

AN OPTIMAL LOWER CURVATURE BOUND FOR CONVEX HYPERSURFACES IN RIEMANNIAN MANIFOLDS

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ABSTRACT. It is proved that a convex hypersurface in a Riemannian manifold of sectional curvature $\geq \kappa$ is an Alexandrov's space of curvature $\geq \kappa$. This theorem provides an optimal lower curvature bound for an older theorem of Buyalo.

The purpose of this paper is to provide a reference for the following theorem:

Theorem 1. *Let M be a Riemannian manifold with sectional curvature $\geq \kappa$. Then any convex hypersurface $F \subset M$ equipped with the induced intrinsic metric is an Alexandrov's space with curvature $\geq \kappa$.*

Here is a slightly weaker statement:

Theorem 2. [Buyalo] *If M is a Riemannian manifold, then any convex hypersurface $F \subset M$ equipped with the induced intrinsic metric is locally an Alexandrov's space.*

In the proof of Theorem 2 in [Buyalo], the (local) lower curvature bound depends on (local) upper as well as lower curvature bounds of M . We show that the approach in [Buyalo] can be modified to give Theorem 1.

Definition 3. *A locally Lipschitz function f on an open subset of a Riemannian manifold is called λ -concave ($\lambda \in \mathbb{R}$) if for any unit-speed geodesic γ , the function*

$$t \mapsto f \circ \gamma(t) - \frac{\lambda}{2}t^2$$

is concave.

Lemma 4. *Let $f: \Omega \rightarrow \mathbb{R}$ be a λ -concave function on an open subset Ω of a Riemannian manifold. Then there is a sequence of nested open domains Ω_i , with $\Omega_i \subset \Omega_j$ for $i < j$ and $\cup_i \Omega_i = \Omega$, and a sequence of smooth λ_i -concave functions $f_i: \Omega_i \rightarrow \mathbb{R}$ such that*

- (i) *on any compact subset $K \subset \Omega$, f_i converges uniformly to f ;*
- (ii) *$\lambda_i \rightarrow \lambda$ as $i \rightarrow \infty$.*

This lemma is a slight generalization of [Greene–Wu, Theorem 2] and can be proved exactly the same way.

Proof of Theorem 1. Without loss of generality one can assume that

- (a) $\kappa \geq -1$,

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- (b) F bounds a compact convex set C in M ,
- (c) there is a (-2) -concave function μ defined in a neighborhood of C and $|\mu(x)| < 1/10$ for any $x \in C$,
- (d) there is unique minimal geodesic between any two points in C .

(If not, rescale and pass to the boundary of the convex piece cut by F from a small convex ball centered at $x \in F$, taking $\mu = -10 \operatorname{dist}_x^2$.)

Consider the function $f = \operatorname{dist}_F$. By Rauch comparison (as in [Petersen, 11.4.8]), for any unit-speed geodesic γ in the interior of C , $(f \circ \gamma)''$ is bounded in the barrier sense by the corresponding value in the model case — when M is Lobachevsky plane and F is a geodesic. In particular,

$$(f \circ \gamma)'' \leq f \circ \gamma.$$

Therefore $f + \varepsilon \mu$ is $(-\varepsilon)$ -concave in $\Omega_\varepsilon = f^{-1}((0, \varepsilon)) \cap C$. Take $K_\varepsilon = f^{-1}([\frac{1}{3}\varepsilon, \frac{2}{3}\varepsilon]) \cap C$. Applying lemma 4, we can find a smooth $(-\frac{\varepsilon}{2})$ -concave function f_ε which is arbitrarily close to $f + \varepsilon \mu$ on K_ε and which is defined on a neighborhood of K_ε . Take a regular value $\vartheta_\varepsilon \approx \frac{1}{2}\varepsilon$ of f_ε . (In fact one can take $\vartheta_\varepsilon = \frac{1}{2}\varepsilon$, but it requires a little work.) Since $|\mu|_C < 1/10$, the level set $F_\varepsilon = f_\varepsilon^{-1}(\vartheta_\varepsilon)$ will lie entirely in K_ε . Therefore F_ε forms a smooth closed convex hypersurface.

Let us denote by ρ and ρ_ε the induced intrinsic metrics on correspondingly F and F_ε . By the Gauss formula, $(F_\varepsilon, \rho_\varepsilon)$ has curvature $\geq \kappa$. Further, F_ε bounds a compact convex set C_ε and $F_\varepsilon \rightarrow F$, $C_\varepsilon \rightarrow C$ in Hausdorff sense as $\varepsilon \rightarrow 0$. By property (d), the restricted metrics from M to C and to C_ε are intrinsic. Thus, C_ε is an Alexandrov space with F_ε as boundary, that converges in Gromov–Hausdorff sense to C . It follows from [Petrunin, Theorem 1.2] (compare [Buyalo, Theorem 1]) that $(F_\varepsilon, \rho_\varepsilon)$ converges in Gromov–Hausdorff sense to (F, ρ) . Therefore (F, ρ) is an Alexandrov space with curvature $\geq \kappa$. \square

Remark. We are not aware of any proof of theorem 1 which is not based on the Gauss formula. (Although if M is Euclidean space, there is a beautiful purely synthetic proof in [Milka].) Finding such a proof would be interesting on its own, and also could lead to the generalization of theorem 1 to the case when M is an Alexandrov space.

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