Spectral Properties of Subordinate Processes in Domains

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Abstract

Suppose that X is an m-symmetric Markov process on a Lusin space E and that S is a subordinator with Laplace exponent ϕ. If X and S are independent, the process X^ϕ defined by $X^\phi_t = X_{S_t}$ is called a subordinate process of X via S. In this note we show that, if S is not a compound Poisson process and the semigroup of X is ultracontractive, then for any $t > 0$ and any open subset $D \subset E$ of finite m-measure, the transition semigroup $P^{\phi,D}_t$ of the subprocess of X^ϕ killed upon leaving D is compact and therefore it has discrete spectrum.

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1 Introduction

Let E be a Lusin space (i.e. a space that is homeomorphic to a Borel sunset of a compact metric space), $\mathcal{B}(E)$ be the Borel σ-algebra on E, which is countably generated, and m be a σ-finite Borel measure on E with $\text{Supp}(m) = E$. Let $X = \{X_t : t \geq 0\}$ be an m-symmetric Markov process on E in the sense that its transition operators $\{P_t : t \geq 0\}$ are symmetric operators in $L^2(E, m)$. Suppose that the semigroup $\{P_t : t \geq 0\}$ of X is ultracontractive; that is, P_t is bounded from $L^2(E, m)$ to $L^\infty(E, m)$ for every $t > 0$. Under this condition, P_t has an integral kernel $p(t, , x, y)$ for all $t > 0$ with respect to the measure m, which satisfies $0 \leq p(t, x, y) \leq c_t$ almost everywhere and with $t \to c_t$ being a decreasing function on $(0, \infty)$ (see, e.g. (2.1.1) and Lemma 2.1.2 of [5]). Conversely, if, for every $t > 0$, the transition operator P_t of X has a bounded integral kernel, then P_t is bounded from $L^2(E, m)$ to $L^\infty(E, m)$ and so $\{P_t : t \geq 0\}$ is ultracontractive. Therefore the semigroups of Brownian motions and spherically symmetric stable processes are ultracontractive.

For any open subset D of E, let X^D be the process on D obtained by killing X upon exiting from D. We shall use $\{P^D_t : t \geq 0\}$ to denote the semigroup of X^D. Under the ultracontractive condition above, one can easily show that if $m(D) < \infty$, then, for any $t > 0$, the operator P^D_t is a Hilbert-Schmidt operator and hence has discrete spectrum. Let $\{e^{-\lambda_n t} : n \geq 1\}$ be the eigenvalues of P^D_t, arranged in decreasing order and repeated according to multiplicity.

Let $S = \{S_t : t \geq 0\}$ be a subordinator with Laplace exponent ϕ, independent of X. Then the process X^ϕ defined by

$$X^\phi_t := X_{S_t}, \quad t \geq 0$$

is called a subordinate process of X via S, which is also an m-symmetric Markov process on E. For any open subset D of E, we shall use $X^{\phi,D}$ to denote the process on D obtained by killing X^ϕ upon exiting from D and use $\{P^{\phi,D}_t : t \geq 0\}$ to denote the semigroup of $X^{\phi,D}$.

The following result on the eigenvalues of $P^{\phi,D}_t$ was proved in [3], which extends the upper bound estimates on the eigenvalues of killed symmetric stable processes in D obtained in [1] and [6].

Theorem A. *Suppose that D is an open subset of E with $m(D) < \infty$ and that S is not a compound Poisson process. Suppose further that, for any $t > 0$, $P^{\phi,D}_t$ has discrete spectrum with eigenvalues $\{e^{-\mu_n t} : n \geq 1\}$, arranged in decreasing order and repeated according to multiplicity.*

(a) Under the assumption above, we have

$$\mu_n \leq 4\phi(\lambda_n), \quad \forall n \geq 1.$$

(b) *When the Laplace exponent* ϕ *of* S *is complete Bernstein, we have*

$$\mu_n \le \phi(\lambda_n), \quad \forall n \ge 1.$$

(c) *When* $(E, m) = (\mathbf{R}^d, dx)$, X *is a spherically symmetric* α*-stable process in* $\mathbf{R}^d$ *with* $\alpha \in (0, 2]$ *and* D *is bounded convex domain in* $\mathbf{R}^d$, *we have*

$$\mu_n \ge \frac{1}{2}\phi(\lambda_n), \quad \forall n \ge 1.$$

For the concept of a complete Bernstein function, one can see, for instance, [3] and [8].

Let $S^{(n)} = \{S_t^{(n)}, t \ge 0\}$ be a subordinator with Laplace exponent ϕ_n and $S = \{S_t, t \ge 0\}$ a subordinator with Laplace exponent ϕ. Suppose that X is independent of S and $S^{(n)}$. Consider the subordinate processes $X^{\phi_n} := \left\{X_{S_t^{(n)}},\ t \ge 0\right\}$ and $X^{\phi} := \{X_{S_t},\ t \ge 0\}$, and their subprocesses $X^{\phi_n,D}$ and $X^{\phi,D}$ killed upon leaving D, where D is an open subset of E. The following continuity result was proved in [4].

Theorem B. *Suppose that* D *is an open subset of* E *with* $m(D) < \infty$ *and that* S *and* S_n, $n \ge 1$, *are not a compound Poisson processes. Suppose further that, for any* $t > 0$ *and* $n \ge 1$, $P_t^{\phi,D}$ *and* $P_t^{\phi_n,D}$ *have discrete spectra with eigenvalues* $\{e^{-\mu_k t} : k \ge 1\}$ *and* $\{e^{-\mu_k^n t} : k \ge 1\}$ *respectively, each arranged in decreasing order and repeated according to multiplicity. If*

$$\lim_{n\to\infty} \phi_n(\lambda) = \phi(\lambda) \qquad \text{for every } \lambda \ge 0,$$

then we have

$$\lim_{n\to\infty} \mu_k^n = \mu_k \quad \text{for every } k \ge 1.$$

In both Theorem A and Theorem B above, we have assumed that $P_t^{\phi,D}$ and $P_t^{\phi_n,D}$ have discrete spectra. This kind of conditions are not very easy to verify in general. These conditions are verified in [3] and [4] for some examples of subordinate spherically symmetric stable processes by showing that, in these cases, $P_t^{\phi,D}$ is a Hilbert-Schmidt operator for some value of t. To show that $P_t^{\phi,D}$ is a Hilbert-Schmidt operator, we need some good estimates on the density $p^{\phi,D}(t, x, y)$ which are not available in general. In fact, in a lot of examples, $P_t^{\phi,D}$ is not a Hilbert-Schmidt operator for every $t > 0$. Moreover, there are examples in which $P_t^{\phi,D}$ is a Hilbert-Schmidt operator for some values of t while it is not for some other values t. See Corollary 3.2 and Remark 3.3 below for such examples.

In this note, we will show that, when D is an open subset of E with $m(D) < \infty$ and S is not compound Poisson, $P_t^{\phi,D}$ is always a compact operator in $L^2(D, m)$ and therefore it has discrete spectrum.

2 Main Result

Let $S = \{S_t : t \geq 0\}$ be a subordinator, that is, an increasing Lévy process taking values in $[0, \infty)$ with $S_0 = 0$. The law of S is characterized by

$$\mathbf{E}[\exp(-\lambda S_t)] = \exp(-t\phi(\lambda)) \qquad \text{for } \lambda > 0. \tag{2.1}$$

The function $\phi : (0, \infty) \to \mathbf{R}_+$ is called the Laplace exponent of S, and has the representation

$$\phi(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda t})\, \nu(dt)\,. \tag{2.2}$$

Here $b \geq 0$, and ν is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_0^\infty (t \wedge 1)\, \nu(dt) < \infty\,.$$

The constant b is called the drift and ν the Lévy measure of the subordinator S. Note that by Proposition I.2, Corollary I.3 and (III.3) in Bertoin [2], ψ is bounded if and only if the subordinator S is compound Poisson.

Let $X = \{X_t : t \geq 0\}$ be an m-symmetric Markov process on a Lusin space E, where m is a σ-finite measure on $\mathcal{B}(E)$ (the Borel σ-field on E) with $\text{Supp}(m) = E$. Suppose that the semigroup $\{P_t, t \geq 0\}$ of X is ultracontractive; that is, P_t is bounded from $L^2(E, m)$ to $L^\infty(E, m)$ for every $t > 0$. Under this condition, P_t has an integral kernel $p(t, , x, y)$ for all $t > 0$ with respect to the measure m, which satisfies

$$0 \leq p(t, x, y) \leq c_t \qquad \text{almost everywhere} \tag{2.3}$$

with $t \to c_t$ being a decreasing function on $(0, \infty)$ (see, e.g. (2.1.1) and Lemma 2.1.2 of [5]). Conversely, if, for every $t > 0$, the transition operator P_t has a bounded density function, i.e., (2.3) holds for some $c_t > 0$, then for a.e. $x \in E$ and $f \in L^2(E, m)$,

$$|P_t f(x)| = \left| \int_E p(t, x, y) f(y) m(dy) \right| \leq \left(\int_E p(t, x, y) f(y)^2 m(dy) \right)^{1/2} \leq c_t^{1/2} \, \|f\|_{L^2(E,m)}.$$

So $\{P_t, t \geq 0\}$ is ultracontractive. Therefore the semigroup $\{P_t, t \geq 0\}$ is ultracontractive if and only if P_t has a bounded integral kernel for every $t > 0$. In particular, the transition semigroups of Brownian motions and spherically symmetric stable processes are ultracontractive.

Suppose that X and S are independent. The transition probability $p^\phi(t, x, dy)$ of the subordinate process $X^\phi := \{X_{S_t}, t \geq 0\}$ is given by

$$p^\phi(t, x, B) = \int_0^\infty p(s, x, B)\, \eta_t(ds) \qquad \text{for } t > 0,\ x \in E \text{ and } B \subset E, \tag{2.4}$$

where η_t is the distribution of S_t. If $\lim_{\lambda\to\infty}\phi(\lambda)=\infty$, i.e., S is not a compound Poisson process, then for any $t>0$, the distribution η_t of S_t does not charge $\{0\}$ and so X^ϕ has a transition density function with respect to the measure m given by

$$p^\phi(t,x,y)=\int_0^\infty p(s,x,y)\eta_t(ds).$$

For any open subset D of E, we will use $X^{\phi,D}$ to denote the process in D obtained by killing X^ϕ upon leaving D. We will use $\left\{P_t^{\phi,D}: t\geq 0\right\}$ to denote the transition semigroup of $X^{\phi,D}$.

The following is the main result of this note. We will prove it by using a result from [7].

Theorem 2.1 *If S is not compound Poisson, then for any open subset D of E with $m(D)<\infty$ and every $t>0$, $P_t^{\phi,D}$ is a compact operator in $L^2(D,m)$ and therefore it has discrete spectrum.*

Proof. For any $s>0$ we define the operator P_s on $L^2(D,m)$ defined by

$$P_sf(x):=\int_D p(s,x,y)f(y)m(dy).$$

For any $f\in L^2(D,m)$ with $\|f\|_{L^2(D,m)}\leq 1$ and any Borel subset A of D, we have

$$\begin{aligned}\int_A (P_sf(x))^2m(dx) &\leq \int_A P_s(f^2)(x)m(dx)=\int_E f(x)^2P_s1_A(x)m(dx)\\ &\leq \|P_s1_A\|_\infty\int_E f(x)^2m(dx)\leq\|P_s1_A\|_\infty\leq\min\{1,\ c_sm(A)\}. \quad (2.5)\end{aligned}$$

Now we fix a $t>0$. For any $f\in L^2(D,m)$ with $\|f\|_{L^2(D,m)}\leq 1$ and any Borel subset A of D, we have

$$\begin{aligned}\int_A\left(\int_0^\infty P_sf(x)\eta_t(ds)\right)^2m(dx) &\leq \int_A\int_0^\infty |P_sf(x)|^2\eta_t(ds)m(dx)\\ &= \int_0^\infty\int_A |P_sf(x)|^2m(dx)\eta_t(ds).\end{aligned}$$

For any $\epsilon>0$, choose $s_0>0$ such that $\eta_t([0,s_0])<\frac{\epsilon}{2}$. Then by (2.3) and (2.5),

$$\int_A\left(\int_0^\infty P_sf(x)\eta_t(ds)\right)^2dx\leq\frac{\epsilon}{2}+c_{s_0}m(A).$$

So for any Borel set $A\subset E$ with $m(A)<\frac{\epsilon}{2c_{s_0}}$, we have

$$\int_A\left(\int_0^\infty P_sf(x)\eta_t(ds)\right)^2m(dx)\leq\epsilon.$$

This shows that the family of functions

$$\left\{\left(\int_0^\infty P_s f(x)\eta_t(ds)\right)^2 : \ \|f\|_{L^2(D,m)} \le 1\right\}$$

is uniformly integrable.

Let $p^{\phi,D}(t,x,y)$ be the density function of $P_t^{\phi,D}$. Clearly,

$$p^{\phi,D}(t,x,y) \le p^{\phi}(t,x,y), \quad (t,x,y) \in (0,\infty)\times D\times D.$$

Thus for any $t>0$ and any Borel function f on D, we have

$$|P_t^{\phi,D} f(x)| \le \int_D p^{\phi}(t,x,y)|f|(y)m(dy) \le \int_0^\infty P_s|f|(x)\eta_t(ds).$$

Consequently, the family of functions

$$\left\{(P_t^{\phi,D} f)^2(x) : \ \|f\|_{L^2(D)} \le 1\right\}$$

is uniformly integrable on $(D, \mathcal{B}(D), m)$. Now we can apply [7, Theorem 1.2] to conclude that $P_t^{\phi,D}$ is a compact operator in $L^2(D,m)$ and therefore it has discrete spectrum. □

By combining the above result with Theorems A and B above in §1, we immediately get the following

Theorem 2.2 *Suppose that D is an open subset of E with $m(D)<\infty$ and that S is not a compound Poisson process. Then, for any $t>0$, $P_t^{\phi,D}$ has discrete spectrum. Let $\{e^{-\mu_n t} : n\ge 1\}$ be the eigenvalues of $P_t^{\phi,D}$, arranged in decreasing order and repeated according to multiplicity.*

(a) In general, we have

$$\mu_n \le 4\phi(\lambda_n), \quad \forall n\ge 1.$$

(b) When the Laplace exponent ϕ of S is complete Bernstein, we have

$$\mu_n \le \phi(\lambda_n), \quad \forall n\ge 1.$$

(c) When $(E,m) = (\mathbf{R}^d, dx)$, X is a spherically symmetric α-stable process in $\mathbf{R}^d$ with $\alpha\in(0,2]$ and D is a bounded convex domain in $\mathbf{R}^d$, we have

$$\mu_n \ge \frac{1}{2}\phi(\lambda_n), \quad \forall n\ge 1.$$

Theorem 2.3 *Suppose that D is an open subset of E with $m(D) < \infty$ and that S and S_n, $n \geq 1$, are not a compound Poisson processes. Then, for any $t > 0$ and $n \geq 1$, $P_t^{\phi,D}$ and $P_t^{\phi_n,D}$ have discrete spectra. Let $\{e^{-\mu_k t} : k \geq 1\}$ and $\{e^{-\mu_k^n t} : k \geq 1\}$ be the eigenvalues of $P_t^{\phi,D}$ and $P_t^{\phi_n,D}$ respectively, each arranged in decreasing order and repeated according to multiplicity. If*

$$\lim_{n\to\infty} \phi_n(\lambda) = \phi(\lambda) \qquad \text{for every } \lambda \geq 0,$$

then we have

$$\lim_{n\to\infty} \mu_k^n = \mu_k \quad \text{for every } k \geq 1.$$

3 Examples

Suppose that $(E, m) = (\mathbf{R}^d, dx)$ and that X is a Brownian motion in $\mathbf{R}^d$. In this section, we give two examples of subordinator S with Laplace exponent ϕ such that, at least for some $t > 0$, the transition operator $P_t^{\phi,D}$ of X_t^ϕ is not a Hilbert-Schmidt operator for any bounded domain $D \subset \mathbf{R}^d$.

In this section we will use the following notation: If f and g are two positive functions, we will write $f \approx g$ if f/g is bounded between two positive numbers.

Let $\phi(\lambda) = \log(1 + \lambda^{\alpha/2}), \alpha \in (0, 2]$. $\phi(\lambda)$ is an unbounded complete Bernstein function, the process X^ϕ is a geometric α-stable process for $\alpha \in (0, 2)$ and a variance gamma process for $\alpha = 2$. Geometric stable distributions were first introduced in [9] and they have played an important role in heavy-tail modeling of economic data. For recent results on the potential theory of geometric stable processes, please see [10]. Since $\phi(\lambda) = \psi_2 \circ \psi_1(\lambda)$, where $\psi_1(\lambda) = \lambda^{\alpha/2}$ and $\psi_2(\lambda) = \log(1 + \lambda)$, it is easy to see that the process X^ϕ can be obtained by subordinating a spherically symmetric α-stable process Y with an independent gamma subordinator. Let $p^Y(t, x, y)$ denote the transition density function of Y and note that the gamma subordinator has transition density function $\frac{1}{\Gamma(t)} u^{t-1} e^{-u}$. Let $p_\alpha(t, x, y)$ be the transition density of the geometric α-stable process in $\mathbf{R}^d$. Then

$$p_\alpha(t, x, y) = \int_0^\infty p^Y(s, x, y) \frac{1}{\Gamma(t)} s^{t-1} e^{-s} ds \qquad \text{for } t, x, y \in \mathbf{R}^d.$$

The following estimates on the density $p_\alpha(t, x, y)$ are also of independent interest.

Theorem 3.1 *For $\alpha \in (0, 2)$, there are positive constants $c_1 < c_2$ such that*

$$c_1 t \left(\frac{1}{|x-y|^{d+\alpha}} \wedge \frac{1}{|x-y|^{d-t\alpha}} \right) \leq p_\alpha(t, x, y) \leq c_2\, t \left(\frac{1}{|x-y|^{d+\alpha}} \wedge \frac{1}{|x-y|^{d-t\alpha}} \right)$$

for $x, y \in \mathbf{R}^d$ and $0 < t \leq 1 \wedge \frac{d}{2\alpha}$.

Proof. Unless otherwise specified, all the constants C_i in this proof depend only on d and α. It is well-known that

$$C_1 s^{-\frac{d}{\alpha}}\left(1\wedge \frac{s^{\frac{d+\alpha}{\alpha}}}{|x-y|^{d+\alpha}}\right)\le p^Y(s,x,y)\le C_2 s^{-\frac{d}{\alpha}}\left(1\wedge \frac{s^{\frac{d+\alpha}{\alpha}}}{|x-y|^{d+\alpha}}\right),\quad \forall s>0 \text{ and } x,y\in \mathbf{R}^d$$

for some positive constants $C_1 < C_2$. Therefore we have

$$\frac{C_1}{\Gamma(t)} I_1(t,|x-y|)\le p_\alpha(t,x,y)\le \frac{C_2}{\Gamma(t)} I(t,|x-y|),\quad \forall t>0 \text{ and } x,y\in R^d$$

with

$$I(t,r)=\int_0^\infty s^{-\frac{d}{\alpha}}\left(1\wedge \frac{s^{\frac{d+\alpha}{\alpha}}}{r^{d+\alpha}}\right) s^{t-1}e^{-s}ds.$$

Note that

$$\begin{aligned} I(t,r) &= \int_0^\infty s^{t-1-\frac{d}{\alpha}}\left(1\wedge \frac{s^{\frac{d+\alpha}{\alpha}}}{r^{d+\alpha}}\right) e^{-s}ds \\ &= \frac{1}{r^{d+\alpha}}\int_0^{r^\alpha} s^t e^{-s}ds + \int_{r^\alpha}^\infty s^{t-1-\frac{d}{\alpha}}e^{-s}ds. \end{aligned}$$

From now on, assume that $0<t\le 1\wedge \frac{d}{2\alpha}$. When $r\le 1$,

$$\begin{aligned} I(t,r) &\approx \frac{1}{r^{d+\alpha}}\int_0^{r^\alpha} s^t ds + \int_{r^\alpha}^1 s^{t-1-\frac{d}{\alpha}}e^{-s}ds + \int_1^\infty s^{t-1-\frac{d}{\alpha}}e^{-s}ds \\ &\approx \frac{1}{t+1} r^{\alpha t-d} + \frac{1}{\frac{d}{\alpha}-t}(r^{\alpha t-d}-1) + \int_1^\infty e^{-s}ds \\ &= C_3 r^{\alpha t-d}. \end{aligned}$$

For any $r>0$, clearly,

$$I(t,r)\le \frac{1}{r^{d+\alpha}}\int_0^\infty s^t e^{-s}ds = \frac{\Gamma(t+1)}{r^{d+\alpha}} = \frac{C_4}{r^{d+\alpha}}.$$

On the other hand, for $r>1$,

$$I(t,r)\ge \frac{1}{r^{d+\alpha}}\int_0^1 s^t e^{-s}\, ds \ge \frac{1}{(1+t)er^{d+\alpha}} \ge \frac{1}{2er^{d+\alpha}}.$$

Combining above, there are positive constants C_5 and C_6 depending only on d and α such that for $0<t\le 1\wedge \frac{d}{2\alpha}$ and $r>0$,

$$C_5\left(\frac{1}{r^{d+\alpha}}\wedge \frac{1}{r^{d-t\alpha}}\right)\le I(t,r)\le C_6\left(\frac{1}{r^{d+\alpha}}\wedge \frac{1}{r^{d-t\alpha}}\right)$$

Note that for $0 < t \le 1$, $\Gamma(t) \approx t^{-1}$. Therefore, we have for $x, y \in \mathbf{R}^d$ and $0 < t \le 1 \wedge \frac{d}{2\alpha}$,

$$C_1 C_5\, t \left(\frac{1}{|x-y|^{d+\alpha}} \wedge \frac{1}{|x-y|^{d-t\alpha}} \right) \le p_\alpha(t,x,y) \le C_2 C_6\, t \left(\frac{1}{|x-y|^{d+\alpha}} \wedge \frac{1}{|x-y|^{d-t\alpha}} \right).$$

This completes the proof. □

Let D be a bounded domain of $\mathbf{R}^d$. Denote by $P_t^{\phi,D}$ the transition semigroup of the subprocess $X^{\phi,D}$ of the geometric α-stable process killed upon leaving D.

Corollary 3.2 *If $\alpha \in (0,2)$, then for $0 < t \le 1 \wedge \frac{d}{2\alpha}$, $P_t^{\phi,D}$ is not a Hilbert-Schmidt operator in $L^2(D,dx)$. However, $P_t^{\phi,D}$ is a Hilbert-Schmidt operator in $L^2(D,dx)$ when $t > \frac{d}{\alpha}$.*

Proof. Let $p_\alpha^D(t,x,y)$ be the density function for $P_t^{\phi,D}$. Then

$$p_\alpha^D(t,x,y) = p_\alpha(t,x,y) - \mathbf{E}_x\left[p_\alpha(t-\tau_D, X^\phi_{\tau_D}, y);\ \tau_D < t\right] \qquad \text{for } x, y \in D.$$

Here

$$\tau_D := \inf\left\{t > 0:\ X_t^\phi \notin D\right\}$$

is the first exit time from D by the geometric α-stable process X^ϕ. Let $x_0 \in D$ and $r \in (0,1)$ be such that $B(x_0, 3r) \subset D$. Then by Theorem 3.1, there is a constant c depending only on d and α such that for $y \in B(x_0, 2r)$,

$$\mathbf{E}_x\left[p_\alpha(t-\tau_D, X_{\tau_D}, y);\ \tau_D < t\right] \le \frac{c}{r^{d+\alpha}}.$$

Therefore for $x \in D$, $y \in B(x_0, 2r)$ and $t \le 1 \wedge \frac{d}{2\alpha}$,

$$p_\alpha^D(t,x,y) \ge c_1 t \left(\frac{1}{|x-y|^{d+\alpha}} \wedge \frac{1}{|x-y|^{d-t\alpha}} \right) - \frac{c}{r^{d+\alpha}}.$$

It follows then for $t \le 1 \wedge \frac{d}{2\alpha}$,

$$\int_D p_\alpha^D(t,x,y)^2 dy \ge \int_{B(x_0,2r)} p_\alpha^D(t,x,y)^2 dy = \infty \qquad \text{for every } x \in B(x_0, r),$$

and, consequently,

$$\int_{D\times D} p_\alpha^D(t,x,y)^2 dx dy = \infty.$$

So $P_t^{\phi,D}$ is not a Hilbert-Schmidt operator in $L^2(D,dx)$ when $t \le 1 \wedge \frac{d}{2\alpha}$. That $P_t^{\phi,D}$ is a Hilbert-Schmidt operator in $L^2(D,dx)$ for $t > \frac{d}{\alpha}$ has been established in Example 5.3 of [3]. □

Remark 3.3 Let X be Brownian motion in $\mathbf{R}^d$ and $\alpha \in (0,2)$. By taking the composition of a geometric $\alpha/2$-stable subordination with a gamma subordination, that is, by taking

$$\phi(\lambda) = \log\left(1 + \log\left(1 + \lambda^{\alpha/2}\right)\right).$$

One can show in a similar way as above, using the heat kernel estimate for geometric α-stable process in Theorem 3.1, that $P_t^{\phi,D}$ is not a Hilbert-Schmidt operator in $L^2(D,dx)$ for every $t > 0$ and every bounded domain $D \subset \mathbf{R}^d$. We leave the details to the interested reader. Note that according to Theorem 2.1, $P_t^{\phi,D}$ is always a compact operator in $L^2(D,dx)$ for every $t > 0$ and every bounded domain $D \subset \mathbf{R}^d$.

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