

Large bounded degree trees in expanding graphs

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Abstract

A remarkable result of Friedman and Pippenger [3] gives a sufficient condition on the expansion properties of a graph to contain all small trees with bounded maximum degree. Haxell [4] showed that under slightly stronger assumptions on the expansion rate, her technique allows one to find arbitrarily large trees with bounded maximum degree. Using a slightly weaker version of Haxell's result we prove that a certain family of expanding graphs, which includes very sparse random graphs and regular graphs with large enough spectral gap, contains all almost spanning bounded degree trees. This improves two recent tree-embedding results of Alon, Krivelevich and Sudakov [1].

1 Introduction

A very well-known folklore result on tree-embedding states that every graph with minimum degree at least k contains all trees with at most k edges and this is best possible (as illustrated by an arbitrarily large disjoint union of $(k+1)$ -vertex complete graphs). A natural question arises – what additional assumptions on a graph can force it to contain certain trees? For an arbitrary graph H and a set $X \subseteq V(H)$, let $N_H(X)$ denote the set of neighbors in H of vertices in X . Extending a path-embedding result of Pósa [6], Friedman and Pippenger [3] proved that all graphs satisfying certain expansion properties contain all small trees with bounded maximum degree.

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Theorem 1 ([3]). *Let m and d be positive integers and let H be a non-empty graph. Moreover, assume that every $X \subseteq V(H)$ with $|X| \leq 2m$ satisfies $|N_H(X)| \geq (d + 1)|X|$. Then H contains every tree with m vertices and maximum degree at most d .*

An apparent shortcoming of Theorem 1 is that it can be helpful in finding only relatively small trees. Namely, in a graph of order n , the size of the largest tree the existence which is guaranteed by Theorem 1 is only about $n/(2d + 2)$, where d is the maximum degree of the tree. Building on the ideas developed by Friedman and Pippenger [3], Haxell [4] managed to overcome this problem.

Theorem 2 ([4]). *Let T be a tree with t edges and maximum degree d . Let $\emptyset = T_0 \subseteq T_1 \subseteq \dots \subseteq T_\ell \subseteq T$ be a sequence of subtrees of T such that T can be obtained by attaching new leaves to T_ℓ . Let $d = d_1 \geq \dots \geq d_\ell$ be a sequence of integers such that for each i with $1 \leq i \leq \ell$ and each $v \in V(T)$ we have*

$$\delta_T(v) - \delta_{T_{i-1}}(v) \leq d_i,$$

where $\delta_S(v)$ denotes the degree of v in the subtree S (if $v \notin V(S)$, then we let $\delta_S(v) = 0$). Let $t_i = |E(T_i)|$. Suppose that $m \geq 1$ is an integer and H is a graph satisfying the following conditions

1. For every subset $X \subseteq V(H)$ with $0 < |X| \leq m$, $|N_H(X)| \geq d|X| + 1$.
2. For every subset $X \subseteq V(H)$ with $m < |X| \leq 2m$, and for each $i \in \{1, \dots, \ell\}$, $|N_H(X)| \geq d_i|X| + t_i + 1$.
3. For every subset $X \subseteq V(H)$ with $|X| = 2m + 1$, $|N_H(X)| \geq t + 1$.

Then G contains T as a subgraph. Moreover, for any vertex x_0 of T_1 and any $y \in V(H)$, there exists an embedding f of T into H such that $f(x_0) = y$.

As an immediate corollary of the somewhat technical Theorem 2, we derive the following statement.

Theorem 3. *Let d , m and M be positive integers, and let $0 \leq L \leq 2dm$. Assume that H is a non-empty graph satisfying the following two conditions.*

1. For every $X \subseteq V(H)$ with $0 < |X| \leq m$, $|N_H(X)| \geq d|X| + 1$.
2. For every $X \subseteq V(H)$ with $m < |X| \leq 2m$, $|N_H(X)| \geq d|X| + M$.

Then H contains every tree T with $M + L$ vertices and maximum degree at most d , provided that T has at least L leaves.

It turns out that Theorem 3 has a few very interesting and yet quite straightforward consequences. First of all, it gives a sufficient condition on the edge probability that almost surely forces the Erdős-Rényi random graph $G(n, p)$ to contain all nearly spanning bounded degree trees.

Theorem 4. *Let $d \geq 2$ and $0 < \varepsilon < 1/2$. If*

$$c > \max \left\{ 1000d \log(20d), \frac{30d}{\varepsilon} \log \frac{4e}{\varepsilon} \right\},$$

then the random graph $G(n, c/n)$ almost surely contains every tree with maximum degree d and order at most $(1 - \varepsilon)n$.

Theorem 4 significantly improves the ‘ $c \geq \frac{10^6 d^3 \log d \log^2(2/\varepsilon)}{\varepsilon}$ ’, lower bound on the edge probability obtained by Alon, Krivelevich and Sudakov [1] with a lengthier and more complex argument making use of Theorem 1. Recently, in his doctoral thesis [5] the third author, using Theorem 2 and a refinement of the piece-by-piece embedding method from [1], obtained an improvement of the above mentioned result of Alon, Krivelevich and Sudakov [1] that is slightly weaker than Theorem 4. Note that in [1] it is suggested that in the statement of Theorem 4 the condition on the constant c could be lowered to $\Theta(d \log(1/\varepsilon))$.

For an n -vertex graph G , let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of its adjacency matrix, where $\lambda_1 \geq \dots \geq \lambda_n$. The second eigenvalue of G is $\lambda(G) := \max_{i \geq 2} |\lambda_i|$. A graph G is called an (n, D, λ) -graph if it is D -regular, has n vertices and its second eigenvalue is at most λ . It is well-known that if λ is much smaller than D , then G has strong expansion properties. The following result, which is another consequence of Theorem 3, shows that an (n, D, λ) -graph G with large spectral gap D/λ contains all almost spanning trees with bounded degree.

Theorem 5. *Let $d \geq 2$ and $0 < \varepsilon < 1/2$. If*

$$\frac{D}{\lambda} > \frac{\sqrt{8d}}{\varepsilon},$$

then every (n, D, λ) -graph contains all trees with maximum degree d and order at most $(1 - \varepsilon)n$.

Theorem 5 is again an improvement over the ‘ $\frac{D}{\lambda} \geq \frac{160d^{5/2} \log(2/\varepsilon)}{\varepsilon}$ ’, lower bound obtained by Alon, Krivelevich and Sudakov [1].

Finally, the lower bounds on c and D/λ in Theorems 4 and 5 can be further improved if we restrict our attention to trees with large number of leaves.

Theorem 6. *Let $d \geq 2$, $0 < \varepsilon < 1/2$ and $0 < \lambda < 1$. If*

$$c > \max \left\{ 1000d \log(20d), \frac{32d}{\lambda} \log \frac{4e}{\varepsilon} \right\},$$

then the random graph $G(n, c/n)$ almost surely contains every tree with maximum degree d and order at most $(1 - \varepsilon)n$, provided that it has at least λn leaves.

Theorem 7. *Let $d \geq 2$, $0 < \varepsilon < 1/2$ and $0 < \lambda < 1$. If*

$$\frac{D}{\lambda} > \sqrt{\frac{18d}{\varepsilon\lambda}},$$

then every (n, D, λ) -graph contains every tree with maximum degree d and order at most $(1 - \varepsilon)n$, provided that it has at least λn leaves.

The remainder of this note is organized as follows. In Section 2 we introduce a notion of graph expansion that gives rise to a certain family of expanding graphs, which we call (ε, b, α) -expanders, and prove that under certain assumptions on the expansion parameters ε, b and α , every such graph contains all almost spanning bounded degree trees. The most technical (but standard) parts Sections 3 and 4, are entirely devoted to the study of expansion properties of random graphs and graphs with large spectral gap. Finally, in Section 5, based on this study, we give very short proofs of our main results – Theorems 4, 5, 6 and 7.

2 Embedding trees in expanding graphs

We start by defining a class of expanding graphs that seems to be most adequate and convenient in our further considerations.

Definition 8. Let $b \geq 2$, $0 < \alpha < 1$ and $0 < \varepsilon < 1/b$. We will say that an n -vertex graph G is an (ε, b, α) -expander if it possesses the following two properties.

1. Every subset $X \subseteq V(G)$ of size at most εn satisfies $|N_G(X)| \geq b|X|$.
2. Every subset $X \subseteq V(G)$ of size at least εn satisfies $|N_G(X)| \geq (1 - \alpha)n$.

As immediate consequences of Theorem 3 we derive the following sufficient conditions on the expansion parameters ε, b and α which guarantee that all (ε, b, α) -expanders contain every almost spanning tree with bounded maximum degree and, additionally, many leaves.

Corollary 9. *Let $d \geq 2$ and $0 < \varepsilon < 1$. Suppose that $\alpha, \varepsilon_0 > 0$ are such that $2d\varepsilon_0 + \alpha \leq \varepsilon$. Then every n -vertex $(\varepsilon_0, d + 1, \alpha)$ -expander contains all trees of order $(1 - \varepsilon)n$ and maximum degree d .*

Proof. Let G be an n -vertex $(\varepsilon_0, d + 1, \alpha)$ -expander. It is straightforward to check that G satisfies assumptions of Theorem 3 with $m := \varepsilon_0 n$, $M := (1 - 2d\varepsilon_0 - \alpha)n$ and $L := 0$. Hence G contains every tree with maximum degree d and order $M \geq (1 - \varepsilon)n$. \square

Corollary 10. *Let $d \geq 2$ and $0 < \varepsilon, \lambda < 1$. Then every n -vertex $(\lambda/(2d), d + 1, \varepsilon)$ -expander contains all trees of order $(1 - \varepsilon)n$ and maximum degree d which contain at least λn leaves.*

Proof. Let G be an n -vertex $(\lambda/(2d), d + 1, \varepsilon)$ -expander. It is straightforward to check that G satisfies assumptions of Theorem 3 with $m := \lambda n/(2d)$, $L := \lambda n$ and $M := (1 - \varepsilon - \lambda)n$. Hence G contains every tree T with maximum degree d and order $M + L = (1 - \varepsilon)n$, provided that T has at least λn leaves. \square

3 Expanding properties of random graphs

Lemma 11. *Let $0 < \beta \leq \gamma \leq 1/2$ and $c \geq \frac{3}{\beta} \log \frac{e}{\gamma}$. Then with high probability the random graph $G(n, c/n)$ does not contain two disjoint sets B, C of size at least βn and γn respectively, such that $e(B, C) = 0$.*

Proof. If $G(n, c/n)$ contains two sets B and C as in the statement of this Lemma, clearly we can also find two disjoint sets B' and C' of size exactly βn and γn respectively, with $e(B', C') = 0$. The probability that such a pair exists is at most

$$\binom{n}{\beta n} \binom{n}{\gamma n} \cdot \left(1 - \frac{c}{n}\right)^{\beta\gamma n^2} \leq \binom{n}{\gamma n}^2 \cdot e^{-c\beta\gamma n} \leq \left(\frac{en}{\gamma n}\right)^{2\gamma n} \cdot \left(\frac{e}{\gamma}\right)^{-3\gamma n} = o(1).$$

□

Lemma 12. *Let $0 < \beta \leq \gamma \leq 1/2$ and let $c \geq \frac{6\gamma}{\beta} \log \frac{e}{\gamma}$. Then almost surely $G(n, c/n)$ does not contain a pair of disjoint sets B and C of sizes at least βn and at least $(1 - \gamma)n$ respectively with $e(B, C) = 0$.*

Proof. As in the proof of Lemma 11, we only need to show that almost surely there is no such pair with sizes exactly βn and $(1 - \gamma)n$. The probability that such a pair exists is at most

$$\binom{n}{\beta n} \binom{n}{(1 - \gamma)n} \left(1 - \frac{c}{n}\right)^{\beta(1 - \gamma)n^2} \leq \binom{n}{\gamma n}^2 \cdot e^{-c\beta n/2} \leq \left(\frac{en}{\gamma n}\right)^{2\gamma n} \cdot \left(\frac{e}{\gamma}\right)^{-3\gamma n} = o(1).$$

□

Lemma 13. *Let $k \geq 2$ and let $c \geq 10k \log_2 k$. Then almost surely every subset A of at most $n/(ek)$ vertices in the random graph $G(n, c/n)$ spans less than $c|A|/k$ edges.*

Proof. Certainly, if a subset A of size a violates the assertion, $a \geq c/k$. The probability that there is a bad subset A of size a , with $c/k \leq a \leq n/ek$, is at most

$$\begin{aligned} \binom{n}{a} \binom{a^2/2}{ac/k} \cdot \left(\frac{c}{n}\right)^{ca/k} &\leq \left(\frac{en}{a}\right)^a \cdot \left(\frac{ea^2}{2} \cdot \frac{k}{ac}\right)^{ca/k} \cdot \left(\frac{c}{n}\right)^{ca/k} = \left(\frac{en}{a} \cdot \left(\frac{eka}{2n}\right)^{c/k}\right)^a \\ &\leq \left(\frac{(ek/2)^{c/k+1}}{(n/a)^{c/k-1}}\right)^a \leq \left(\frac{(ek/2)^{11}}{(n/a)^9}\right)^a = o(n^{-1}), \end{aligned}$$

where the last estimate can be easily seen by considering two cases – $c/k \leq a \leq \sqrt{n}$ and $\sqrt{n} \leq a \leq n/ek$. Summing the estimates over all values of a yields the result. □

Lemma 14. *Let $0 < \alpha < 1/2$. If $c > 64 \log \frac{e}{\alpha}$, then with high probability the random graph $G(n, c/n)$ contains an induced subgraph G' with at least $(1 - \alpha)n$ vertices and minimum degree at least $c/4$.*

Proof. Let G be our random graph $G(n, c/n)$. While G contains a vertex with degree less than $c/4$, delete that vertex. Denote the remaining induced subgraph of G by G' . If G' has at least $(1 - \alpha)n$ vertices, we have found the subgraph we were looking for. It suffices to show that the probability of G' having less than $(1 - \alpha)n$ vertices is small. First observe that if we were forced to delete more than αn vertices, then the original graph G contained a set A of size αn such that $e_A := e(A, V(G) - A) < \alpha cn/4$. Note that $\mathbb{E}[e_A] = \alpha(1 - \alpha)cn \geq \alpha cn/2$. By standard Chernoff-type estimates, the probability of this event in our random graph is at most

$$\mathbb{P}(e_A < \alpha cn/4) \leq \mathbb{P}(e_A - \mathbb{E}[e_A] < -\alpha cn/4) \leq e^{-\frac{(\alpha cn/4)^2}{2\alpha cn}} = e^{-\alpha cn/32}.$$

Hence the probability that such a set A exists in our graph G is bounded by

$$\binom{n}{\alpha n} \cdot e^{-\alpha cn/32} \leq \left(\frac{en}{\alpha n}\right)^{\alpha n} \cdot \left(\frac{e}{\alpha}\right)^{-2\alpha n} = o(1).$$

□

Theorem 15. *Let $b \geq 2$ and $0 < \rho \leq \varepsilon \leq \alpha < 1/2$, where $\varepsilon < 1/(2b + 4)$. If*

$$c > \max \left\{ 500b \log(12b), \frac{6}{\varepsilon} \log \frac{2e}{\alpha}, 64 \log \frac{e}{\rho} \right\},$$

then with high probability the random graph $G(n, c/n)$ contains an induced subgraph G' of order at least $(1 - \rho)n$ that is an (ε, b, α) -expander.

Proof. By Lemma 14, almost surely $G(n, c/n)$ contains an induced subgraph G' of order n' , with $n' \geq (1 - \rho)n$ and $\delta(G') \geq c/4$. Conditioning on that event, we will show that G' is an (ε, b, α) -expander with high probability.

Suppose that G' fails to possess property 1 from Definition 8. Then there is a set $X \subseteq V(G')$ of size t , with $t \leq \varepsilon n'$ and $|N_{G'}(X)| \leq bt$. Let $A := X \cup N_{G'}(X)$. Clearly $|A| \leq (b + 1)t$. We consider three cases, depending on the order of t .

Case 1. $t \leq \frac{n}{8e(b+1)^2}$.

Let $k := 8(b + 1)$. Since edges incident to vertices in X are contained in A , $e(A) \geq \delta(G')|X|/2 \geq ct/8 \geq c|A|/k$. By our assumptions, $|A| \leq n/(ek)$, and $c > 10k \log_2 k$. By Lemma 13, such non-expanding set X almost surely does not exist.

Case 2. $\frac{n}{8e(b+1)^2} \leq t \leq \frac{n}{20e(b+1)}$.

Since G' is an induced subgraph, in G there are no edges between X and $Y := V(G') - A$. By our assumptions on t and ε , the latter set has at least

$$n' - |A| \geq (1 - \rho)n - (b + 1)t \geq n - n/(b + 1) - (b + 1)t \geq n - (8e + 1)(b + 1)t$$

vertices. Let $\beta := t/n$ and $\gamma := (8e + 1)(b + 1)\beta$. Since $\beta \geq \frac{1}{8e(b+1)^2}$, then $c > 6\frac{\gamma}{\beta} \log \frac{e}{\gamma}$ and, by Lemma 12, such non-expanding set X almost surely does not exist.

Case 3. $\frac{n}{20e(b+1)} \leq t \leq \varepsilon n'$.

Again, in G there are no edges between X and $Y := V(G') - A$. By our assumptions on t and ε , the latter set has at least

$$n' - |A| \geq (1 - (b + 1)\varepsilon)n' \geq (1 - (b + 1)\varepsilon)(1 - \varepsilon)n \geq (1 - (b + 2)\varepsilon)n \geq \frac{n}{2}$$

vertices. Let $\beta := \frac{1}{20e(b+1)}$ and $\gamma := 1/2$. Clearly $c > (3/\beta) \log(e/\gamma)$. By Lemma 11, such non-expanding set X almost surely does not exist.

Hence almost surely the graph G' satisfies property 1 from definition 8. Finally, suppose that G' fails to possess the other property. Then there is a set X of size exactly $\varepsilon n'$ with $|N_{G'}(X)| \leq (1 - \alpha)n'$. It follows that in G there are no edges between X and $Y := V(G') - X - N_G(X)$. Clearly Y contains at least $\alpha n' \geq \alpha n/2$ vertices. Let $\beta := \varepsilon/2$ and $\gamma := \alpha/2$. Since $c > (3/\beta) \log \frac{\varepsilon}{\gamma}$, by Lemma 11, this almost surely does not happen. \square

4 Expanding properties of quasi-random graphs

For two not necessarily disjoint subsets of the set of vertices of a graph G , let

$$e(A, B) := |\{(a, b) \in A \times B : \{a, b\} \in E(G)\}|.$$

In [2], it is proved that for every two subsets A and B of the set of vertices of an (n, D, λ) -graph G ,

$$\left| e(A, B) - \frac{|A||B|D}{n} \right| \leq \lambda \sqrt{|A||B|}. \quad (1)$$

Theorem 16. *Let $b \geq 2$, $\alpha > 0$ and $0 < \varepsilon < 1/b$. If*

$$\frac{D}{\lambda} > \max \left\{ \frac{\sqrt{b}}{1 - b\varepsilon}, \frac{1}{\sqrt{\alpha\varepsilon}} \right\} \quad (2)$$

then every (n, D, λ) -graph G is an (ε, b, α) -expander.

Proof. Suppose that G fails to possess property 1 from Definition 8. Then there is a set $X \subseteq V(G')$ of size t , with $t \leq \varepsilon n$ and $Y := N_G(X) \leq bt$. Since G is D -regular, clearly $e(X, Y) \geq Dt$. On the other hand, by (1) and (2),

$$\begin{aligned} e(X, Y) &\leq \frac{|X||Y|D}{n} + \lambda \sqrt{|X||Y|} = \frac{bt^2 D}{n} + \lambda \sqrt{bt} = Dt \left(\frac{bt}{n} + \frac{\lambda \sqrt{b}}{D} \right) \\ &\leq Dt \left(b\varepsilon + \frac{\lambda \sqrt{b}}{D} \right) < Dt(b\varepsilon + (1 - b\varepsilon)) = Dt, \end{aligned}$$

which is a clear contradiction.

Finally, suppose that G fails to have property 2 from Definition 8. Then there are sets $X, Y \subseteq V(G')$ with sizes εn and αn respectively such that $e(X, Y) = 0$. But, by (1) and (2),

$$e(X, Y) \geq \frac{|X||Y|D}{n} - \lambda\sqrt{|X||Y|} = \alpha\varepsilon Dn - \sqrt{\alpha\varepsilon\lambda}n > 0.$$

Again, this is a contradiction. □

5 Proofs of Theorems 4, 5, 6 and 7

Proof of Theorem 4. Let $\varepsilon_0 := \frac{\varepsilon}{4d+2}$. By Theorem 15, (substituting with $\alpha := \varepsilon/2$, $b := d+1$, $\rho = \varepsilon_0$ and $\varepsilon := \varepsilon_0$), $G(n, c/n)$ almost surely contains a subgraph G' of order at least $(1 - \varepsilon_0)n$, which is an $(\varepsilon_0, d+1, \varepsilon/2)$ -expander. By Corollary 9, G' contains every tree with maximum degree d and order

$$(1 - 2d\varepsilon_0 - \varepsilon/2)|V(G')| \geq (1 - (4d+1)\varepsilon_0) \cdot (1 - \varepsilon_0)n \geq (1 - \varepsilon)n.$$

□

Proof of Theorem 6. Let $\varepsilon_0 := \frac{\lambda}{2d}$. By Theorem 15, (substituting with $\alpha := \varepsilon/2$, $\rho := \min\{\varepsilon_0, \varepsilon/2\}$, $\varepsilon := \varepsilon_0$, and $b := d+1$), $G(n, c/n)$ almost surely contains a subgraph G' of order at least $(1 - \rho)n$, which is an $(\varepsilon_0, d+1, \alpha)$ -expander. By Corollary 10, G' contains every tree T with maximum degree d and order

$$(1 - \varepsilon/2)|V(G')| \geq (1 - \varepsilon/2) \cdot (1 - \varepsilon/2)n \geq (1 - \varepsilon)n,$$

provided that T has at least λn leaves. □

Proof of Theorem 5. Let $\varepsilon_0 := \frac{\varepsilon}{4d}$. By Theorem 16, (substituting with $\alpha := \varepsilon/2$, $b := d+1$ and $\varepsilon := \varepsilon_0$), every (n, D, λ) -graph G is an $(\varepsilon_0, d+1, \varepsilon/2)$ -expander. By Corollary 9, G contains every tree with maximum degree d and order $(1 - 2d\varepsilon_0 - \varepsilon/2)n = (1 - \varepsilon)n$. □

Proof of Theorem 7. Let $\varepsilon_0 := \frac{\lambda}{2d}$. By Theorem 16, (substituting with $\alpha := \varepsilon$, $b := d+1$ and $\varepsilon := \varepsilon_0$), every (n, D, λ) -graph G is an $(\varepsilon_0, d+1, \varepsilon)$ -expander. By Corollary 10, G contains every tree T with maximum degree d and order $(1 - \varepsilon)n$, provided that T has at least λn leaves. □

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