The number of $K_{m,m}$-free graphs

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Abstract

A graph is called *H-free* if it contains no copy of H. Denote by $f_n(H)$ the number of (labeled) H-free graphs on n vertices. Erdős conjectured (see [7]) that $f_n(H) \leq 2^{(1+o(1))\,\mathrm{ex}(n,H)}$. This was first shown to be true for cliques [9] and then Erdős, Frankl and Rödl [8] proved it for all graphs H with $\chi(H) \geq 3$. For most bipartite H, the question is still wide open, and even the correct order of magnitude of $\log_2 f_n(H)$ is not known. We prove that for every $m \geq 2$, $f_n(K_{m,m}) \leq 2^{O(n^{2-1/m})}$, extending the result of Kleitman and Winston [15] and answering a question of Erdős. This bound is asymptotically sharp for $m \in \{2,3\}$, and possibly for other values of m, for which the order of $\mathrm{ex}(n, K_{m,m})$ is not known, but it is conjectured to be $\Theta(n^{2-1/m})$. Our method also yields a bound on the number of $K_{m,m}$-free graphs with fixed order and size, extending the result of Füredi [11]. Using this bound, we prove a relaxed version of a conjecture due to Haxell, Kohayakawa and Łuczak [13] and show that almost all $K_{3,3}$-free graphs of order n have more than $1/20 \cdot \mathrm{ex}(n, K_{3,3})$ edges.

1 Introduction

Let H be an arbitrary graph. We say that a graph G is *H-free*, if G does not contain H as a (not necessarily induced) subgraph. Denote by $\mathcal{F}_n(H)$ the family of labeled H-free graphs with vertex set $[n] = \{1, \ldots, n\}$, and let $f_n(H) = \left|\mathcal{F}_n(H)\right|$. Let $\mathrm{ex}(n, H)$ denote the Turán number for H, i.e., the maximum number of edges that an H-free graph on n vertices may have. The celebrated Turán's theorem [20] states that

$$\mathrm{ex}(n, K_m) = \left(1 - \frac{1}{m-1}\right)\frac{n^2}{2} + O(n),$$

and the unique K_m-free graph with $\mathrm{ex}(n, K_m)$ edges is the complete $(m-1)$-partite graph with all parts as equal as possible. Generalizing this, Erdős and Stone [10] showed that the chromatic number of H determines the order of magnitude of $\mathrm{ex}(n, H)$, i.e.,

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H)-1}\right)\frac{n^2}{2} + o(n^2). \tag{1}$$

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Since every subgraph of an H-free graph is also H-free, $\mathcal{F}_n(H)$ contains at least $2^{\mathrm{ex}(n,H)}$ graphs. Erdős, Kleitman and Rothschild [9] proved that this crude lower bound is in fact tight for complete graphs, obtaining an asymptotic formula for $\log_2 f_n(K_m)$, namely

$$\mathrm{ex}(n, K_m) \leq \log_2 f_n(K_m) \leq (1 + o(1))\, \mathrm{ex}(n, K_m). \tag{2}$$

Later Kolaitis, Prömel and Rothschild [17] obtained an asymptotic formula for $f_n(K_m)$ by proving that almost all K_m-free graphs are m-colorable. Erdős asked if (2) is also true when one replaces K_m by any graph H. The question was resolved in the affirmative by Erdős, Frankl and Rödl [8], in the case $\chi(H) \geq 3$. For a brief survey and some related results see, e.g., [1], [2], [3] and [19].

The picture is very different when one drops the $\chi(H) \geq 3$ assumption. For the remainder of this discussion assume that H is a bipartite graph that contains a cycle. For most such H the problem of determining $f_n(H)$ remains wide open, especially that even not much is known about the order of magnitude of $\mathrm{ex}(n, H)$. Unlike the non-bipartite case, the trivial lower and upper bounds for $f_n(H)$, i.e.,

$$2^{\mathrm{ex}(n,H)} \leq f_n(H) \leq \sum_{s=0}^{\mathrm{ex}(n,H)} \binom{\binom{n}{2}}{s}, \tag{3}$$

do not even determine the order of magnitude of $\log_2 f_n(H)$. The only nontrivial bipartite graphs, for which an estimate stronger than (3) is known, are cycles. Kleitman and Winston [15] proved that $\log_2 f_n(C_4) \leq 2.16384 \cdot \mathrm{ex}(n, C_4)$, and later Kleitman and Wilson [14] proved $\log_2 f_n(C_6) = \Theta(\mathrm{ex}(n, C_6))$. For a little stronger estimates of the number of graphs with large (even) girth, i.e., graphs with no short (even) cycles, see [14], [16]. Our main result extends that of Kleitman and Winston from $K_{2,2}$ to all complete bipartite graphs with equal class sizes.

Definition 1. The binary entropy function $H : [0, 1] \to \mathbb{R}$ is defined as

$$H(x) := -x \log_2 x - (1 - x) \log_2(1 - x).$$

For every positive integer m, let

$$C_m := \sup_{x \in (0,1)} \left(x^{-1+1/m} H(x) \right) \in [1, 0.55m + 1.5).$$

Theorem 2. *The number of labeled $K_{m,m}$-free graphs on n vertices satisfies*

$$\log_2 f_n(K_{m,m}) \leq (1 + o(1)) \frac{m(m-1)^{1/m}}{2m-1} C_m \cdot n^{2-1/m}.$$

This is known to be asymptotically sharp for $m \leq 3$. For other values of m, Erdős conjectured that $\mathrm{ex}(n, K_{m,m}) = \Theta(n^{2-1/m})$, i.e., the $O(n^{2-1/m})$ upper bound, proved by Kövári, Sós and Turán [18], is optimal (see [7]). If this conjecture is true, Theorem 2 would be also sharp for all $m \geq 4$.

In 1966, Brown [6] gave an algebraic construction estimating $\mathrm{ex}(p^3, K_{3,3}) \geq (p^5 - p^4)/2$ for all primes $p \equiv 3 \pmod 4$, and Füredi [12] proved that this construction is asymptotically optimal, that is, $\mathrm{ex}(n, K_{3,3}) = (1 + o(1))\frac{1}{2} n^{5/3}$. Together with Theorem 2, this implies the following.

Corollary 3. *The number of labeled $K_{3,3}$-free graphs of order n is bounded as follows:*

$$(1+o(1))\frac{1}{2}n^{5/3} \le \log_2 f_n(K_{3,3}) \le (1.64618\ldots)n^{5/3}.$$

Let $f_{n,s}(H)$ denote the number of H-free graphs with exactly s edges. Our methods give an upper bound on $f_{n,s}(K_{m,m})$, which extends the result in [11].

Theorem 4. *There is an $n_0 = n_0(m)$, such that for all $n \ge n_0$ and $s \ge n^{2-m/(m^2-m+1)}(\log n)^2$, the number of labeled $K_{m,m}$-free graphs of order n and size s is*

$$f_{n,s}(K_{m,m}) \le \left(3m\frac{n^{2m-1}}{s^m}\right)^s. \tag{4}$$

Let H be a fixed non-bipartite graph. Then for every positive $\varepsilon > 0$, almost all H-free graphs of order n have at least $(\frac{1}{2}-\varepsilon)\,\mathrm{ex}(n,H)$ and at most $(\frac{1}{2}+\varepsilon)\,\mathrm{ex}(n,H)$ edges. It is not known if a similar concentration around a half occurs when H is bipartite. Still, one should expect that the number of edges in a "typical" H-free grah is at least bounded away from the extremal values, 0 and $\mathrm{ex}(n,H)$. Balogh, Bollobás and Simonovits [3] formalized this intuition by conjecturing that for every bipartite graph H that contains a cycle, there is a positive constant c such that almost all H-free graphs of order n have at least $c \cdot \mathrm{ex}(n,H)$ and at most $(1-c)\cdot\mathrm{ex}(n,H)$ edges. So far this has been proved only for C_4 [4], [11] and partially (only the lower bound) for C_6 [11], [14]. An immediate corollary of Theorem 4 proves the lower bound in the case $H = K_{3,3}$.

Corollary 5. *Almost all $K_{3,3}$-free graphs of order n have more than $1/20 \cdot \mathrm{ex}(n,K_{3,3})$ edges.*

Given graphs G and H, let us define $\mathrm{ex}(G,H) := \max\{e(K) : H \not\subseteq K \subseteq G\}$, where $e(K) = |E(K)|$ denotes the size of K. As $\mathrm{ex}(n,H) = \mathrm{ex}(K_n,H)$, where K_n denotes the complete graph on n vertices, the above definition is a natural generalization of the Turán number. If we fix an H and any graph sequence $(G_n)_n$, a simple averaging argument implies that

$$\liminf_{n\to\infty} \frac{\mathrm{ex}(G_n,H)}{e(G_n)} \ge 1 - \frac{1}{\chi(H)-1}. \tag{5}$$

Haxell, Kohayakawa and Łuczak [13] conjectured that if $e(G_n)\to\infty$, the number of copies $N_G(H)$ of H in G_n is larger than $e(G_n)$ and these copies are "uniformly" distributed in G_n, one has equality in (5) with lim inf replaced by lim.

Definition 6. A graph H is balanced if

$$\max_{H'\subseteq H} \frac{e(H')}{v(H')} = \frac{e(H)}{v(H)}.$$

Conjecture 7 ([13])**.** *Let H be a fixed balanced graph and let $G(n,p)$ denote the usual binomial random graph of order n with edge probability p. Let $\omega = \omega(n) \to \infty$ as $n\to\infty$ and suppose that $p = p(n)$ is such that $\mathbb{E}\left[N_{G(n,p)}(H)\right] \ge \omega p n^2$. Then with probability tending to 1 as $n \to \infty$,*

$$\mathrm{ex}(G(n,p),H) = \left(1 - \frac{1}{\chi(H)-1} + o(1)\right)e(G(n,p)).$$

We prove the above conjecture for $H = K_{m,m}$, under an additional assumption on the growth rate of ω.

Theorem 8. *Fix a real number* $0 < \gamma \leq 1$. *There is a constant* $C = C(\gamma)$ *such that, if* $p = p(n) \geq Cn^{-m/(m^2-m+1)}(\log n)^2$, *with probability tending to* 1,

$$\mathrm{ex}(G(n,p), K_{m,m}) < \gamma \cdot e(G(n,p)).$$

In particular, if $pn^{m/(m^2-m+1)} \to \infty$, *then asymptotically almost surely*

$$\mathrm{ex}(G(n,p), K_{m,m}) = o\big(e(G(n,p))\big). \tag{6}$$

Remark 9. Note that in order to prove Conjecture 7, one would have to show that (6) is still true if we only assume that $pn^{2/(m+1)} \to \infty$. Still, unless $pn^{1/m} \to \infty$, and hence $\mathrm{ex}(n, K_{m,m}) = o\left(\mathbb{E}\big[e(G(n,p))\big]\right)$, the result proved by Theorem 8 is non-trivial.

In particular, proving Conjecture 7 for $H = K_{3,3}$ would require showing that (6) holds with high probability whenever $p \gg n^{-1/2}$. Note that the assumptions on p in the statement of Theorem 8 fall only a little short of that threshold.

Corollary 10. *If* $p = p(n) \gg n^{-3/7}(\log n)^2$, *then a.a.s.*

$$\mathrm{ex}(G(n,p), K_{3,3}) = o\big(e(G(n,p))\big).$$

For a graph G, we denote its vertex and edge sets by $V(G)$ and $E(G)$ respectively. The number of edges in G is $e(G) = |E(G)|$. For a vertex $v \in V(G)$, we denote the set of its neighbors by $N_G(v)$. The degree of v in G is the size of its neighborhood, $d_G(v) = d(v) = |N_G(v)|$. The minimum degree of G is the quantity $\delta(G) = \min_{v \in V(G)} d_G(v)$. For a set A of vertices of G, by $N^*_G(A) = \bigcap_{v \in A} N_G(v)$ we will denote the set of common neighbors of all vertices in A. Given an arbitrary set X and a nonnegative integer k, the power set of X, i.e., the family of all subsets of X is denoted by $\mathcal{P}(X)$. The subfamily of $\mathcal{P}(X)$ containing all k-element subsets is denoted by $\binom{X}{k}$. Finally, the term *k-set* abbreviates the phrase *k-element set*. Also, throughout the paper log will always denote natural logarithm.

The paper is organized as follows. In Section 2 we formulate and prove a general counting lemma, which is one of the basic building blocks of the proof of Theorem 2, which is given in Section 3. Theorems 4 and 8 are proved in Sections 4 and 5 respectively. Finally, Section 6 contains a few concluding remarks.

2 Counting complete bipartite subgraphs

One of the most important ingredients in our proof of Theorem 2 is Lemma 14 – an estimate on the number of copies of the complete bipartite graph $K_{m-1,m}$ in a larger graph with bounded minimum degree. Lemma 14 is a straightforward corollary of a more general statement that we prove below. The proof of Lemma 11 relies on a classic double counting argument in the spirit of Kövári, Sós and Turán [18].

Lemma 11. *Fix two integers* $1 \leq s \leq t$ *and a positive real* $\varepsilon > 0$. *Let* G *be an* n*-vertex graph with minimum degree at least* d, *and* A *be any set of* $a \geq (1+\varepsilon)(t-1)\binom{n}{s}/\binom{d}{s}$ *vertices* *of G. Then the number of copies of $K_{s,t}$ in G with the larger partite set completely contained in A,*

$$N_{s,t}(A) \geq \beta \cdot a^t, \tag{7}$$

where

$$\beta = \beta(s,t,d,\varepsilon) := \frac{\varepsilon^t}{t!}\binom{d}{s}^t \Big/ \binom{n}{s}^{t-1}.$$

Proof. For a set $U = \{u_1, \ldots, u_s\}$ of vertices of G, let

$$c(U) := \left|N_G^*(U) \cap A\right|$$

be the number of common neighbors of $u_1, \ldots, u_s$ in the set A. Clearly

$$\sum_U c(U) = \sum_{w \in A} \binom{d_G(w)}{s} \geq a\binom{\delta(G)}{s} \geq a\binom{d}{s}.$$

The number of copies of $K_{s,t}$ in G with the larger partite set contained in A is

$$N_{s,t}(A) = \sum_U \binom{c(U)}{t} \geq \binom{n}{s}\binom{a\binom{d}{s}/\binom{n}{s}}{t},$$

where the above inequality (Jensen's inequality) follows from convexity of the function

$$B_t(x) \ := \ \begin{cases} 0, & x \leq t-1, \\ \binom{x}{t}, & x > t-1, \end{cases}$$

and the assumption that $a\binom{d}{s}/\binom{n}{s} > t-1$. It follows that

$$\begin{aligned} N_{s,t}(A) \ &\geq \ \binom{n}{s} \cdot \frac{1}{t!}\prod_{i=0}^{t-1}\left(\frac{a\binom{d}{s}}{\binom{n}{s}} - i\right) = \binom{n}{s} \cdot \left(\frac{a\binom{d}{s}}{\binom{n}{s}}\right)^t \cdot \frac{1}{t!}\prod_{i=0}^{t-1}\left(1 - i\frac{\binom{n}{s}}{a\binom{d}{s}}\right) \\ &\geq \ \frac{a^t}{t!}\binom{d}{s}^t \Big/ \binom{n}{s}^{t-1} \cdot \prod_{i=0}^{t-1}\left(1 - \frac{i}{(1+\varepsilon)(t-1)}\right) \\ &\geq \ \frac{a^t}{t!}\binom{d}{s}^t \Big/ \binom{n}{s}^{t-1} \cdot \left(1 - \frac{1}{1+\varepsilon}\right)^t \geq \frac{\varepsilon^t}{t!}\binom{d}{s}^t \Big/ \binom{n}{s}^{t-1} \cdot a^t. \end{aligned}$$

□

3 Proof of Theorem 2

Let G be a $K_{m,m}$-free graph on n vertices and let v be a vertex of minimum degree in G. We let $d = d(v) - 1 = \delta(G) - 1$. Clearly the graph $G' = G - \{v\}$ is $K_{m,m}$-free and $\delta(G') \geq \delta(G) - 1 \geq d$. Arguing along these lines one can find an ordering $v_1, \ldots, v_n$ of all the vertices of G, such that if we denote the subgraph induced on $\{v_1, \ldots, v_i\}$ by G_i, then

$$\delta(G_i) \geq d_{G_{i+1}}(v_{i+1}) - 1 \ \text{ for all } \ i \in \{1, \ldots, n-1\}.$$

In other words, every n-vertex $K_{m,m}$-free graph can be obtained from a single vertex by succesively adjoining a vertex of degree $d+1$ to a graph with minimum degree at least d, for some d (which can obviously change as the graph grows). The idea of the proof is showing that the number of ways in which one can obtain a $K_{m,m}$-free graph of order $i+1$ from some i-vertex $K_{m,m}$-free graph in the above process of adjoining vertices of minimum degree is $2^{O(i^{1-1/m})}$, and therefore the number of labeled $K_{m,m}$-free graphs on n vertices is at most

$$f_n(K_{m,m}) \le n! \cdot \prod_{i=1}^{n-1} 2^{O(i^{1-1/m})} = 2^{O(n^{2-1/m})}.$$

For the remainder of the proof, fix some positive integer d and an n-vertex $K_{m,m}$-free graph G with minimum degree $\delta(G) \ge d$. In the sequel we will give an $2^{O(n^{1-1/m})}$ bound on $f(G;d,m)$ – the number of ways to adjoin to G a vertex v of degree $d+1$, so that the resulting graph is still $K_{m,m}$-free. Clearly,

$$f(G;d,m) \le \binom{n}{d+1} \le n^{d+1} = 2^{(d+1)\log_2 n}, \tag{8}$$

and so if $d+1 \le n^{1-1/m}/\log_2 n$, then $f(G;d,m) \le 2^{n^{1-1/m}}$. Therefore from now on we can assume that d is "large", i.e., $d > n^{1-1/m}/(2\log n)$.

Since $\delta(G) \ge d \gg n^{1-1/(m-1)}$, G contains numerous and evenly distributed copies of $K_{m-1,m}$. More precisely, larger partite sets of copies of $K_{m-1,m}$ in G constitute a big proportion of m-subsets of every large enough $A \subseteq V(G)$. Obviously we cannot make v adjacent to all vertices in any such m-set, since that would create a copy of $K_{m,m}$ in the graph $G \cup \{v\}$. Hence it is clear that making v adjacent to some of the vertices in G will forbid many other adjacencies. In fact, we will prove that choosing as few as $O\big((\log n)^{m^2+1}\big)$ neighbors for v restricts the remaining choices (for neighbors of v) to a set of rather small size. Now we will formalize these intuitions.

Definition 12. Let $B = \{w_1, \ldots, w_m\}$ be a set of m vertices of G and let $N^*_G(B) = \bigcap_{w\in B} N_G(w)$ be the set of their common neighbors. We say that B is *dangerous* if $|N^*_G(B)| \ge m-1$, i.e., G contains a copy of $K_{m-1,m}$, in which B forms the larger partite set. For a set $A \subseteq V(G)$, we denote the number of its dangerous m-subsets by

$$D_m(A) = \big|\{B \subseteq A : |B| = m \text{ and } B \text{ is dangerous}\}\big|.$$

Observation 13. *Let $B \subseteq V(G)$ be a dangerous m-set. Then the adjoined vertex v can be connected to at most $m-1$ vertices in B.*

Lemma 14. *Fix some $\varepsilon > 0$ and let A be any set of $a \ge (1+\varepsilon)(m-1)\binom{n}{m-1}/\binom{d}{m-1}$ vertices in G. If $d \ge d_0 = d_0(m)$, then the number of dangerous m-sets in A,*

$$D_m(A) \ge \alpha \cdot a^m,$$

where

$$\alpha = \alpha(m,d,\varepsilon) \ := \ \frac{\varepsilon^m}{(m!)^2} \cdot \frac{d^{m(m-1)}}{n^{(m-1)^2}}. \tag{9}$$

Proof. Since G' is $K_{m,m}$-free, every dangerous m-set is the larger partite set of exactly one copy of $K_{m-1,m}$ in G', and therefore by Lemma 11,

$$D_m(A) = N_{m-1,m}(A) \geq \beta(m-1, m, d, \varepsilon) \cdot a^m,$$

where $\beta = \beta(m-1, m, d, \varepsilon)$ is defined in the statement of Lemma 11. It suffices to prove that $\beta \geq \alpha$. First let us observe that

$$\lim_{d\to\infty} (1 - m/d)^{m-1} = 1,$$

and hence there is a $d_0 = d_0(m)$, such that for all $d \geq d_0$,

$$m \cdot (d-m)^{m(m-1)} \geq d^{m(m-1)}.$$

It follows that for all $d \geq d_0$,

$$\begin{aligned} \beta &= \frac{\varepsilon^m}{m!}\binom{d}{m-1}^m \Big/ \binom{n}{m-1}^{m-1} \geq \frac{\varepsilon^m}{m!} \cdot \left(\frac{(d-m)^{m-1}}{(m-1)!}\right)^m \cdot \left(\frac{(m-1)!}{n^{m-1}}\right)^{m-1} \\ &\geq \frac{\varepsilon^m}{m!} \cdot \frac{d^{m(m-1)}}{m(m-1)!n^{(m-1)^2}} = \alpha. \end{aligned}$$

□

Fix some function ε with $\lim_{n\to\infty} \varepsilon(n) = 0$, such that $\varepsilon(n) \gg (\log n)^{-1}$, and let $t_0 = t_0(n) := (\log n)/\alpha$. The key step in the proof is to show that there is a map

$$\psi : \binom{V(G)}{(m-1)t_0} \to \mathcal{P}\big(V(G)\big),$$

satisfying $|\psi(X)| \leq (1+2\varepsilon)(m-1)(n/d)^{m-1}$, such that the following holds.

Claim 15. *Let G' be a $K_{m,m}$-free graph obtained from G by adjoining a vertex v of degree $d+1$. Then there is an $X \subseteq N_{G'}(v)$ of size $(m-1)t_0$, such that $N_{G'}(v) \subseteq \psi(X)$.*

Before we start proving Claim 15, let us first show how it implies an upper bound on the number of ways to connect a vertex v of degree $d+1$ to our graph G.

Corollary 16. *With our assumptions on G, d and ε,*

$$\log_2 f(G; d, m) \leq ((1+2\varepsilon)(m-1))^{1/m} C_m \cdot n^{1-1/m} + o(n^{1-1/m}), \tag{10}$$

where C_m is defined as in Definition 1.

Proof. By Claim 15, for every G' counted by $f(G; d, m)$, we can find some $X \subseteq N_{G'}(v) \subseteq V(G)$ of size $(m-1)t_0$, such that $N_{G'}(v) \subseteq \psi(X)$. Since $\psi(X)$ does not depend on G',

$$f(G; d, m) \leq \binom{n}{(m-1)t_0} \cdot \binom{|\psi(X)|}{d+1}. \tag{11}$$

Since we assumed that $d > n^{1-1/m}/(2\log n)$, we have

$$t_0 = \frac{\log n}{\alpha} = \frac{\log n \cdot (m!)^2 n^{(m-1)^2}}{\varepsilon^m d^{m(m-1)}} \leq (m!)^2 \cdot (2\log n)^{m^2+1}. \tag{12}$$

Using (12), we can bound the first term in (11) as follows:

$$\binom{n}{(m-1)t_0} \le n^{(m-1)t_0} \le 2^{(\log_2 n)\cdot(m-1)(m!)^2(2\log n)^{m^2+1}} \ll 2^{n^{1-1/m}}. \tag{13}$$

Bounding the second term in (11) requires a little more work. First we note that

$$\binom{|\psi(X)|}{d+1} \le n \cdot \binom{|\psi(X)|}{d} \le n \cdot \binom{(1+2\varepsilon)(m-1)(n/d)^{m-1}}{d},$$

and then, using the well-known estimate relating binomial coefficients with the binary entropy function,

$$\frac{1}{n+1} \cdot 2^{nH(k/n)} \le \binom{n}{k} \le 2^{nH(k/n)},$$

we further estimate

$$\log_2 \binom{|\psi(X)|}{d+1} \le \log_2 n + (1+2\varepsilon)(m-1)(n/d)^{m-1} \cdot H\left(\frac{d^m}{(1+2\varepsilon)(m-1)n^{m-1}}\right). \tag{14}$$

Substituting $x = d^m/\big((1+2\varepsilon)(m-1)n^{m-1}\big)$ in (14) yields

$$\log_2 \binom{|\psi(X)|}{d+1} \le \log_2 n + \big((1+2\varepsilon)(m-1)\big)^{1/m} \cdot \frac{H(x)}{x^{1-1/m}} \cdot n^{1-1/m}. \tag{15}$$

Recall that $C_m = \sup_x \big(x^{-1+1/m}H(x)\big)$. Clearly, (13) and (15) imply (10). □

In order to complete the proof, we show the existence of a map ψ satisfying Claim 15. Recall that d is an integer and G is a fixed $K_{m,m}$-free graph of order n with minimum degree at least d. We are going to describe an algorithm $\mathcal{A}$ that works as follows:

- INPUT: A set $N \subseteq V(G)$ of size $d+1$, such that joining a new vertex v to all vertices in N yields a $K_{m,m}$-free graph of order $n+1$.
- OUTPUT: A pair of sets (A, X), such that A contains $N - X$ and has size at most $(1+\varepsilon)(m-1)\binom{n}{m-1}/\binom{d}{m-1}$, and X is a subset of N with exactly $(m-1)t_0$ elements

Most importantly, A will depend solely on X, i.e., if for two inputs N_1 and N_2 our algorithm $\mathcal{A}$ outputs the same set X, it also produces the same A. Hence putting $\psi(X) := A \cup X$ for every output (A, X) of $\mathcal{A}$ uniquely defines an appropriate map ψ, as by the assumption $d > n^{1-1/m}/(2\log n)$ and (12),

$$\begin{aligned}
|\psi(X)| &\le (m-1)t_0 + (1+\varepsilon)(m-1)\binom{n}{m-1}\Big/\binom{d}{m-1} \\
&\le (m-1)\cdot(m!)^2(2\log n)^{m^2+1} + (1+\varepsilon)(m-1)n^{m-1}/(d-m)^{m-1} \\
&\le (1+2\varepsilon)(m-1)(n/d)^{m-1},
\end{aligned}$$

whenever $n \ge n_0(m)$.

We now describe the algorithm $\mathcal{A}$:

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1. Set $A_0 := V(G)$ and $X_0 := \emptyset$.
2. For $t = 0, \dots, t_0 - 1$, do the following:
    (a) Set $A_t^0 := A_t$ and $S_t^0 := \emptyset$.
    (b) For $i = 0, \dots, m-2$, do the following:
        i. List all the vertices in $A_t^i$ as $w_{t,i}^1, \dots, w_{t,i}^{|A_t^i|}$ in a unique way so that for each $j$, the vertex $w_{t,i}^{j+1}$ is the vertex with the minimum label among all vertices in $A_t^i - \{w_{t,i}^1, \dots, w_{t,i}^j\}$ belonging to the maximum number of dangerous sets $B$ that contain $S_t^i$ and the remaining $m - i$ vertices of $B$ all come from the set $A_t^i - \{w_{t,i}^1, \dots, w_{t,i}^j\}$.
        ii. Let $j(t,i)$ be the smallest $j$ such that $w_{t,i}^j \in N$.
        iii. Set $A_t^{i+1} := A_t^i - \{w_{t,i}^1, \dots, w_{t,i}^{j(t,i)}\}$ and $S_t^{i+1} := S_t^i \cup \{w_{t,i}^{j(t,i)}\}$.
    (c) Let $F_t$ be the set of all vertices $w \in A_t^{m-1}$ such that $\{w\} \cup S_t^{m-1}$ is a dangerous set.
        Set $A_{t+1} := A_t^{m-1} - F_t$ and $X_{t+1} := X_t \cup S_t^{m-1}$.
3. Set $A := A_{t_0}$ and $X := X_{t_0}$. Return $(A, X)$.
```

To make the analysis of $\mathcal{A}$ a little clearer, let us make one more definition. For fixed $t \in \{0, \dots, t_0 - 1\}$ and $i \in \{0, \dots, m-1\}$, let us say that an $(m-i)$-set $C \subseteq A_t^i$ is *dangerous at step t* if the m-set $C \cup \{w_{t,i}^{j(t,0)}, \dots, w_{t,i}^{j(t,i-1)}\}$ is dangerous. For a subset $A' \subseteq A_t^i$, define

$$D_t^i(A') := \big|\{C \subseteq A' : |C| = m - i \text{ and } C \text{ is dangerous at step } t\}\big|.$$

Suppose we run the algorithm $\mathcal{A}$ on some input N. An easy induction on t and i proves the following statement.

Claim 17. *For every $0 \le t \le t_0 - 1$ and $0 \le i \le m-1$, the following assertions are satisfied:*

- $S_t^i \subseteq N$,
- $N - X_t - S_t^i \subseteq A_t^i$,
- *F_t is disjoint from N and*
- $|X_t| = (m-1)t$.

It follows that $X \subseteq N$, $|X| = (m-1)t_0$ and $N - X \subseteq A$. □

Since, given a fixed graph G, the sequence $\big(j(t,i)\big)_{t,i}$ uniquely determines both X and A, it should be clear that $\mathcal{A}$ cannot output two pairs (X, A) and (X, A') with $A \neq A'$. As we have already mentioned, this allows us to define $\psi(X) := A \cup X$, where (X, A) ranges over all possible outputs of $\mathcal{A}$. In order to complete the proof of Claim 15, it remains to prove the following claim.

Claim 18. *Suppose we run the algorithm $\mathcal{A}$ on some input N. Then*

$$|A| + |X| \le (1 + 2\varepsilon)(m-1)(n/d)^{m-1}. \tag{16}$$

The key step in proving Claim 18 is the following estimate.

Lemma 19. *For every fixed* $0 \le t < t_0$ *and* $0 \le i < m$*, the following holds. Suppose that* $D_t^i(A_t^i) \ge \gamma |A_t^i|^{m-i}$ *for some* $0 < \gamma \le 1$*. Then*

$$|F_t| + \sum_{k=i}^{m-1} j(t,k) \ge \gamma |A_t^i|. \tag{17}$$

Proof. For a fixed t, we prove the Lemma by reverse induction on i. Since $|F_t| = D_t^{m-1}(A_t^{m-1})$, the inequality (17) is vacuously true for $i = m-1$. Suppose that $i < m-1$ and (17) holds for $i+1$. For the sake of brevity, let $a := |A_t^i|$. Each of $w_{t,i}^1, \ldots, w_{t,i}^{j(t,i)-1}$ belongs to at most a^{m-i-1} $(m-i)$-subsets of A_t^i, and hence

$$\begin{aligned} D_t^i(A_t^i - \{w_{t,i}^1, \ldots, w_{t,i}^{j(t,i)-1}\}) &\ge D_t^i(A_t^i) - (j(t,i)-1) \cdot a^{m-i-1} \\ &\ge \gamma a^{m-i} - (j(t,i)-1) \cdot a^{m-i-1}. \end{aligned} \tag{18}$$

If $j(t,i) \ge \gamma a$, then (17) holds, so we may suppose that the reverse inequality is true, and therefore the rightmost term in (18) is positive. Since we have selected $w_{t,i}^{j(t,i)}$ to maximize $D_t^{i+1}(A_t^i - \{w_{t,i}^1, \ldots, w_{t,i}^{j(t,i)-1}, w\})$ over all $w \in A_t^i - \{w_{t,i}^1, \ldots, w_{t,i}^{j(t,i)-1}\}$,

$$\begin{aligned} D_t^{i+1}(A_t^{i+1}) &\ge \frac{m-i}{a - j(t,i) + 1} \cdot D_t^i(A_t^i - \{w_{t,i}^1, \ldots, w_{t,i}^{j(t,i)-1}\}) \\ &\ge \frac{m-i}{a - j(t,i) + 1} \cdot (\gamma a^{m-i} - (j(t,i)-1) \cdot a^{m-i-1}) \\ &\ge \frac{\gamma a - j(t,i) + 1}{a - j(t,i) + 1} \cdot a^{m-i-1} \ge \frac{\gamma a - j(t,i)}{a - j(t,i)} \cdot |A_t^{i+1}|^{m-(i+1)}, \end{aligned} \tag{19}$$

where the last inequality holds since $|A_t^{i+1}| \le |A_t^i| = a$ and $\gamma \le 1$. Hence, by the inductive assumption, with '$\gamma = (\gamma a - j(t,i))/(a - j(t,i))$',

$$|F_t| + \sum_{k=i+1}^{m-1} j(t,k) \ge \frac{\gamma a - j(t,i)}{a - j(t,i)} \cdot |A_t^{i+1}| = \gamma a - j(t,i).$$

□

Corollary 20. *If* $|A_t| \ge (1+\varepsilon)(m-1)\binom{n}{m-1}/\binom{d}{m-1}$*, then* $|A_{t+1}| \le (1-\alpha)|A_t|$*.*

Proof. Recall that $A_{t+1} = A_t^{m-1} - F_t$ and hence

$$|A_{t+1}| = |A_t^0| - \sum_{i=1}^{m-1} \left(|A_t^{i-1}| - |A_t^i|\right) - |F_t| = |A_t| - \sum_{i=1}^{m-1} j(t,i) - |F_t|. \tag{20}$$

The assumed lower bound on $|A_t|$ guarantees that Lemma 14 can be applied and hence

$$D_t^0(A_t^0) = D_m(A_t) \ge \alpha |A_t|^m.$$

By (20) and Lemma 19, where we set $\gamma = \alpha$ and $i = 0$, we get

$$|A_{t+1}| \le |A_t| - \alpha |A_t^0| = (1-\alpha)|A_t|.$$

□

Proof of Claim 18. Note that by Corollary 20,

$$|A_{t_0}| \leq \max\left\{(1-\alpha)^{t_0}|A_0|, (1+\varepsilon)(m-1)\binom{n}{m-1}/\binom{d}{m-1}\right\}, \tag{21}$$

and recall that $t_0 = (\log n)/\alpha$. Therefore

$$(1-\alpha)^{t_0}|A_0| \leq \exp(-\alpha t_0)\cdot|V(G)| = \exp(-\log n)\cdot n = 1.$$

This implies that the second term in the maximum in (21) is larger than the first, and so

$$\begin{aligned}|A_{t_0}| &\leq (1+\varepsilon)(m-1)\binom{n}{m-1}/\binom{d}{m-1} \leq (1+\varepsilon)(m-1)\frac{n^{m-1}}{(d-m)^{m-1}}\\ &\leq (1+2\varepsilon)(m-1)(n/d)^{m-1},\end{aligned}$$

provided that $n \geq n_0(m)$ – recall that $d > n^{1-1/m}/(2\log n)$. □

To complete the proof of Theorem 2, observe that, since G is $K_{m,m}$-free, $\delta(G) \leq c_m n^{1-1/m}$ for some absolute constant c_m. By (8) and Corollary 16, the number of ways to adjoin to G a vertex of degree $d+1 \leq \delta(G)+1$, so that the resulting graph is $K_{m,m}$-free, is

$$\begin{aligned}f(G;m) &= \sum_{d\leq\delta(G)} f(G;d,m) \leq \sum_{d+1\leq\frac{n^{1-1/m}}{\log_2 n}} f(G;d,m) + \sum_{d>\frac{n^{1-1/m}}{2\log n}} f(G;d,m)\\ &\leq \frac{n^{1-1/m}}{\log_2 n}\cdot 2^{n^{1-1/m}} + c_m n^{1-1/m}\cdot 2^{(1+o(1))(m-1)^{1/m}C_m\cdot n^{1-1/m}}\\ &\leq 2^{(1+o(1))(m-1)^{1/m}C_m\cdot n^{1-1/m}}.\end{aligned}$$

Hence,

$$\begin{aligned}\log_2 f_n(K_{m,m}) &\leq \log_2(n!) + (1+o(1))(m-1)^{1/m}C_m\cdot\sum_{k=1}^{n} k^{1-1/m}\\ &\leq (1+o(1))\cdot\frac{m(m-1)^{1/m}}{2m-1}C_m\cdot n^{2-1/m}.\end{aligned}$$

□

4 Proof of Theorem 4

For the sake of brevity let $\mu = m/(m^2-m+1)$. As it was remarked at the beginning of the proof of Theorem 2, every n-vertex graph G can be constructed from an isolated vertex v_1 by succesively connecting a vertex v_{i+1} to some d_i+1 vertices in $G[\{v_1,\ldots,v_i\}]$ in such a way that

$$d_i + 1 = \delta\big(G[\{v_1,\ldots,v_{i+1}\}]\big) \leq \delta\big(G[\{v_1,\ldots,v_i\}]\big) + 1$$

for all $i = 1,\ldots,n-1$. Call the sequence $(d_i)_{i=1}^{n-1}$ a *degeneracy sequence* of G and note that $e(G) = \sum_{i=1}^{n-1} d_i$.

Recall from the proof of Theorem 2, that $f(G;d,m)$ is the number of ways one can adjoin to a $K_{m,m}$-free graph G, which has minimum degree at least d, a new vertex of degree $d+1$, so that the graph remains $K_{m,m}$-free. Clearly all subgraphs of a $K_{m,m}$-free graph are also $K_{m,m}$-free, and hence, if we let

$$f(i; d_i, m) := \sup \big\{ f(G; d_i, m) : G \text{ is a } K_{m,m}\text{-free graph of order } i, \text{ with } \delta(G) \geq d_i \big\},$$

then

$$f_{n,s}(K_{m,m}) \leq n! \cdot \sum_{(d_i)} \prod_{i=1}^{n-1} f(i; d_i, m) \tag{22}$$

where the above sum is taken over all degeneracy sequences with sum s.

If $d \leq n^{1-\mu}(\log n)^{2/3}$, and $n \geq n_0$, then we give a rather crude bound

$$f(i; d, m) \leq \binom{i}{d+1} \leq n \binom{n}{d} \leq n \left(\frac{en}{d} \right)^d \leq \exp\big(n^{1-\mu}(\log n)^{5/3}\big). \tag{23}$$

Suppose now that $d > n^{1-\mu}(\log n)^{2/3}$, and let $\alpha = \alpha(m, d, 1/(2m-2))$ be as in Lemma 14. Since

$$t_0 = \frac{\log n}{\alpha} = \frac{\log n \cdot (m!)^2 n^{(m-1)^2}}{(2m-2)^{-m} d^{m(m-1)}} \leq m^{4m} \cdot n^{1-\mu} (\log n)^{1-\frac{2}{3}m(m-1)} \ll n^{1-\mu} \leq d,$$

Claim 15 can be applied, and reasoning along the lines of Corollary 16, see (11), we show that for large enough n,

$$\begin{aligned} f(i; d, m) &\leq i^{(m-1)t_0} \cdot \binom{m(i/d)^{m-1}}{d} \leq n^{n^{1-\mu}} \cdot \left(\frac{emn^{m-1}}{d^m} \right)^d \\ &\leq \exp\left(n^{1-\mu} \log n + d \log \frac{emn^{m-1}}{d^m} \right). \end{aligned} \tag{24}$$

Finally, fix some degeneracy sequence $(d_i)_{i=1}^{n-1}$ with sum s. Combining inequalities (23) and (24) yields

$$\prod_{i=1}^{n-1} f(i; d_i, m) \leq \exp\left(n^{2-\mu} (\log n)^{5/3} + \sum_{i=1}^{n-1} d_i \log \frac{emn^{m-1}}{d_i^m} \right). \tag{25}$$

The function $[0, \infty) \ni x \mapsto x \log x \in \mathbb{R}$ is convex, and so for every degeneracy sequence with sum s (putting $d_0 = 0$),

$$\sum_{i=1}^{n-1} d_i \log d_i = \sum_{i=0}^{n-1} d_i \log d_i \geq n \cdot (s/n) \log(s/n).$$

This yields

$$\sum_{i=1}^{n-1} d_i \log \frac{emn^{m-1}}{d_i^m} \leq s \log(emn^{m-1}) - ms \log(s/n) = s \log \frac{emn^{2m-1}}{s^m},$$

which combined with (25) gives

$$\prod_{i=1}^{n-1} f(i; d_i, m) \leq \exp\left(n^{2-\mu} (\log n)^{5/3} + s \log \frac{emn^{2m-1}}{s^m} \right). \tag{26}$$

Since we assumed that $s \gg n^{2-\mu}(\log n)^{5/3}$, $e < 3$, $s \le \mathrm{ex}(n, K_{m,m}) \le n^{2-1/m}$, and there are at most $n!$ degeneracy sequences, combining (22) with (26) yields

$$f_{n,s}(K_{m,m}) \le \left(\frac{3mn^{2m-1}}{s^m}\right)^s.$$

$\square$

5 Proof of Theorem 8

The proof is a rather straightforward application of Theorem 4 and the first moment method. We let $C = C(\gamma) = 3/\gamma$ and $s = (\gamma/3)pn^2 \ge n^{2-m/(m^2-m+1)} \log^2 n$. Recall that for any fixed $\varepsilon > 0$, with probability tending to 1, the random graph $G(n,p)$ has at least $(1/2 - \varepsilon)pn^2$ edges. Hence

$$s < \gamma \cdot e(G(n,p)) \tag{27}$$

holds asymptotically almost surely. Conditioning on (27), the event

$$\mathrm{ex}(G(n,p), K_{m,m}) \ge \gamma \cdot e(G(n,p)) \tag{28}$$

implies that $G(n,p)$ contains a $K_{m,m}$-free subgraph with s edges. But the expected number of copies of such a graph in $G(n,p)$ is

$$\begin{aligned} f_{n,s}(K_{m,m})p^s &\le \left(3m\frac{n^{2m-1}}{s^m}p\right)^s = \left(\frac{3^{m+1}m}{\gamma^m} \cdot \frac{p}{np^m}\right)^s \\ &\le \left(\frac{3^{m+1}m}{\gamma^m} \cdot \frac{1}{n^{1/(m^2-m+1)}}\right)^s = o(1). \end{aligned}$$

We conclude that

$$\mathbb{P}\big(\mathrm{ex}(G(n,p), K_{m,m}) \ge \gamma \cdot e(G(n,p))\big) = o(1).$$

$\square$

6 Concluding remarks

Unfortunately, the technique used in the proof of Theorem 2 fails to yield an $2^{O(n^{2-1/s})}$ bound on the number of $K_{s,t}$-free graphs when we assume that $s < t$. If we were to directly transfer the ideas from the proof of Theorem 2 to this new setting, we would similarly try to bound the number of ways to adjoin a vertex of degree $d+1$ to an n-vertex $K_{s,t}$-free graph G with minimum degree $\delta(G) \ge d$, so that the new graph is still $K_{s,t}$-free. The case when $d + 1 \le n^{1-1/s}/(\log_2 n)$ can be dealt with easily, the main problem is to give an $2^{O(n^{1-1/s})}$ bound in the case $d \ge n^{1-1/s}/(2\log n)$. One can again introduce the notion of a dangerous set, which now is the larger partite set in a copy of $K_{s-1,t}$ in G (the other possibility, i.e., looking for copies of $K_{s,t-1}$, can be ruled out quite easily – under our assumptions on d, the double counting argument used in Lemma 11 cannot even prove existence of a single copy of $K_{s-1,t}$ in G; this should not come at a surprise, as we know that $\mathrm{ex}(n, K_{s-1,t}) \ll n^{2-1/s}$ and most likely $\mathrm{ex}(n, K_{s,t-1}) = \Theta(n^{2-1/s})$). Using Lemma 11, we prove that every set of a

vertices of G contains at least $\alpha \cdot a^t \approx d^{(s-1)t}/n^{(s-1)(t-1)} \cdot a^t$ dangerous sets, provided that $a \geq t\binom{n}{s-1}/\binom{d}{s-1}$. Then with the help of an algorithm very similar to $\mathcal{A}$, one could try to reprove versions of Claim 15 and Corollary 16, which would imply the desired upper bound. Here lies the difficulty – the set $X \subseteq N_{G'}(v)$ would have to be of size $(t-1) \cdot (\log n)/\alpha$, and one can see that this is optimal, since one iteration of $\mathcal{A}$ adds $(t-1)$ elements to X, shrinks the set A by multiplicative factor $(1-\alpha)$, and in the end we clearly want $|A| = o(n)$. A simple computation shows that now $|X| \gg (t-1)d^{t-s} \geq (t-1)d \geq |N_{G'}(v)|$, which is impossible.

Let H be a bipartite graph obtained from the complete bipartite graph $K_{m,m}$ by growing a tree out of each vertex, such that all the trees are pairwise vertex-disjoint. Since in a graph G with large minimum degree, one can find a copy of any fixed-size tree T, even requiring of T to be rooted at a specified vertex and of the vertex set of T to avoid a specified small subset of the set of vertices of G, it is straightforward to reprove Lemma 11 with $K_{m,m}$ replaced with H. Consequently, one can reprove Lemma 14 with appropriately defined dangerous sets. Following the proof of Theorem 2 from there on gives

$$\log_2 f_n(H) \leq (1+o(1))\frac{m(m-1)^{1/m}}{2m-1}C_m \cdot n^{2-1/m}.$$

Finally, in [3] it is said that any bound on the number of $K_{3,3}$-free graphs of small size that is similar to the one we obtained as Corollary 5 seems to be the only missing ingredient needed to prove Conjecture 31 from [3] with $a_0 = a_1 = 3$. The conjecture says that given $a_0 \leq \ldots \leq a_p$, the vertex set of almost every $K(a_0, \ldots, a_p)$-free graph G of order n admits a partition $(U_1, \ldots, U_p)$ where $G[U_1]$ is $K(a_0, a_1)$-free, and for all $i > 1$, $G[U_i]$ is a graph with maximum degree less than a_1.

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