge 3$, we have by Theorem 5.1 and (5.8) that

$$\begin{aligned}
&\int_{D\times D}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|z-x|^{d+\alpha}}\right)G^m_{H_1}(z,w)\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{d+\alpha}}\right)dzdw \\
&\le c_7\int_{D\times D}\left(1\wedge\frac{1}{|x-z|^{d+\alpha}}\right)\frac{1}{|z-w|^{d-\alpha}}\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dzdw \\
&\quad + c_7\int_{D\times D}\left(1\wedge\frac{1}{|x-z|^{d+\alpha}}\right)\frac{m^{(2-\alpha)/\alpha}}{|z-w|^{d-2}}1_{\{|z-w|>3m^{-1/\alpha}\}}\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dzdw \\
&\le c_7\int_{D\times D}\left(1\wedge\frac{1}{|x-z|^{d+\alpha}}\right)\frac{1}{|z-w|^{d-\alpha}}\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dzdw \\
&\quad + c_8\,m^{(d-\alpha)/\alpha}\int_D\left(1\wedge\frac{1}{|x-z|^{d+\alpha}}\right)dz\int_D\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dw \\
&\le c_7\int_{D\times D}\left(1\wedge\frac{1}{|x-z|^{d+\alpha}}\right)\frac{1}{|z-w|^{d-\alpha}}\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dzdw \;+\; c_9\,m^{(d-\alpha)/\alpha} \\
&\le c_7\int_{D\times D}\left(1\wedge\frac{1}{|x-z|^{d+\alpha}}\right)\frac{1}{|z-w|^{d-\alpha}}\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dzdw \;+\; c_{10}|x-y|^{-d+\alpha}.
\end{aligned}$$

Note that, by (2.11)

$$\begin{aligned}
&\int_{D\times D}\left(1\wedge\frac{1}{|x-z|^{d+\alpha}}\right)\frac{1}{|z-w|^{d-\alpha}}\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dzdw\\
=\;&\int_D\left(\int_D\left(1\wedge\frac{1}{|x-z|^{d+\alpha}}\right)\frac{1}{|z-w|^{d-\alpha}}\,dz\right)\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dw\\
\le\;& c_{11}\int_D\frac{1}{|x-w|^{d-\alpha}}\left(\int_D\left(1\wedge\frac{1}{|x-z|^{2\alpha}}\right)\left(\frac{1}{|x-z|^{d-\alpha}}+\frac{1}{|z-w|^{d-\alpha}}\right)dz\right)\\
&\qquad\cdot\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dw\\
\le\;& c_{12}\int_D\frac{1}{|x-w|^{d-\alpha}}\left(1\wedge\frac{1}{|w-y|^{d+\alpha}}\right)dw\\
\le\;& c_{13}\int_1^\infty\int_{\mathbb{R}^d}p(t-1,x,w)p(1,w,y)dwdt\;=\;c_{13}\,G(x,y)\;\le\;\frac{c_{14}}{|x-y|^{d-\alpha}}.
\end{aligned}\tag{5.9}$$

Here in the second to the last inequality of (5.9), we used the fact that

$$\sup_{x,w\in D}\int_D\left(1\wedge\frac{1}{|x-z|^{2\alpha}}\right)\frac{1}{|z-w|^{d-\alpha}}\,dz\le\sup_{w\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left(1\wedge\frac{1}{|z|^{2\alpha}}\right)\frac{1}{|z-w|^{d-\alpha}}\,dz<\infty.$$

Thus when $d\ge 3$, by (5.2), (5.7) and (5.8),

$$\begin{aligned}
&\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-d}\\
\le\;& G^m_D(x,y)\;=\;\int_0^1 p^m_D(t,x,y)dt+\int_1^\infty p^m_D(t,x,y)dt\\
\le\;& c_{15}\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-d}\;+\;c_{16}\,\frac{(1\wedge\delta_D(x))^{\alpha/2}(1\wedge\delta_D(y))^{\alpha/2}}{|x-y|^{d-\alpha}}\\
\asymp\;&\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-d}.
\end{aligned}$$

Now consider the case $d=2$. If $1\le\delta_D(x)\le\delta_D(y)$, by (5.3) and Theorem 5.1, we have

$$\begin{aligned}
&\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-2}+m^{(2-\alpha)/\alpha}\ln(1+m^{1/\alpha}(\delta_D(x)\wedge\delta_D(y)))\\
\asymp\;& G^m_{H_{1/4}}(x,y)\;\le\;G^m_D(x,y)\;\le\;G^m_{H_{-1/4}}(x,y)\\
\asymp\;&\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-2}+m^{(2-\alpha)/\alpha}\ln(1+m^{1/\alpha}(\delta_D(x)\wedge\delta_D(y))).
\end{aligned}$$

Thus in the rest of this proof, without loss of generality, we assume that $\delta_D(x)\le\delta_D(y)$, $\delta_D(x)\le 1$ and $|x-y|\le 3m^{-1/\alpha}$. Observe that for $z,w\in D$, by Theorem 5.1, when $|z-w|>3m^{-1/\alpha}$,

$$G^m_{H_1}(z,w)\le c_{17}\,m^{(2-\alpha)/\alpha}\,\ln\left(1+\frac{\left(m^{1/\alpha}\delta_D(z)+\left(m^{1/\alpha}\delta_D(z)\right)^{\alpha/2}\right)\left(m^{1/\alpha}\delta_D(w)+\left(m^{1/\alpha}\delta_D(w)\right)^{\alpha/2}\right)}{\left(m^{1/\alpha}|z-w|\right)^2}\right).$$

Hence by Theorem 5.1

$$\begin{aligned}
&\int_{D\times D}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|z-x|^{2+\alpha}}\right)G^m_{H_1}(z,w)\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{2+\alpha}}\right)dzdw\\
&\leq\ I+II+III, \qquad (5.10)
\end{aligned}$$

where

$$\begin{aligned}
I\ &\leq\ c_{18}\int_{D\times D}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|z-x|^{2+\alpha}}\right)\frac{1}{|z-w|^{2-\alpha}}\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{2+\alpha}}\right)dzdw\\
&\leq\ \frac{c}{|x-y|^{2-\alpha}} \qquad (5.11)
\end{aligned}$$

by the same argument as that for (5.9),

$$\begin{aligned}
II\ &=\ c_{17}\int_{D\times D}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|z-x|^{2+\alpha}}\right)m^{(2-\alpha)/\alpha}\,1_{\{m^{1/\alpha}|z-w|>3\}}\\
&\qquad\cdot\ln\left(1+\frac{\left(m^{1/\alpha}\delta_D(z)+\left(m^{1/\alpha}\delta_D(z)\right)^{\alpha/2}\right)\left(m^{1/\alpha}\delta_D(w)+\left(m^{1/\alpha}\delta_D(w)\right)^{\alpha/2}\right)}{\left(m^{1/\alpha}|z-w|\right)^2}\right)\\
&\qquad\cdot\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{2+\alpha}}\right)dzdw, \qquad (5.12)
\end{aligned}$$

while

$$\begin{aligned}
III\ &\leq\ c_{19}\int_{D\times D}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|z-x|^{2+\alpha}}\right)m^{(2-\alpha)/\alpha}\,1_{\{m^{1/\alpha}|z-w|\leq 3\}}\\
&\qquad\cdot\ln(1+m^{1/\alpha}(\delta_D(z)\wedge\delta_D(w)))\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{2+\alpha}}\right)dzdw. \qquad (5.13)
\end{aligned}$$

As $m\in(0,1]$, $\delta_D(x)\leq 1$ and $m^{1/\alpha}|x-y|\leq 3$, we have $m^{1/\alpha}\delta_D(x)\leq 1$ and

$$m^{1/\alpha}\delta_D(y)\leq m^{1/\alpha}\delta_D(x)+m^{1/\alpha}|x-y|\leq 4.$$

Thus we have

$$\begin{aligned}
II\ &\leq\ c_{20}\,m^{(2-\alpha)/\alpha}\int_{D\times D}\left(1\wedge\frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|x-z|^{2+\alpha}}\right)1_{\{m^{1/\alpha}|z-w|>3\}}\\
&\qquad\cdot(\ln(1+|x-z|)+\ln(1+|y-w|))\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{2+\alpha}}\right)dzdw\\
&\leq\ c_{21}\,m^{(2-\alpha)/\alpha}\left(\sup_{x\in D}\int_D\left(1\wedge\frac{1}{|x-z|^{2+\alpha}}\right)\ln(1+|x-z|)dz\right)\left(\sup_{y\in D}\int_D\left(1\wedge\frac{1}{|w-y|^{2+\alpha}}\right)dw\right)\\
&\leq\ c_{22}\,m^{(2-\alpha)/\alpha}\leq\frac{c_{23}}{|x-y|^{2-\alpha}}. \qquad (5.14)
\end{aligned}$$

and

$$\begin{aligned} III &\le c_{24}\, m^{(2-\alpha)/\alpha} \int_{D\times D} \left(1\wedge \frac{\phi(m^{1/\alpha}|x-z|/C_2)}{|z-x|^{2+\alpha}}\right) 1_{\{m^{1/\alpha}|z-w|\le 3\}} \\ &\qquad \cdot \ln\left(1+\left(1+m^{1/\alpha}|x-z|\right)\wedge\left(1+m^{1/\alpha}|w-y|\right)\right)\\ &\qquad\cdot\left(1\wedge\frac{\phi(m^{1/\alpha}|w-y|/C_2)}{|w-y|^{2+\alpha}}\right) dzdw \\ &\le c_{25}\, m^{(2-\alpha)/\alpha}\left(\int_D \left(1\wedge \frac{1}{|x-z|^{2+\alpha}}\right)\ln(1+|x-z|)dz\right)\left(\int_D\left(1\wedge\frac{1}{|w-y|^{2+\alpha}}\right)dw\right)\\ &\le c_{26}\, m^{(2-\alpha)/\alpha} \le \frac{c_{27}}{|x-y|^{2-\alpha}}. \end{aligned} \tag{5.15}$$

Therefore by (5.7) and (5.10)-(5.15), we have

$$\int_1^\infty p_D^m(t,x,y)dt \le c_{27}\,\frac{(1\wedge\delta_D(x)^{\alpha/2})(1\wedge\delta_D(y)^{\alpha/2})}{|x-y|^{2-\alpha}} \le c_{28}\left(1\wedge\frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha}}\right)\frac{1}{|x-y|^{2-\alpha}}.$$

This together with (5.2) yields that

$$G_D^m(x,y)\asymp\left(1\wedge\frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|^{\alpha}}\right)\frac{1}{|x-y|^{2-\alpha}}.$$

Observe that in the case of $\delta_D(x)\le\delta_D(y)$, $\delta_D(x)\le 1$ and $|x-y|\le 3m^{-1/\alpha}$, we have

$$\begin{aligned} &\left(1\wedge\frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha/2}|x-y|^{\alpha-2} \ge c_{29}\,\delta_D(x)^{\alpha/2} \ge c_{30}\ln\left(1+(\delta_D(x)\wedge\delta_D(y)\right) \\ &\ge c_{30}m^{(2-\alpha)/\alpha}\ln\left(1+m^{1/\alpha}(\delta_D(x)\wedge\delta_D(y)\right). \end{aligned}$$

We conclude from this that $G_D^m(x,y)\asymp\widetilde{V}_D^{\alpha,m}(x,y)$. This completes the proof for the case $d\ge 2$.

As we mentioned in the beginning of this proof, the above argument can be modified to give a proof for the case $d=1$. For example, by following the arguments in parts (ii) and (iii) of the proof of [9, Corollary 1.2], our Theorem 1.1(i) gives that for $|x-y|\le 3m^{-1/\alpha}$,

$$\int_0^1 p_D^m(t,x,y)dt\asymp\begin{cases}\left(\delta_D(x)\delta_D(y)\right)^{(\alpha-1)/2}\wedge\frac{\delta_D(x)^{\alpha/2}\delta_D(y)^{\alpha/2}}{|x-y|} & \text{when } d=1<\alpha,\\ \log\left(1+\frac{\delta_D(x)^{1/2}\delta_D(y)^{1/2}}{|x-y|}\right) & \text{when } d=1=\alpha.\end{cases}\tag{5.16}$$

The remaining details are left to the interested reader. □

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