On suprema of Lévy processes and application in risk theory

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Abstract

Let $\widehat{X} = C - Y$ where Y is a general one-dimensional Lévy process and C an independent subordinator. Consider the times when a new supremum of $\widehat{X}$ is reached by a jump of the subordinator C. We give a necessary and sufficient condition in order for such times to be discrete. When this is the case and $\widehat{X}$ drifts to $-\infty$, we decompose the absolute supremum of $\widehat{X}$ at these times, and derive a Pollaczek-Hinchin-type formula for the distribution function of the supremum.

Soit Y un processus de Lévy réel quelconque et C un subordinateur indépendant de Y. On considère les temps en lesquels le processus $\widehat{X} = C - Y$ atteint un nouveau maximum par un saut de C. Nous donnons une condition nécessaire et suffisante pour que l'ensemble de ces temps soit discret. Lorsque tel est le cas et que le processus $\widehat{X}$ dérive vers $-\infty$, nous décomposons son maximum absolu en cette suite de temps. Nous déduisons alors de cette décomposition une formule du type Pollaczek-Hinchin pour la loi du maximum absolu de $\widehat{X}$.

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1 Introduction

Several papers in recent years ([4], [5], [6], [7]) formulated the insurance risk processes in a general Lévy setting, and addressed questions about ruin probabilities of such processes. The model proposed in [4] assumed that the insurance risk process $X = (X(t) : t \geq 0)$ is equal to $X(t) = ct + Z(t) - C(t)$, where $c > 0$ is the premium rate, $C = (C(t) : t \geq 0)$ is the claim process modeled by a subordinator with finite mean $\mathbb{E}C(1) < c$, and $Z = (Z(t) : t \geq 0)$ is a general spectrally negative Lévy process with mean zero serving as a perturbation. Relying on the fluctuation theory and the explicit formula for the infimum of a spectrally negative Lévy process, the authors derived the following Pollaczek-Hinchin-type formula for the survival probability with the initial capital $x \geq 0$:

$$\theta(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n (G^{(n+1)*} * H^{n*})(x) . \tag{1.1}$$

Here $\rho = \mathbb{E}C(1)/c$, G is the distribution function of the absolute infimum of $-ct - Y(t)$, and H is the integrated tail of the Lévy measure of the subordinator. The following explanation of formula (1.1) is given: Consider the dual process $\widehat{X} = -X$, and let $S^{\widehat{X}} = \sup_{0 \leq t < \infty} \widehat{X}(t)$. Then $\theta(x) = \mathbb{P}(S^{\widehat{X}} \leq x)$. By considering times when a new supremum of $\widehat{X}$ is reached by a jump of the subordinator C, $S^{\widehat{X}}$ may be written as a geometric sum of two types of i.i.d. random variables: one type with the distribution of the supremum of $\widehat{X}$ before the first such time, and the second type with the distribution of the overshoot. This leads to the formula (1.1) for $\theta(x)$. A non-obvious fact is that the times when a new supremum of $\widehat{X}$ is reached by a jump of the subordinator C are indeed discrete (which makes the above mentioned decomposition possible). For the spectrally negative process X (i.e., spectrally positive $\widehat{X}$), this fact was proved by using that the distribution of the supremum of X at an independent exponential time has exponential distribution.

In this paper we consider a general Lévy process Y, an independent subordinator C, and define $X = Y - C$. We consider the times when a new supremum of $\widehat{X} = -X$ is reached by a jump of the subordinator C. Our main result gives a necessary and sufficient condition for such times to be discrete. The precise setting and the result are described in the next section. Section 3 contains proofs, corollaries and examples. Finally, in Section 4, following closely the decomposition arguments from [4], we sketch a proof of a Pollaczek-Hinchin-type formula (1.1) for the distribution of the supremum of the process $\hat{X}$ in the case when X drifts to $+\infty$ and the times when a new supremum of $\widehat{X} = -X$ is reached by a jump of the subordinator C are discrete.

2 Setting and the main result

Let $Y = (Y(t) : t \geq 0)$ be a one dimensional Lévy process with characteristic exponent

$$\Psi(\lambda) = ib\lambda + \frac{1}{2}a^2\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} - i\lambda x 1_{|x|\leq 1})\,\Pi(dx)\,,$$

where $b \in \mathbb{R}$, $a \geq 0$, and Π is a measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}}(x^2 \wedge 1)\,\Pi(dx) < \infty$ called the Lévy measure of Y. Let $C = (C(t) : t \geq 0)$ be a subordinator independent of Y with the Laplace exponent

$$\Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda x})\,\nu(dx)\,.$$

Here ν is a measure on $(0, \infty)$ satisfying $\int_{(0,\infty)}(x \wedge 1)\,\nu(dx) < \infty$, called the Lévy measure of C. We assume that both Y and C have right continuous paths with left limits. Let $\Delta C(t) = C(t) - C(t-)$ be the jump of C at time t. Then $(\Delta C(t) : t \geq 0)$ is a Poisson point process with characteristic measure ν.

For $t \geq 0$ let $\mathcal{F}^0(t) = \sigma(Y_s, C_s : 0 \leq s \leq t)$. The filtration that we are going to work with is $\mathcal{F} = (\mathcal{F}(t) : t \geq 0)$, the usual augmentation of $\mathcal{F}^0(t)$.

Let $X(t) := Y(t) - C(t)$, $t \geq 0$. Then $X = (X(t) : t \geq 0)$ is a Lévy process with respect to the filtration $\mathcal{F}$. Further, let $\widehat{Y}$, respectively $\widehat{X}$, denote the dual processes of Y, respectively X. Thus, $\widehat{Y}(t) = -Y(t)$ and $\widehat{X}(t) = -X(t) = C(t) - Y(t)$. The supremum process of $\widehat{X}$, respectively $\widehat{Y}$ will be denoted by $S^{\widehat{X}}$, respectively $S^{\widehat{Y}}$. Thus

$$S^{\widehat{X}}(t) = \sup_{0\leq s\leq t} \widehat{X}(s)\,, \qquad S^{\widehat{Y}}(t) = \sup_{0\leq s\leq t} \widehat{Y}(s)\,.$$

Similarly, let S^X (respectively S^Y) denote the supremum process of X (respectively Y).

We will be interested in times when a new supremum of $\widehat{X}$ is attained by a jump of the subordinator C. To be more precise, define

$$\sigma = \inf\{t > 0 : \Delta C(t) > S^{\widehat{X}}(t-) - \widehat{X}(t-)\}\,.$$

Clearly, σ is an $\mathcal{F}$-stopping time, and by the Blumenthal 0-1 law, $\mathbb{P}(\sigma = 0) = 0$ or 1. If $\sigma > 0$ a.s., we think of σ as the first time when a new supremum of $\widehat{X}$ is attained by a jump of the subordinator C. If $\sigma = 0$ a.s., then the number of times a new supremum of $\widehat{X}$ is attained by a jump of the subordinator C is infinite and time 0 is an accumulation point of such times.

The goal of the paper is to give a necessary and sufficient condition for $\mathbb{P}(\sigma > 0) = 1$. It is clear that if C is a compound Poisson process, then $\sigma > 0$ a.s. Therefore, from now on we assume that $\nu((0, \infty)) = \infty$, i.e, C is not compound Poisson.

Let $\tau_0 = \inf\{t > 0 : X(t) > 0\}$ be the first passage time of X over the level 0. Recall that 0 is said to be regular for $(0,\infty)$ (with respect to the process X), if $\mathbb{P}(\tau_0 = 0) = 1$, and irregular otherwise. Assume that 0 is regular for $(0,\infty)$ with respect to X. Let $L^X = (L^X(t) : t \geq 0)$ be the local time of the reflected process $S^X - X$ at 0, and denote by $T^X = (T^X(t) : t \geq 0)$ the inverse local time, $T^X(t) = \inf\{s \geq 0 : L^X(s) > t\}$. Let $H^X = (H^X(t) : t \geq 0)$ be the ladder height process of X defined by $H^X(t) = S^X(T^X(t)) = X(T^X(t))$. By the regularity of 0 for $(0,\infty)$, H^X is not a compound Poisson process. The two-dimensional process (T^X, H^X) is a bivariate (possibly killed) subordinator. We denote its bivariate Laplace exponent by $\kappa^X(\cdot,\cdot)$. The renewal function of H^X is defined by

$$\mathcal{V}^X(x) = \mathbb{E}\int_0^\infty 1_{H^X(t)\leq x}\,dt = \mathbb{E}\int_0^\infty 1_{S^X(t)\leq x}\,dL^X(t)\,, \quad x \geq 0\,.$$

Note that 0 is regular for $(0,\infty)$ also with respect to Y. Let L^Y, T^Y, H^Y, $\mathcal{V}^Y$ and κ^Y be respectively the local time of $S^Y - Y$ at 0, the inverse local time, the ladder height process of Y, the renewal function of Y, and the bivariate Laplace exponent of (T^Y, H^Y). Here is our main result:

Theorem 2.1 *Assume that 0 is regular for $(0,\infty)$ with respect to X. Then the following are equivalent:*

(a) $\sigma > 0$ *a.s.,*

(b) $\int_0^1 \mathcal{V}^X(x)\,\nu(dx) < \infty$*,*

(c) $\int_0^1 \mathcal{V}^Y(x)\,\nu(dx) < \infty$*.*

Condition (b) is not satisfactory, because, except in some special cases, it is difficult to check. It serves only as an intermediary step towards condition (c) which is better since the processes Y and C are separated. It is easy to see that (b) implies (c), while the reverse implication relies on the following result from [2].

Proposition 2.2 *Let $Z = Z^+ - Z^-$ where Z^+ and Z^- are independent subordinators with no drift. Let μ^+ and μ^- be their respective Lévy measures, and $\mathcal{V}^+$ and $\mathcal{V}^-$ their respective renewal functions. Then*

$$\lim_{t\downarrow 0}\frac{Z^+(t)}{Z^-(t)} = 0 \quad a.s$$

if and only if

$$\int_0^1 \mathcal{V}^-(x)\,\mu^+(dx) < \infty\,.$$

A consequence of condition (c) is that $\lim_{t\downarrow 0} C(t)/S^Y(t) = 0$ a.s. (See the proof of Theorem 2.1 given in Section 3 below.) By Proposition 3.5 below this implies that $\mathcal{V}^X(x)$ and $\mathcal{V}^Y(x)$ are comparable for small x. We show by an example that $\lim_{t\downarrow 0} C(t)/S^Y(t) = 0$ a.s. is not, in general, equivalent to the conditions in Theorem 2.1.

We end this section by discussing briefly the case when 0 is irregular for $(0,\infty)$ with respect to X. This is possible only if X, and thus consequently Y, is of bounded variation. Let $Y = Y^1 - Y^2$ be the difference of two subordinators. Then $\widehat{X} = (C + Y^2) - Y^1$. Since 0 is irregular for $(-\infty, 0)$ with respect to $\widehat{X}$, the process $\widehat{X}$ stays above zero for a positive amount of time. Upward jumps come from independent subordinators C and Y^2, hence at different times. This suggests that times at which new a supremum of $\widehat{X}$ is reached by a jump of C accumulate at zero. Indeed, in Proposition 3.4 we prove that in this case $\sigma = 0$ a.s.

3 Proofs, corollaries and examples

At this point we do not make any assumptions about regularity of 0 for $(0,\infty)$. The following lemma was essentially proved in [4].

Lemma 3.1 *Let* $\mathbf{e}_q$ *be an exponential time with parameter* $q \in (0,\infty)$ *independent of* Y *and* C. *Then*

$$\mathbb{E}\left(\sum_{0<t\le \mathbf{e}_q} 1_{\Delta C(t) > S^{\widehat{X}}(t-) - \widehat{X}(t-)}\right) = \frac{1}{q}\int_0^\infty \mathbb{P}(S^X(\mathbf{e}_q) \le x)\,\nu(dx)\,.$$

Proof. By using the compensation formula we get

$$\begin{aligned}
&\mathbb{E}\left(\sum_{0<t\le \mathbf{e}_q} 1_{\Delta C(t) > S^{\widehat{X}}(t-) - \widehat{X}(t-)}\right)\\
&= \mathbb{E}\left(\int_0^{\mathbf{e}_q}\int_{(0,\infty)} 1_{(S^{\widehat{X}}(t-) - \widehat{X}(t-),\infty)}(\epsilon)\,\nu(d\epsilon)\,dt\right)\\
&= \mathbb{E}\left(\int_0^{\mathbf{e}_q} \nu((S^{\widehat{X}}(t) - \widehat{X}(t),\infty))\,dt\right)\\
&= \mathbb{E}\left(\int_0^{\infty} qe^{-qs}\int_0^s \nu((S^{\widehat{X}}(t) - \widehat{X}(t),\infty)\,dt\,ds\right)\\
&= \mathbb{E}\left(\int_0^{\infty} e^{-qt}\nu((S^{\widehat{X}}(t) - \widehat{X}(t),\infty))\,dt\right).
\end{aligned}$$

Now by using the fact that $S^{\widehat{X}}(t)-\widehat{X}(t)\stackrel{d}{=}S(t)$, we arrive at

$$\begin{aligned}
&\mathbb{E}\left(\sum_{0<t\le \mathbf{e}_q} 1_{\Delta C(t)>S^{\widehat{X}}(t-)-\widehat{X}(t-)}\right)\\
&= \int_0^\infty e^{-qt}\mathbb{E}(\nu(S^X(t),\infty))\,dt\\
&= \frac{1}{q}\mathbb{E}(\nu(S^X(\mathbf{e}_q),\infty))\\
&= \frac{1}{q}\mathbb{E}\int_0^\infty 1_{S^X(\mathbf{e}_q)\le x}\,\nu(dx)\\
&= \frac{1}{q}\int_0^\infty \mathbb{P}(S^X(\mathbf{e}_q)\le x)\,\nu(dx).
\end{aligned}$$

□

Lemma 3.2 *It holds that $\sigma>0$ a.s. if and only if*

$$\int_0^\infty \mathbb{P}(S^X(\mathbf{e}_q)\le x)\,\nu(dx)<\infty\,. \tag{3.1}$$

Proof. For $t\ge 0$, let $A(t)=\{\Delta C(t)>S^{\widehat{X}}(t-)-\widehat{X}(t-)\}$. Assume that $\int_0^\infty \mathbb{P}(S^X(\mathbf{e}_q)\le x)\,\nu(dx)<\infty$. Then by Lemma 3.1, $\mathbb{E}\sum_{0<t\le\mathbf{e}_q}1_{A(t)}<\infty$ and therefore $\sum_{0<t\le\mathbf{e}_q}1_{A(t)}<\infty$ a.s. This proves that $\sigma>0$ a.s. Conversely, if $\sigma>0$ a.s., define $\sigma_1=\sigma$, and inductively $\sigma_n=\inf\{t>\sigma_{n-1}:\Delta C(t)>S^{\widehat{X}}(t-)-\widehat{X}(t-)\}$ for $n\ge 2$. Let $p=\mathbb{P}(\sigma<\mathbf{e}_q)$. By the memoryless property of $\mathbf{e}_q$ and the strong Markov property of X, we have $\mathbb{P}(\sigma_n<\mathbf{e}_q)=p^n$, $n\ge 1$. Therefore,

$$\mathbb{E}\sum_{0<t\le\mathbf{e}_q}1_{A(t)}=\mathbb{E}\sum 1_{\sigma_n<\mathbf{e}_q}=\sum_n p^n<\infty\,.$$

□

Remark 3.3 *We record here the fact that $\mathbb{P}(\sigma=\infty)<1$. Indeed, if $\sigma=\infty$ a.s., then for any $q>0$, $\sum_{0<t\le\mathbf{e}_q}1_{\Delta C(t)>S^{\widehat{X}}(t-)-\widehat{X}(t-)}=0$ a.s., implying by Lemma 3.1 that $\int_0^\infty \mathbb{P}(S^X(\mathbf{e}_q)\le x)\,\nu(dx)=0$. On the other hand, since $\lim_{t\downarrow 0}S^X(t)=0$ by right continuity, we have $\mathbb{P}(S^X(\mathbf{e}_q)\le x)=\int_0^\infty qe^{-qt}\mathbb{P}(S^X(t)\le x)dt>0$ for every $x>0$, hence $\int_0^\infty \mathbb{P}(S^X(\mathbf{e}_q)\le x)\,\nu(dx)>0$.*

Proposition 3.4 *Assume that 0 is irregular for $(0,\infty)$ with respect to X. Then $\sigma=0$ a.s.*

Proof. For every $x \geq 0$,

$$\mathbb{P}(S^X(\mathbf{e}_q) \leq x) \geq \mathbb{P}(S^X(\mathbf{e}_q) \leq x, \mathbf{e}_q < \tau_0) = \mathbb{P}(\mathbf{e}_q < \tau_0) > 0\,,$$

since $S^X(t) = 0$ for $t < \tau_0$. Therefore,

$$\int_0^\infty \mathbb{P}(S^X(\mathbf{e}_q) \leq x)\,\nu(dx) \geq \int_0^\infty \mathbb{P}(\mathbf{e}_q < \tau_0)\,\nu(dx) = \infty\,,$$

and the claim follows from the previous lemma. □

¿From now on we assume that 0 is regular for $(0,\infty)$ for X. We want to express the condition (3.1) in terms of the renewal function for the ladder height process of X. By Lemma 21 on p.177 of [1], there exist constants $c_1 > 0$ and $x_1 > 0$ such that

$$c_1 \mathcal{V}^X(x) \leq \mathbb{P}(S^X(\mathbf{e}_q) \leq x) \leq c_1^{-1} \mathcal{V}^X(x) \tag{3.2}$$

for all $x \leq x_1$.

Proof of Theorem 2.1: (a) ⇔ (b): Immediately from (3.2) and Lemma 3.2. □

Note that similarly as for (3.2), there exist constants $c_2 > 0$ and $x_2 > 0$ such that

$$c_2 \mathcal{V}^Y(x) \leq \mathbb{P}(S^Y(\mathbf{e}_q) \leq x) \leq c_2^{-1} \mathcal{V}^Y(x) \tag{3.3}$$

for all $x \leq x_2$. Since clearly $\mathbb{P}(S^Y(\mathbf{e}_q) \leq x) \leq \mathbb{P}(S^X(\mathbf{e}_q) \leq x)$, (3.2) and (3.3) imply that

$$\mathcal{V}^Y(x) \leq c_3 \mathcal{V}^X(x)\,, \quad 0 \leq x \leq x_3\,, \tag{3.4}$$

where $c_3 = c_1^{-1} c_2^{-1}$ and $x_3 = x_1 \wedge x_2$. We would like to find a condition on Y and C which ensures the validity of the reverse inequality.

Let M^Y (respectively M^X) denote the random set of times when Y (respectively X) attains its maximum. Suppose that $t \in M^X$. Then $X(s) \leq X(t)$ for all $s \leq t$. This reads $Y(s) - C(s) \leq Y(t) - C(t)$ for all $s \leq t$. Since C is increasing we obtain that $Y(s) \leq Y(t)$ for all $s \leq t$. Therefore, $M^X \subset M^Y$. From this and the strong Markov property of $S^X - X$ we conclude that M^X is regeneratively embedded in M^Y (see [3]). In particular, there exists a constant $k > 0$ such that

$$1_{t \in M^X}\, dL^Y(t) = k\, dL^X(t)\,. \tag{3.5}$$

Proposition 3.5 *Assume that* $\lim_{t\downarrow 0} C(t)/S^Y(t) = 0$ *a.s. Then there exist constants* $c_4 > 0$ *and* $x_4 > 0$ *such that*

$$\mathcal{V}^X(x) \leq c_4 \mathcal{V}^Y(x)\,, \quad 0 \leq x \leq x_4\,, \tag{3.6}$$

Proof. Let $\tau = \inf\{t > 0 : S^X(t) < S^Y(t)/2\}$. Then τ is an $\mathcal{F}$-stopping time. We claim that $\tau > 0$ a.s. If not, then for each ω in a set of positive probability, there exists a decreasing sequence $(t_n : n \geq 1)$ (depending on ω) such that $t_n \to 0$, $t_n \in M^X$ and $S^X(t_n) < S^Y(t_n)/2$. Since $t_n \in M^X \subset M^Y$, we have that $Y(t_n)/2 > X(t_n) = Y(t_n) - C(t_n)$. Hence, $S^Y(t_n)/2 = Y(t_n)/2 < C(t_n)$ implying that $\limsup_n C(t_n)/S^Y(t_n) \geq 1/2$ which contradicts the assumption of the proposition. Therefore, $\tau > 0$ a.s. Hence

$$S^X(t) \geq \frac{1}{2} S^Y(t) , \quad 0 \leq t < \tau . \tag{3.7}$$

¿From now on we follow the idea of the proof of Proposition 21, p.177, of [1]. Let $F^Y(x) = \mathbb{P}(S^Y(\mathbf{e}_q) \leq x)$ denote the distribution function of $S^Y(\mathbf{e}_q)$. By [1], p.177,

$$F^Y(x) = \kappa^Y(q,0) \mathbb{E} \int_0^\infty 1_{S^Y(t) \leq x} \, e^{-qt} \, dL^Y(t) . \tag{3.8}$$

Then

$$\begin{aligned} F^Y(x) &\geq \kappa^Y(q,0) \, \mathbb{E} \int_0^{\tau \wedge 1} 1_{S^Y(t) \leq x} \, e^{-qt} \, dL^Y(t) \\ &\geq \kappa^Y(q,0) \, e^{-q} \mathbb{E} \int_0^{\tau \wedge 1} 1_{S^X(t) \leq x/2} \, dL^Y(t) \\ &\geq \kappa^Y(q,0) \, e^{-q} k \, \mathbb{E} \int_0^{\tau \wedge 1} 1_{S^X(t) \leq x/2} \, dL^X(t), \end{aligned}$$

where the last inequality follows from (3.5). By use of the previous display we get

$$\begin{aligned} \mathcal{V}^X(x/2) &= \mathbb{E} \int_0^\infty 1_{S^X(t) \leq x/2} \, dL^X(t) \\ &= \mathbb{E} \int_0^{\tau \wedge 1} 1_{S^X(t) \leq x/2} \, dL^X(t) + \mathbb{E} \int_{\tau \wedge 1}^\infty 1_{S^X(t) \leq x/2} \, dL^X(t) \\ &\leq \frac{e^q}{k \kappa^Y(q,0)} \, F^Y(x) + \mathbb{E} \int_{\tau \wedge 1}^\infty 1_{S^X(t) \leq x/2} \, dL^X(t) \\ &\leq \frac{e^q}{k \kappa^Y(q,0)} \, F^Y(x) + \mathbb{P}(S^X(\tau \wedge 1) \leq x/2) \, \mathcal{V}^X(x/2) , \end{aligned}$$

where the last line follows by the strong Markov property. This implies that

$$\mathcal{V}^X(x/2) \leq \frac{e^q}{k \kappa^Y(q,0)} \, \frac{1}{\mathbb{P}(S^X(\tau \wedge 1) > x/2)} \, F^Y(x) .$$

Let $x_5 > 0$ be such that for all $x \in [0, x_5]$, $\mathbb{P}(S^X(\tau \wedge 1) > x/2) > 1/2$. Then

$$\mathcal{V}^X(x/2) \leq c_5 F^Y(x) \quad 0 \leq x \leq x_5 , \tag{3.9}$$

with $c_5 = 2e^q/\kappa^Y(q,0)$. By the subadditivity of $\mathcal{V}^X$ we have $\mathcal{V}^X(x) \le 2\,\mathcal{V}^X(x/2)$. Now (3.9) and (3.4) imply (3.6) with $c_4 = c_5 c_2^{-1}$ and $x_4 = x_2 \wedge x_5$. □

Proof of Theorem 2.1 (b) ⇔ (c): By (3.4), (b) implies (c). Assume that (c) holds. Recall that $\mathcal{V}^Y$ denotes the renewal function of the ladder height process $H^Y = S^Y(T^Y)$. By Proposition 2.2, the finiteness of the integral in (c) implies that $\limsup_{t\downarrow 0} C(t)/H^Y(t) = 0$ a.s. We discuss separately the case when 0 is regular for $(-\infty,0)$ with respect to Y and the case when 0 is irregular for $(-\infty,0)$ with respect to Y.

In the first case, 0 is regular for both $(0,\infty)$ and $(-\infty,0)$ with respect to Y. This implies that the inverse local time T^Y does not have a drift (see, e.g., [8], Corollary 6.11, p.153). Therefore, $\lim_{t\downarrow 0} T^Y(t)/t = 0$ a.s., implying that $\limsup_{t\downarrow 0} S^Y(T^Y(t))/S^Y(t) < \infty$ a.s.

In the case when 0 is irregular for $(-\infty,0)$ with respect to Y, the inverse local time T^Y has a drift (e.g., [8], Corollary 6.11, p.153). By appropriately choosing the normalization constant in the definition of the local time L^Y, we could make the drift d of the inverse local time T^Y less than 1. Therefore, $\lim_{t\downarrow 0} T^Y(t)/t = d < 1$, a.s., implying again that $\limsup_{t\downarrow 0} S^Y(T^Y(t))/S^Y(t) < \infty$ a.s.

Hence, in both cases we have

$$\limsup_{t\downarrow 0} \frac{C(t)}{S^Y(t)} = \limsup_{t\downarrow 0} \frac{C(t)}{S^Y(T^Y(t))} \frac{S^Y(T^Y(t))}{S^Y(t)} = 0 \quad \text{a.s.}$$

By Proposition 3.5, $\mathcal{V}^X(x) \le c_4 \mathcal{V}^Y(x)$ for small x, implying that the integral in (b) is finite. □

Corollary 3.6 *Assume that Y is of unbounded variation. Then $\sigma > 0$ a.s. if and only if $\int_0^1 \mathcal{V}^Y(x)\,\nu(dx) < \infty$.*

Example 3.7 *Let Y be a strictly α-stable process, $\alpha \in (0,2)$, such that $|Y|$ is not a subordinator, and let $\varrho = \mathbb{P}(Y(t) \ge 0)$ be the positivity parameter. Then $\varrho \in (0,1)$ for $0 < \alpha \le 1$, and $\varrho \in [1-1/\alpha, 1/\alpha]$ for $1 < \alpha < 2$. The ladder height process H^Y is an $\alpha\varrho$-stable subordinator, hence $\mathcal{V}^Y$ is proportional to $x^{\alpha\varrho}$. Assume that C is a β-stable subordinator. Its Lévy measure has a density proportional to $x^{-1-\beta}$. Hence $\int_0^1 \mathcal{V}^Y(x)\,\nu(dx) < \infty$ if and only if $\int_0^1 x^{\alpha\varrho} x^{-\beta-1}\,dx < \infty$. This implies that $\sigma > 0$ a.s. if and only if $\alpha\varrho > \beta$.*

By the proof of Theorem 2.1 we know that $\int_0^1 \mathcal{V}^Y(x)\,\nu(dx) < \infty$ implies that $\lim_{t\downarrow 0} C(t)/S^Y(t) = 0$ a.s. We now give examples which show that in general the converse is not true neither in the unbounded variation case nor in the bounded variation case (but see Corollary 3.9 below).

Let $1 < \alpha < 2$. *Then by [1], Theorem 6, p. 224., we have that* $\liminf_{t\downarrow 0} S^Y(t)/t = +\infty$ *a.s. Since* $\lim_{t\downarrow 0} C(t)/t = 0$ *a.s., it follows that* $\lim_{t\downarrow 0} C(t)/S^Y(t) = 0$ *a.s. regardless whether* $\alpha\varrho > \beta$ *or not.*

Similarly, when $0 < \alpha \le 1$, *choose* $\beta < \alpha$ *and* $\gamma > 0$ *such that* $1/\alpha + \gamma < 1/\beta$. *By the same result from [1] mentioned above, it follows that* $\liminf_{t\downarrow 0} S^Y(t)/t^{1/\alpha+\gamma} = +\infty$ *a.s. Further,* $\int_0^1 t^{-\beta(1/\alpha+\gamma)}\, dt < \infty$, *which by [1], Proposition 10, p. 87, implies that* $\lim_{t\downarrow 0} C(t)/t^{1/\alpha+\gamma} = 0$ *a.s. Therefore,* $\lim_{t\downarrow 0} C(t)/S^Y(t) = 0$ *a.s. If, further,* $\alpha\varrho \le \beta$, *then* $\int_0^1 \mathcal{V}^Y(x)\, \nu(dx) = \infty$.

Let us denote by μ^Y the Lévy measure of the ladder height process H^Y. It is known (see [2]) that condition (c) from Theorem 2.1 is equivalent to

$$\int_0^1 \frac{x\, \nu(dx)}{\int_0^x \mu^Y(y,\infty)\, dy} < \infty\,.$$

On the other hand, in view of the previous example it is unlikely that condition (c) can be expressed in terms of the tails of Lévy measures ν and Π of given processes C and Y (as is the case for conditions for creeping [9], or regularity of the half line for bounded variation process [2]).

Example 3.8 *Assume that* Y *a Lévy process of unbounded variation that creeps upwards (i.e., with positive probability it crosses every level from below continuously). Then there exists a constant* $c > 0$ *such that* $\mathcal{V}^Y(x) \le cx$ *(see, e.g., [1], Theorem 19, pp.174-175). Therefore,* $\int_0^1 \mathcal{V}^Y(x)\, \nu(dx) < \infty$ *for any subordinator* C, *implying* $\sigma > 0$ *a.s. By noting that every spectrally negative process creeps upwards, we obtain an extension of Theorem 4.1 from [4].*

When Y is of bounded variation with no drift and with 0 being irregular for $(-\infty, 0)$ with respect to Y, we will give a more explicit condition equivalent to those in Theorem 2.1.

Assume now that Y is of bounded variation with no drift. Then $Y = Y^1 - Y^2$ where Y^1 and Y^2 are pure jump subordinators. Let ν^i, respectively $\mathcal{V}^i$, denote the Lévy measure, respectively the renewal function, of Y^i, $i = 1, 2$. If 0 is irregular for $(0,\infty)$ with respect to X, then $\sigma = 0$ a.s. by Proposition 3.4. So we assume that 0 is regular for $(0,\infty)$ with respect to X, implying that 0 is regular for $(0,\infty)$ with respect to Y.

Corollary 3.9 *Assume that* Y *is of bounded variation with no drift, that 0 is regular for* $(0,\infty)$ *with respect to* X, *and that 0 is irregular for* $(-\infty, 0)$ *with respect to* Y. *Then the following are equivalent:*

(a) $\sigma > 0$ *a.s.*

(b) $\int_0^1 \mathcal{V}^X(x)\,\nu(dx) < \infty$,

(c) $\int_0^1 \mathcal{V}^Y(x)\,\nu(dx) < \infty$,

(d) $\int_0^1 \mathcal{V}^1(x)\,\nu(dx) < \infty$.

(e) $\lim_{t\downarrow 0} C(t)/S^Y(t) = 0$ *a.s.*

Proof. Equivalence of (a), (b) and (c) follows from Theorem 2.1. Consider $\widehat{X} = C - Y = C + Y^2 - Y^1$. Since 0 is irregular for $(0,\infty)$ with respect to $-Y$, we have that $\sigma > 0$ a.s. if and only if 0 is irregular for $(0,\infty)$ with respect to $\widehat{X}$. By Proposition 2.2, this last condition is equivalent to $\int_0^1 \mathcal{V}^1(x)(\nu(dx) + \nu^2(dx)) < \infty$. The assumption that 0 is irregular for $(0,\infty)$ with respect to $-Y$ implies that $\int_0^1 \mathcal{V}^1(x)\,\nu^2(dx) < \infty$. Hence, $\sigma > 0$ a.s. if and only if $\int_0^1 \mathcal{V}^1(x)\,\nu(dx) < \infty$, proving the equivalence of (a) and (d). That (c) implies (e) was shown in the proof of Theorem 2.1. Finally, assume that (e) holds true. Since $Y \leq Y^1$, we have $S^Y \leq Y^1$, and therefore $\lim_{t\downarrow 0} C(t)/Y^1(t) = 0$ a.s. By Proposition 2.2, this is equivalent to $\int_0^1 \mathcal{V}^1(x)\,\nu(dx) < \infty$, hence (d) holds. □

Finally, we discuss the case when Y is of bounded variation with a drift d: $Y(t) = Y^1(t) - Y^2(t) + dt$. Recall that $\lim_{t\downarrow 0} Y(t)/t = d$ a.s. If $d < 0$, then the point 0 is irregular for $(0,\infty)$ with respect to the process X, hence $\sigma = 0$ a.s. by Lemma 3.4. Assume that $d > 0$. Note that $-Y^2(t) + dt$ is a spectrally negative process, hence its renewal function $\mathcal{V}^{2,d}(x) = cx$ for a positive constant c. Since $Y(t) \geq -Y^2(t) + dt$, we have that $\mathcal{V}^Y(x) \leq cx$ (with a possibly different constant $c > 0$). Hence, $\int_0^1 \mathcal{V}^Y(x)\,\nu(dx) < \infty$. This proves the following

Corollary 3.10 *Assume that Y is of bounded variation with a non-zero drift. When the drift is negative we have $\sigma = 0$ a.s. and when the drift is positive we have $\sigma > 0$ a.s.*

We also note that in the case when Y is of bounded variation with a positive drift it always holds that $\lim_{t\downarrow 0} C(t)/S^Y(t) = 0$ a.s.

4 Decomposition of the supremum

In this section we assume that $\lim_{t\to\infty} X(t) = \infty$ a.s., i.e., that X drifts to infinity. This implies that $S^{\widehat{X}}(+\infty) = \sup_{t\geq 0} \widehat{X}(t) < \infty$ a.s. Moreover, we also assume that $\mathbb{P}(\sigma > 0) = 1$. By Theorem 2.1, this is equivalent to $\int_0^1 \mathcal{V}^X(x)\,\nu(dx) < \infty$.

Lemma 4.1 *It holds that* $\mathbb{P}(\sigma < \infty) < 1$.

Proof. Assume, on the contrary, that $\mathbb{P}(\sigma < \infty) = 1$. Recall the notation $\sigma_1 = \sigma$ and $\sigma_n = \inf\{t > \sigma_{n-1} : \Delta C(t) > S^{\widehat{X}}(t-) - \widehat{X}(t-)\}$ for $n \geq 2$. By the strong Markov property, $\mathbb{P}(\sigma_n < \infty) = 1$. For $n \geq 1$, let $J_n = S^{\widehat{X}}(\sigma_n) - S^{\widehat{X}}(\sigma_n -)$. Again by the strong Markov property, $(J_n : n \geq 1)$ is a sequence of i.i.d. strictly positive random variables. Therefore, $\sum_{n=1}^{\infty} J_n = \infty$ a.s. Since $S^{\widehat{X}}(+\infty) \geq \sum_{n=1}^{\infty} J_n$, this clearly contradicts the assumption that $S^{\widehat{X}}(+\infty) = \sup_{t\geq 0} \widehat{X}(t) < \infty$ a.s. □

For $y > 0$, let $\widehat{\tau}_y := \inf\{t > 0 : \widehat{X}(t) > y\}$ be the entrance time of $\widehat{X}$ in (y, ∞), and, similarly, $\tau_y := \inf\{t > 0 : X(t) > y\}$. Note that $S^{\widehat{X}}(t-) \leq y$ if and only if $t \leq \widehat{\tau}_y$. We need the expected occupation time formula for the reflected process $S^{\widehat{X}} - \widehat{X}$ before $\sigma \wedge \widehat{\tau}_y$.

Proposition 4.2 *There exists a constant $k > 0$ such that for $x > 0$ and $y > 0$ the following formula is valid:*

$$\mathbb{E} \int_0^{\sigma\wedge\widehat{\tau}_y} 1_{(S^{\widehat{X}}(t)-\widehat{X}(t)\leq x)}\, dt = k\, \mathbb{P}(\sigma = \infty, \widehat{\tau}_y = \infty)\mathcal{V}^X(x)\,. \tag{4.1}$$

Proof. Let us compute the expected occupation time of $S^{\widehat{X}} - \widehat{X}$ below x:

$$\begin{aligned} \mathbb{E} \int_0^{\infty} 1_{(S^{\widehat{X}}(t)-\widehat{X}(t)\leq x)}\, dt &= \int_0^{\infty} \mathbb{P}(S^{\widehat{X}}(t) - \widehat{X}(t) \leq x)\, dt \\ &= \int_0^{\infty} \mathbb{P}(S^X(t) \leq x)\, dt \\ &= \mathbb{E}\tau_x = k\mathcal{V}^X(x)\,, \end{aligned} \tag{4.2}$$

for a positive constant $k > 0$. Here the last equality follows from Proposition 17.(ii) of [1] (see [1], p.172). The rest of the proof follows exactly as the proof of Proposition 4.3 from [4]. □

Let $J := (\Delta C(\sigma) - (S^{\widehat{X}}(\sigma-) - \widehat{X}(\sigma-)))1_{(\sigma<\infty)}$ be the overshoot at time σ. The next proposition is a preliminary version of the joint distribution of the vector $(S^{\widehat{X}}(\sigma-), J, S^{\widehat{X}}(\sigma-) - \widehat{X}(\sigma-))$ on $\{\sigma < \infty\}$. We omit the proof which can be found in [4].

Proposition 4.3 *For* $x, y, z > 0$

$$\begin{aligned} &\mathbb{P}(S^{\widehat{X}}(\sigma-) \leq y, J > x, S^{\widehat{X}}(\sigma-) - \widehat{X}(\sigma-) > z,\, \sigma < \infty) = \\ &\qquad = k\, \mathbb{P}(\sigma = \infty, \widehat{\tau}_y = \infty) \int_{x+z}^{\infty} \nu(u, \infty)\, \mathcal{V}^X(du)\,. \end{aligned} \tag{4.3}$$

By letting $x \to 0$, $z \to 0$ and $y \to \infty$ in (4.3), we obtain that

$$\mathbb{P}(\sigma < \infty) = k\,\mathbb{P}(\sigma = \infty) \int_0^\infty \nu(u,\infty)\,\mathcal{V}^X(du) = k\,\mathbb{P}(\sigma = \infty) \int_0^\infty \mathcal{V}^X(u)\,\nu(du)\,.$$

By Lemma 4.1, $\mathbb{P}(\sigma = \infty) > 0$. As a consequence, $\int_0^\infty \mathcal{V}^X(u)\,\nu(du) < \infty$. We record this fact in the following

Corollary 4.4 *Assume that X drifts to ∞ and $\mathbb{P}(\sigma > 0) = 1$. Then $\int_0^\infty \mathcal{V}^X(u)\,\nu(du) < \infty$.*

Let

$$\rho = \frac{k \int_0^\infty \mathcal{V}^X(u)\,\nu(du)}{1 + k \int_0^\infty \mathcal{V}^X(u)\,\nu(du)}\,.$$

Then $\mathbb{P}(\sigma < \infty) = \rho$. By letting $z \to 0$ and $y \to \infty$ in (4.3), it follows that $\mathbb{P}(J > x, \sigma < \infty) = k\,\mathbb{P}(\sigma = \infty) \int_x^\infty \nu(u,\infty)\,\mathcal{V}^X(du)$. Therefore,

$$\mathbb{P}(J > x \mid \sigma < \infty) = \frac{\int_x^\infty \nu(u,\infty)\,\mathcal{V}^X(du)}{\int_0^\infty \nu(u,\infty)\,\mathcal{V}^X(du)} = 1 - H(x)\,,$$

where

$$H(x) = \frac{\int_0^x \nu(u,\infty)\,\mathcal{V}^X(du)}{\int_0^\infty \nu(u,\infty)\,\mathcal{V}^X(du)}$$

can be thought of as the integrated tail distribution.

By letting $x \to 0$ and $z \to 0$ in (4.3), it easily follows that random variables $1_{\sigma<\infty}$ and $S^{\widehat{X}}(\sigma-)$ are independent. In case $\sigma = \infty$, we interpret $S^{\widehat{X}}(\sigma-)$ as $S^{\widehat{X}}(\infty)$. In particular, $\mathbb{P}(\sigma = \infty, \widehat{\tau}_y = \infty) = \mathbb{P}(\sigma = \infty)\mathbb{P}(S^{\widehat{X}}(\sigma-) \le y)$. Let $G(x) = \mathbb{P}(S^{\widehat{X}}(\sigma-) \le x)$.

Proposition 4.3 can be now improved to

Theorem 4.5 *The distribution of the vector $(\widehat{S}^X(\sigma-), J, S^{\widehat{X}}(\sigma-) - \widehat{X}(\sigma-))$ on the set $\{\sigma < \infty\}$ is given by*

$$\begin{aligned}
&\mathbb{P}(S^{\widehat{X}}(\sigma-) \le y, J > x, S^{\widehat{X}}(\sigma-) - \widehat{X}(\sigma-) > z,\ \sigma < \infty) \\
&= \mathbb{P}(S^{\widehat{X}}(\sigma-) \le y)\, \frac{\int_{x+z}^\infty \nu(u,\infty)\,\mathcal{V}^X(du)}{\int_0^\infty \nu(u,\infty)\,\mathcal{V}^X(du)}\, \mathbb{P}(\sigma < \infty)\,.
\end{aligned} \tag{4.4}$$

Moreover, $S^{\widehat{X}}(\sigma-)$ and J are conditionally independent given $\sigma < \infty$, and

$$\mathbb{P}(S^{\widehat{X}}(\sigma-) \le y, J > x \mid \sigma < \infty) = G(y)(1 - H(x))\,. \tag{4.5}$$

It is now possible to write the absolute maximum of $\widehat{X}$ as a random sum of modified ladder heights. Let $L_0 := S^{\widehat{X}}(\sigma_1-)$, $J_1 := S^{\widehat{X}}(\sigma_1) - S^{\widehat{X}}(\sigma_1-)$ and $L_1 := S^{\widehat{X}}(\sigma_2-) - S^{\widehat{X}}(\sigma_1)$ on $\{\sigma_1 < \infty\}$, etc: $L_0, J_1, L_1, \ldots$ are called the modified ladder heights. Let also $N := \max\{n : \sigma_n < \infty\}$. By the strong Markov property of $\widehat{X}$, N has a geometric distribution with parameter $\mathbb{P}(\sigma_1 = \infty) = 1 - \rho$. Clearly,

$$S^{\widehat{X}}(\infty) = L_0 + J_1 + L_1 + \cdots + J_N + L_N\,. \tag{4.6}$$

Note that $\mathbb{P}(L_0 \le x, N = 0) = \mathbb{P}(S^{\widehat{X}}(\sigma-) \le x, \sigma = \infty) = G(x)(1-\rho)$. For every $n \in \mathbb{N}$, by the strong Markov property at σ_n, and by equality (4.5) we have

$$\mathbb{P}(L_0 + J_1 + L_1 + \cdots + J_n + L_n \le x, N = n) = (1-\rho)\rho^n (G^{(n+1)*} * H^{n*})(x)\,.$$

This leads to the Pollaczek-Hinchin-type formula for the distribution function of $S^{\widehat{X}}(\infty)$.

Theorem 4.6 *For $x \ge 0$,*

$$\mathbb{P}(S^{\widehat{X}}(\infty) \le x) = (1-\rho)\sum_{n=0}^{\infty} \rho^n (G^{(n+1)*} * H^{n*})(x)\,. \tag{4.7}$$

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