# TOPOLOGY OF NONNEGATIVELY CURVED HYPERSURFACES WITH PRESCRIBED BOUNDARY IN $\mathbf{R}^n$

STEPHANIE ALEXANDER, MOHAMMAD GHOMI, AND JEREMY WONG

Abstract. We prove that a smooth compact immersed submanifold of codimension 2 in $\mathbf{R}^n$, $n \geq 3$, bounds at most finitely many topologically distinct compact nonnegatively curved hypersurfaces. Analogous results for noncompact fillings are obtained as well. On the other hand, we show that these topological finiteness theorems may not hold if the prescribed boundary is not sufficiently regular. In particular we construct a simple closed differentiable and rectifiable curve in $\mathbf{R}^3$ which bounds infinitely many topologically distinct smooth positively curved surfaces. The proofs employ, among other methods, theorems of Gromov and Perelman on Alexandrov spaces with curvature bounded below.

Contents



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## 1. Introduction

Recently there has been considerable interest in studying the structure of locally convex hypersurfaces with boundary in Euclidean space [2, 3, 12, 13, 14, 16, 18, 21, 26, 30]. A fundamental open problem in this area is deciding when a simple closed curve $\Gamma$ in $\mathbf{R}^3$ bounds a surface of positive curvature [36, Problem 26]. The primary question we address in this work is whether $\Gamma$, or more generally a closed submanifold of codimension 2 in $\mathbf{R}^n$, may bound infinitely many topologically distinct types of nonnegatively curved hypersurfaces. We begin by examining the key role played by the regularity of the prescribed boundary.

**Theorem 1.1.** *There exists a simple closed rectifiable curve in $\mathbf{R}^3$ that is differentiable in its arclength parameter, is $\mathcal{C}^\infty$ in the complement of two points, and bounds infinitely many topologically distinct, compact, embedded, positively curved $\mathcal{C}^\infty$ surfaces.*

In contrast, the following theorem shows that if the regularity of the bounding curve is increased merely to *finite turn*, then there are only finitely many such "fillings" or "spanning surfaces", even if their regularity is relaxed to the locally convex category. Finite turn corresponds to rectifiability of the tangent indicatrix, and is considerably weaker than, for example, piecewise $\mathcal{C}^{1,1}$. By a *locally convex immersed* hypersurface $M$ with boundary, we mean a map $f : M \to \mathbf{R}^{n+1}$ that has an extension to a continuous map $\widetilde{f}$ of a manifold without boundary $\widetilde{M}$, where every point of $M$ has a neighborhood in $\widetilde{M}$ that is embedded by $\widetilde{f}$ into the boundary of some convex body.

**Theorem 1.2.** *A finite collection of closed curves of finite turn immersed in $\mathbf{R}^3$ bounds at most finitely many topologically distinct, compact, locally convex immersed surfaces.*

Assuming greater regularity, one can prove a much stronger finiteness result for spanning surfaces. In the following theorem, the ambient space is an arbitrary Riemannian manifold, of arbitrary codimension, and the spanning surfaces need not be compact.

**Theorem 1.3.** *A finite collection of closed $\mathcal{C}^3$ curves immersed in a given Riemannian manifold bounds at most finitely many topologically distinct, complete, immersed $\mathcal{C}^3$ surfaces whose total curvature is uniformly bounded below.*

The proofs of Theorems 1.2 and 1.3 draw on the classical resource of Gauss-Bonnet. To investigate to what extent Theorem 1.3 extends to higher dimenions, we turn to the Gromov-Hausdorff convergence theory. We start with the following finiteness theorem for Riemannian fillings of arbitrary dimension and codimension in the presence of bounds on curvature and diameter:

**Theorem 1.4.** *Let $\Gamma$ be a compact (but not necessarily connected) submanifold of a given Riemannian manifold. Then $\Gamma$ bounds finitely many distinct topological types of compact immersed submanifolds with uniform bounds below on curvature and above on intrinsic diameter.*

The diameter bound in Theorem 1.4 cannot in general be removed, even for positively curved fillings in Euclidean spaces, as the following example shows.

**Theorem 1.5.** *There is a* 2*-dimensional submanifold of* $\mathbf{R}^{19}$ *that bounds infinitely many topologically distinct smooth compact* 3*-dimensional submanifolds, each of which has constant positive curvature.*

However, it is possible to remove the diameter bound in Theorem 1.4 when $\Gamma$ is a submanifold of codimension 2 in Euclidean space and the filling hypersurface is nonnegatively curved. In this setting, we exploit convexity as an additional resource in order to prove our next two theorems.

**Theorem 1.6.** *A compact (but not necessarily connected),* $\mathcal{C}^3$ *immersed submanifold of codimension* 2 *in* $\mathbf{R}^{n+1}$, $n > 2$, *bounds at most finitely many topologically distinct, compact, nonnegatively curved* $\mathcal{C}^3$ *immersed hypersurfaces.*

Moreover, topological finiteness extends to *noncompact* positively curved fillings, according to the following theorem:

**Theorem 1.7.** *A compact (but not necessarily connected),* $\mathcal{C}^3$ *immersed submanifold of codimension* 2 *in* $\mathbf{R}^{n+1}$, $n > 2$, *bounds at most finitely many topologically distinct, complete, positively curved* $\mathcal{C}^3$ *immersed hypersurfaces.*

The proofs of these results are presented in the following sections. Theorem 1.1 is proved in Section 2 by means of a direct construction which develops certain gluing techniques, including a bridge principal, for positively curved surfaces with boundary. Theorems 1.2 and 1.3, which are concerned with 2-dimensional fillings and draw on Gauss-Bonnet theory, are proved in Section 3. Theorems 1.4 and 1.6 concern higher-dimensional compact fillings. They are proved in Section 4 with the aid of results of Gromov and Perelman on compactness and stability of Alexandrov spaces with curvature bounded below, together with extension results for manifolds with boundary and some diameter estimates. Theorem 1.5 is also proved in this section, using a relative version of the Nash isometric embedding theorem. Theorem 1.7, which concerns complete open fillings, is proved in Section 5 via a clipping procedure, and a theorem of Cai on finiteness of the number of ends, which are used to reduce the proof to that of the compact case considered in Section 4.

## 2. The Example: Proof of Theorem 1.1

In this section we construct a differentiable curve in $\mathbf{R}^3$ with infinitely many topologically distinct positively curved filling surfaces. The proof is presented in the setting of a surgery procedure for locally convex surfaces. In particular we use a gluing result (Theorem 2.1) which in turn is proved with the aid of a bridge principal (Proposition 2.7).

2.1. **Overview.** Figure 1 shows a picture of the curve $\Gamma$, together with three of its positively curved fillings, which we will construct here. Each of these filling surfaces is symmetric with respect to a horizontal plane. The first surface on the left, say $M_0$, is homeomorphic to a disk. It consists of two spherical pieces which are connected by a strip of positive curvature. The surface in the middle, $M_1$, is obtained by adding a handle to $M_0$ (without perturbing $\Gamma$) and is thus homeomorphic to a punctured torus. Similarly, the surface on the right, $M_2$, is obtained by adding a handle to

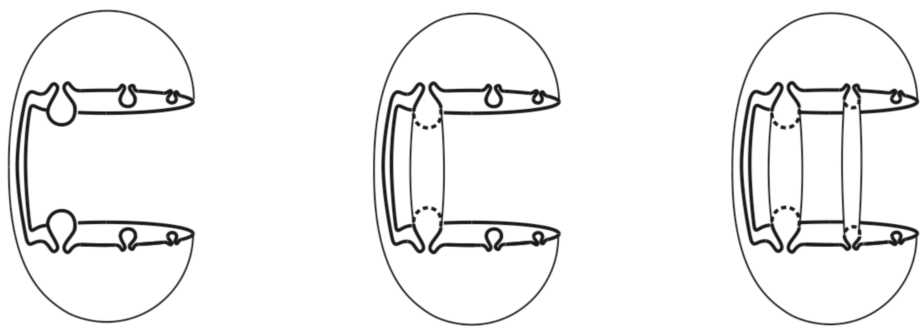

FIGURE 1

$M_1$. This process of addition of handles, or more precisely surgery where we remove a pair of disks from a surface and glue an annulus in their place, may be continued indefinitely because $\Gamma$ has an infinite pair of corrugations where handles may be added. Thus we obtain an infinite sequence of surfaces $M_i$, with $\partial M_i = \Gamma$ such that $g(M_i) = i$, where $g$ denotes the topological genus. Further we note that the corrugations of $\Gamma$ converge to a pair of points on $\Gamma$ with a sufficiently fast rate so that $\Gamma$ is rectifiable, is differentiable at the accumulation points with respect to the arclength parameter, and is $\mathcal{C}^\infty$ elsewhere. Furthermore, each of the surfaces $M_i$ is $\mathcal{C}^\infty$ and has positive curvature up to the boundary; this means that for every $p \in \Gamma$ there exists an open neighborhood $U \subset M_i$ containing $p$, and a $\mathcal{C}^\infty$-smooth positively curved surface $\widetilde{U}$ without boundary such that $\widetilde{U} \cap M_i = U$.

2.2. **Construction of $\Gamma$.** The general strategy here is to construct one half of the surface $M_0$ mentioned above, reflect it, and let $\Gamma$ be the boundary of the resulting surface. This is achieved in the following steps:

*Step 1.* Let $S \subset \mathbf{R}^3$ be a sphere whose center lies below the $xy$-plane, e.g. at $(0, 0, -1)$. Let $p_0 \in S$ be a point on the equator $\Gamma_0$ of $S$. Then for every $r > 0$ we may find an ellipsoid of revolution, symmetric with respect to the $xy$-plane, and with a vertical axis of revolution, which is tangent to exactly one point of $S$, say $q_0$, and intersects $S$ in a figure-eight curve which is contained entirely in $B_r(p_0)$, a ball of radius $r$ centered at $p_0$. Let the half of this ellipsoid which lies below the $xy$-plane be denoted by $E_0$. See the picture on the left in Figure 2.

*Step 2.* We deform $E_0$ to obtain a smooth surface of positive curvature $\overline{E}_0$ such that (i) $\overline{E}_0$ coincides with $S$ in an open neighborhood $U$ of $q_0$, (ii) $\overline{E}_0$ coincides with $E_0$ outside of $B_r(p_0)$, and (iii) $E_0 \cap S \subset \overline{E}_0$. This follows immediately from the following result, which will be proved later.

**Theorem 2.1.** *Let $\Sigma_1$ and $\Sigma_2$ be smooth strictly convex compact disks of positive curvature in $\mathbf{R}^3$ which are tangent to each other at a common interior point $p$.*

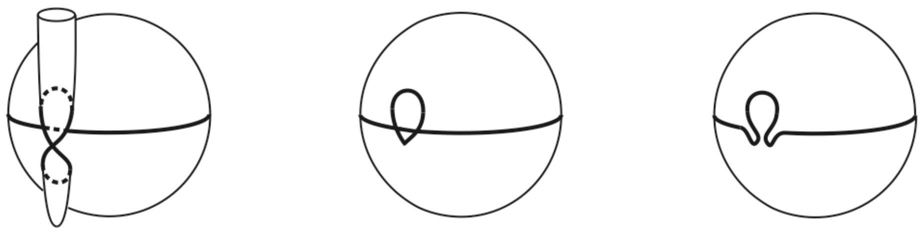

FIGURE 2

*Suppose that $\Sigma_1$ and $\Sigma_2$ lie locally on the same side of their common tangent plane at $p$, and that their intersection near $p$ consists of a finite number of smooth curve segments each of which emanates from $p$ in a distinct direction. Then for every open neighborhood $U_1$ of $p$ in $\Sigma_1$ there exists an open neighborhood $U_2$ of $p$ in $\Sigma_2$ and a smooth positively curved compact disk $\Sigma$ such that $(\Sigma_1 - U_1) \cup U_2 \subset \Sigma$, and $\Sigma$ contains $\Sigma_1 \cap \Sigma_2$.*

*Step 3.* We take the top loop from the figure-eight curve $E_0 \cap S$ and merge it with the equator $\Gamma_0$ of $S$ to obtain a smooth simple closed curve $\Gamma_1 \subset S$, as shown in the middle and right of Figure 2. See also Figure 3. Here the shaded region indicates the neighborhood $U$ mentioned in Step 2 above. The important point here is that $\Gamma_1$ contain a portion of the top loop of the figure-eight which begins and ends inside $U$.

FIGURE 3

*Step 4.* Now take a sequence of points $p_0, p_1, \dots$ on the equator $\Gamma_0$ of $S$ which converge monotonically to a point $p_\infty$ of $\Gamma_0$ with respect to the intrinsic distance in $\Gamma_0$. Let $r_0, r_1, \dots$ be a corresponding sequence of positive numbers so that the closures $B_{r_i}(p_i)$ are pairwise disjoint. Then repeating the first three steps, we obtain a sequence of smooth simple closed curves $\Gamma_i \subset S$. This sequence will converge, with respect to the Hausdorff metric, to a rectifiable curve $\Gamma_\infty$ which is differentiable everywhere, with respect to an arclength parametrization, and is smooth on $\Gamma_\infty - \{p_\infty\}$, provided that $r_i$ converge to zero at a sufficiently fast rate. In particular, we need to choose $r_i$ so small that $B_{r_i}(p_i)$ is disjoint from a pinched solid torus $T$ which wraps around $\Gamma_0$ at $p_\infty$, see Figure 4, such that the interior of $T$ is disjoint from $\Gamma_0$ (since $\Gamma_0$ is smooth, such a torus exists at any point of it). By a pinched solid torus we mean the set of points in $\mathbf{R}^3$ whose distance from a circle of radius $r$ is at most $r$.

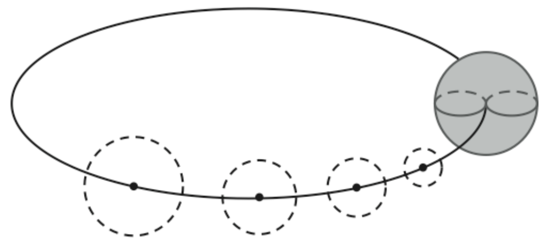

FIGURE 4

*Step 5.* By Jordan's curve theorem, $\Gamma_\infty$ divides $S$ into a pair of disks. Let $D$ be the bottom disk, see Figure 5. Now, using the same procedure described in Steps 1 and 2, we may glue to $D$ a piece of an ellipsoid, as depicted in Figure 5. Reflecting the boundary of the resulting surface with respect the $xy$-plane yields the desired curve $\Gamma$, which already bounds a smooth positively curved surface $M_0$ with positive curvature. Further, note that at the neck of each corrugation we have an open neighborhood where $M_0$ is tangent to an ellipsoid of revolution which contains the corrugation. So we may delete a pair of disks from $M_0$ which are bounded in part by a pair of corrugations and replace them with an annular piece of an ellipsoid. Thus we may construct a sequence of positively curved surfaces $M_i$ with $g(M_i) = i$ bounded by $\Gamma$. Note that each $M_i$ is smooth because, by construction, each boundary point of $M_i$ has an open neighborhood in $M_i$ which is part of an open neighborhood of a sphere or a slightly deformed ellipsoid.

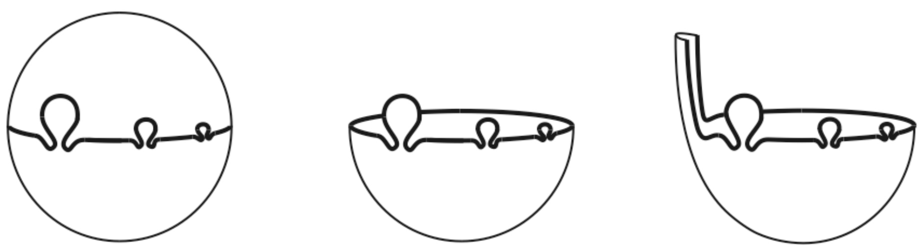

FIGURE 5

### 2.3. **Proof of Theorem 2.1.**

**Lemma 2.2.** *The space of positive definite $n \times n$ matrices is convex, i.e., if $A$ and $B$ are positive definite $n \times n$ matrices, then*

$$C := \lambda A + (1 - \lambda)B$$

*is also positive definite for all $0 \leq \lambda \leq 1$.*

*Proof.* A matrix $A$ is positive definite if and only if the associated quadratic form $Q_A(v) := v^T A v$ is positive definite. Further note that $Q_C(v) = \lambda Q_A(v) + (1 - \lambda) Q_B(v)$. Thus if $Q_A$ and $Q_B$ are positive definite, then so is $Q_C$. □

**Lemma 2.3.** *Let $U \subset \mathbf{R}^n$ be an open subset, $f$, $g$, $\phi\colon U \to \mathbf{R}$ be $\mathcal{C}^2$ functions, and set*

$$h := \phi f + (1-\phi)g.$$

*Suppose that $f(p) = g(p)$ and $\operatorname{grad} f(p) = \operatorname{grad} g(p)$ at some point $p \in U$. Then*

$$\operatorname{Hess} h(p) = \phi(p) \operatorname{Hess} f(p) + (1 - \phi(p)) \operatorname{Hess} g(p).$$

*In particular, if $f$ and $g$ have positive definite Hessians at $p$, and $0 \leq \phi(p) \leq 1$, then $h$ has positive definite Hessian at $p$ as well.*

*Proof.* Noting that $h = g + \phi(f - g)$, we easily compute the second partial derivatives

$$h_{ij} = g_{ij} + \phi_{ij}(f-g) + \phi_i(f_j - g_j) + \phi(f_{ij} - g_{ij}).$$

Thus, since $f(p) = g(p)$ and $f_i(p) = g_i(p)$, it follows that

$$h_{ij}(p) = g_{ij}(p) + \phi(p)(f_{ij}(p) - g_{ij}(p)),$$

which completes the proof. □

Let $B^n_r$ denote an open ball of radius $r$ centered at the origin of $\mathbf{R}^n$. Closure will be denoted by cl.

**Lemma 2.4.** *Let $f$, $g$: $\operatorname{cl}(B^2_r) \to \mathbf{R}$ be $\mathcal{C}^{k\geq 2}$ functions with positive definite Hessians, and $\Gamma \subset \operatorname{cl}(B^2_r)$ be a connected curve which meets $\partial B^2_r$ only at its boundary points $\partial\Gamma = \{(x_1, y_1), (x_2, y_2)\}$. Suppose that $x_1 < 0$, $x_2 > 0$, and $f(p) = g(p)$, $\operatorname{grad} f(p) = \operatorname{grad} g(p)$ for all $p \in \Gamma$. Then there exists an open neighborhood $U$ of $\Gamma$ in $\operatorname{cl}(B^2_r)$, an $\epsilon > 0$, and a $\mathcal{C}^k$ function $h\colon U \to \mathbf{R}$ with positive definite Hessian such that $f(p) = h(p) = g(p)$ for all $p \in \Gamma$, $h(x, y) = f(x, y)$ when $x \leq x_1 + \epsilon$, and $h(x, y) = g(x, y)$ when $x \geq x_2 - \epsilon$.*

*Proof.* Choose $\epsilon$ so that $x_1 + \epsilon < x_2 - \epsilon$. Then there exists a smooth function $\overline{\phi}\colon \mathbf{R} \to \mathbf{R}$ such that $0 \leq \phi \leq 1$, $\phi = 1$ when $x \leq x_1 + \epsilon$, and $\phi = 0$ when $x \geq x_2 - \epsilon$. Now define $\phi\colon \mathbf{R}^2 \to \mathbf{R}$ by $\phi(x, y) := \overline{\phi}(x)$ and set $h := \phi f + (1 - \phi)g$. □

**Lemma 2.5.** *Let $\Gamma \subset \mathbf{R}^3$ be a compact connected embedded curve with endpoints $\partial\Gamma = \{p, q\}$. Suppose that there exist $\mathcal{C}^k$ positively curved surfaces $M_i \subset \mathbf{R}^3$, $i = 1, \ldots, n$ which cover $\Gamma$, with $p \subset M_1$, $q \subset M_n$, such that each $\Gamma_i := M_i \cap \Gamma$ is a connected nonempty open subset of $\Gamma$, and whenever $\Gamma_i \cap \Gamma_j \neq \emptyset$, there exists an open neighborhood $U_{ij} \subset \Gamma_i \cap \Gamma_j$ where $M_i$ and $M_j$ are tangent for all $p \in U_{ij}$. Then there exists a $\mathcal{C}^k$ connected positively curved surface $M \subset \mathbf{R}^3$ which contains $\Gamma$. Further we may require that $M$ contain an open neighborhood of $p$ in $M_1$ and an open neighborhood of $q$ in $M_n$.*

*Proof.* After replacing $M_i$ by a subcollection and a reindexing we may assume that no subcollection of $\Gamma_i$ covers $\Gamma$, and $\Gamma_i \cap \Gamma_j \neq \emptyset$ if and only if $j = i \pm 1$. Then for $i = 1, \ldots, n-1$ there exists a connected open neighborhood $U_i \subset \Gamma_i \cap \Gamma_{i+1}$ where $M_i$ and $M_{i+1}$ are tangent. Now take a point $p_i \in U_i$. Then in a neighborhood of $p_i$, $M_i$ and $M_{i+1}$ may be represented as the graphs of functions with positive definite Hessians over a small ball centered at $p_i$ in the tangent plane $T_{p_i}M_i = T_{p_i}M_{i+1}$. Then using the previous lemma, we may smoothly glue $M_i$ and $M_{i+1}$ near $p_i$. Then if we let $\overline{M}_i$ be small enough open neighborhoods (or strips) of $\Gamma$ in $M_i$, $M := \cup_i \overline{M}_i$ is the desired surface. □

We say that a $\mathcal{C}^{k\geq 2}$ embedded curve $\Gamma \subset \mathbf{R}^3$ is strictly convex if $\Gamma$ has nonvanishing curvature and through every point $p \in \Gamma$ there passes a strict support plane $\Pi$, i.e., $\Gamma$ lies on one side of $\Pi$ and intersects $\Pi$ only at $p$. By a nonsingular support vector field $n\colon \Gamma \to \mathbf{S}^2$ we mean a vector field such that if $\Pi_{p,n}$ denotes the plane which passes through $p$ and is orthogonal to $n$, then $\Pi_{p,n}$ is a strict support plane of $\Gamma$ at $p$, and furthermore, $n(p)$ is not orthogonal to the principal normal of $\Gamma$ at $p$.

**Lemma 2.6** ([12]). *Let $\Gamma \subset \mathbf{R}^3$ be a compact $\mathcal{C}^k$ strictly convex curve. Then $\Gamma$ lies on a $\mathcal{C}^k$ surface $M$ of positive curvature without boundary. Furthermore, for any $\mathcal{C}^k$ nonsingular support vector field $n\colon \Gamma \to \mathbf{S}^2$ along $\Gamma$ we may require that $M$ be normal to $n$.*

Combining the last two lemmas now immediately yields the following bridge principal for positively curved surfaces:

**Proposition 2.7.** *Let $M \subset \mathbf{R}^3$ be a $\mathcal{C}^k$ compact (but not necessarily connected) embedded surface of positive curvature, and $\Gamma \subset \mathbf{R}^3$ be a connected locally strictly convex curve segment with $\partial\Gamma = \{p_1, p_2\} \subset \partial M$, $p_1 \neq p_2$. Suppose that $\Gamma$ meets $\partial M$ transversally, and that a neighborhood in $\Gamma$ of each $p_i$ lies in a local extension of $M$ at $p_i$. Then there exists a $\mathcal{C}^k$ compact surface of positive curvature $\overline{M}$ which contains $M$ and $\Gamma$. Further, if $n\colon \Gamma \to \mathbf{S}^2$ is a local nonsingular $\mathcal{C}^k$ support vector field along $\Gamma$, which is orthogonal to the local extensions of $M$ at $p_i$, then we may require that $\overline{M}$ be orthogonal to $n$ along $\Gamma$.*

We only need to recall one more result from [12].

**Lemma 2.8** ([12]). *Every compact (but not necessarily connected) strictly convex $\mathcal{C}^k$ surface of positive curvature $\Sigma \subset \mathbf{R}^3$ lies on a $\mathcal{C}^k$ closed surface of positive curvature.*

Using the last two results, we may now prove the main result of this section.

*Proof of Theorem 2.1.* Let us denote the intersection curves which emanate from $p$ and are contained in a small neighborhood of $p$ by $\Gamma_i$, $i = 1, \dots, k$. Since each $\Gamma_i$ is $\mathcal{C}^1$ and has a distinct tangent at $p$, we may assume that the given neighborhood $U_1$ of $p$ in $\Sigma_1$ is so small that $\Gamma_i$ and $\Gamma_j$ intersect each other in $U_1$ only at $p$ for $i \neq j$, each $\Gamma_i \cap U_1$ is connected, and if we parametrize each $\Gamma_i \cap U_1$ by $\gamma_i(t)$, $t \in [0, \epsilon_i)$, $\gamma_i(0) = p$, then the distance between $\gamma_i(t)$ and $p$ is an increasing function of $t$. Now let $U_2$ be an open neighborhood of $p$ in $\Sigma_2$ which is chosen so small that $U_2 \cap \Gamma_i$ is connected, and $(\Sigma_1 - U_1) \cup \mathrm{cl}(U_2)$ is strictly convex. Note that $\Sigma_1 - U_1$ and $\mathrm{cl}(U_2)$ are connected by a portion $\widetilde{\Gamma}_i$ of $\Gamma_i$, see Figure 6. Using the bridge principle mentioned in the above proposition we may connect $\Sigma_1 - U_1$ and $\mathrm{cl}(U_2)$ by thin strips of positive curvature containing $\widetilde{\Gamma}_i$ to obtain a compact positively curved surface $\widetilde{\Sigma}$ with $k + 1$ boundary components (including $\partial\Sigma_1$). We can make sure that $\widetilde{\Sigma}$ is strictly convex by prescribing appropriate nonsingular support vector fields along each $\widetilde{\Gamma}_i$. Then by the previous lemma, $\widetilde{\Sigma}$ may be extended to a closed surface of positive curvature $O$.

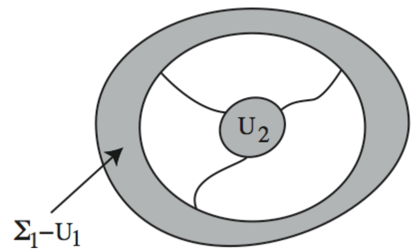


FIGURE 6

By the Jordan curve theorem, $\partial\Sigma_1$ divides $O$ into two components. The component which contains $\widetilde{\Sigma}$ is our desired surface $\Sigma$. □

## 3. Spanning Surfaces

In this section we prove Theorems 1.2 and 1.3, which concern two-dimensional fillings. Theorem 1.2 gives topological finiteness for locally convex, compact spanning surfaces in $\mathbf{R}^3$, under minimal regularity assumptions. Here we use the theory of two-dimensional manifolds of bounded curvature due to Alexandrov and Zalgaller. Under greater regularity, Theorem1.3 offers a far stronger finiteness statement for possibly noncompact spanning surfaces immersed in any given Riemannian manifold and having a uniform bound below on total curvature. This proof draws on classical results of Cohn-Vossen and Huber.

3.1. **Proof of Theorem 1.2.** Now we prove the topological finiteness result for the class $\mathcal{F}_\Gamma$ of compact, immersed, locally convex spanning surfaces of a finite collection $\Gamma$ of closed curves of finite turn immersed in $\mathbf{R}^3$.

The *turn* of a curve in $\mathbf{R}^3$ (also called the *spatial turn*) is the supremum of

$$\Sigma(\pi - \alpha_i')$$

over inscribed Euclidean polygons, where the $\alpha_i'$ are the Euclidean angles at the vertices. See [5, Ch. V] for a discussion of the properties of curves of finite spatial turn. In particular, a curve of finite spatial turn has one-sided tangents and hence is rectifiable, and its turn is equal to the length of its tangent indicatrix on the unit sphere.

A locally convex surface without boundary is an example of a *two-dimensional manifold of bounded curvature*. These spaces were studied by Alexandrov and Zalgaller in [6], which is our basic reference throughout this section. The class consists of, besides two-dimensional Riemannian spaces, all two-dimensional metrized manifolds $M$ whose metric in a neighborhood of each point may be uniformly approximated by Riemannian metrics such that the integrals of the absolute values of the Gaussian curvatures are uniformly bounded. Equivalently, one can use polyhedral metrics for the approximation. The intrinsic defining property is that every point has a neighborhood in which the sum of positive excesses of nonoverlapping triangles is bounded. In this setting, the positive part $\omega^+$ of curvature, and the negative part

$\omega^-$, are defined as nonnegative, completely additive Borel set functions on $M$, and the curvature $\omega = \omega^+ - \omega^-$ is defined for any set for which at least one of $\omega^+$ and $\omega^-$ is finite. In the case of locally convex surfaces, $\omega^- = 0$ and so $\omega$ is nonnegative.

The *righthand turn of a geodesic polygon* in $M$ is

$$\Sigma(\pi - \alpha_i),$$

where the $\alpha_i$ are the righthand sector angles between the directions at the interior vertices. Now consider a curve in $M$ with one-sided tangents at each point. The *righthand turn* of any open subarc $\gamma$ of this curve is the limit of the righthand turns of simple polygons in $D$ with the same endpoints as $\gamma$, converging to $\gamma$ and disjoint from $\gamma$ except at the endpoints, and whose angles with $\gamma$ at the endpoints converge to 0. Under sufficient regularity, the righthand turn is equal to $\int \kappa_g \, dl$, where $\kappa_g$ is the geodesic curvature relative to the righthand normal. For a trivial subarc, namely, a single interior point, the turn is defined as $(\pi - \alpha)$ where $\alpha$ is the righthand sector angle between the directions at that point; when an endpoint is adjoined to $\gamma$, this quantity is added to the turn. The turn is additive under decomposition of a curve into disjoint subarcs, and is said to have *bounded variation* if, for a curve parameter $0 \le t \le 1$, the turn on the interval $(0, t)$ has bounded variation as a function of $t$.

The righthand turn of a curve $\gamma$ in a convex surface $M$ in $\mathbf{R}^3$ is bounded above in absolute value by the spatial turn of $\gamma$, as is the variation of righthand turn of $\gamma$. This fact is well-known in the Alexandrov-Pogorelov-Zalgaller theory of convex surfaces (see, for example, [8, p. 42]), but the argument is less well-known in the literature in English. The main idea (as described in [31, p.1045]) is that it suffices by approximation to consider a polygonal curve $\gamma_k$ in a convex polyhedron $M_k$. Then the variation of righthand turn of $\gamma_k$ in $M_k$ is at most equal to the variation of spatial turn of $\gamma_k$ because angles in $M_k$ are no smaller than spatial angles. And since the spatial turn restricted to a parameter interval $(0, t)$ is monotone in $t$, the variation of spatial turn equals the spatial turn.

With this preparation, we are ready to prove our finiteness theorem for locally convex surfaces $M$ that span a given collection $\Gamma_i$, $1 \le i \le l$ of curves of finite turn:

*Proof of Theorem 1.2.* Denote the spatial turn of $\Gamma_i$ by $\kappa(\Gamma_i)$. For any locally convex spanning surface $M$, denote the one-sided turn of $\Gamma_i$ in $M$ by $\kappa_M(\Gamma_i)$. Since $|\kappa_M(\Gamma_i)| \le \kappa(\Gamma_i)$, then

$$\kappa_M(\Gamma_i) \ge -\kappa(\Gamma_i).$$

By the Gauss-Bonnet theorem [6, p.192], and since $\omega(M) \ge 0$,

$$2\pi\chi(M) = \omega(M) + \sum_{i=1}^{l} \kappa_M(\Gamma_i) \ge -\sum_{i=1}^{l} \kappa(\Gamma_i).$$

Thus the Euler characteristics of all such spanning surfaces $M$ are uniformly bounded, and there are only finitely many possible topological types for $M$. □

3.2. **Proof of Theorem 1.3.** Here we show that a finite collection of closed $\mathcal{C}^3$ curves immersed in a given Riemannian manifold bounds at most finitely many topologically distinct, complete immersed $\mathcal{C}^3$ surfaces whose total curvature is uniformly bounded below. The proof follows from classical results of Cohn-Vossen and Huber on Gauss-Bonnet type theorems for noncompact surfaces, which have been proved in greater generality in [28, p. 46].

A surface is said to be *finitely connected* if it is homeomorphic to a compact surface minus finitely many points. A smooth immersed surface is (metrically) *complete* if it is a complete metric space with respect to the induced intrinsic metric. Thus, in our terminology, a complete surface may have boundary (it need not be *geodesically* complete).

**Lemma 3.1** (Cohn-Vossen)**.** *Let $M$ be a connected, finitely connected complete 2-dimensional Riemannian manifold. Suppose that $\int_M K$ exists in $[-\infty,\infty]$, and $\partial M$ is compact. Then*

$$\int_M K + \int_{\partial M} \kappa_g \leq 2\pi\chi(M).$$

**Lemma 3.2** (Huber)**.** *Let $M$ be a connected, infinitely connected, complete 2-dimensional Riemannian manifold without boundary. Suppose that $\int_M K$ exists in $[-\infty,\infty]$. Then*

$$\int_M K = -\infty.$$

These results yield:

**Proposition 3.3.** *Let $\mathcal{F}$ be a family of complete 2-dimensional manifolds $M$ with compact boundary, where the total curvature of $M$ and total geodesic curvature of $\partial M$ are both uniformly bounded below. Then $\mathcal{F}$ contains only finitely many topological types.*

*Proof.* For $M \in \mathcal{F}$, let $D(M)$ be the doubling of $M$ across its boundary, and $\widetilde{D}(M)$ be a smoothing of $D(M)$ which perturbs the metric only in a compact neighborhood of the boundary $\partial M$. Then the total curvature of $\widetilde{D}(M)$ is bounded below because it is bounded below on the complement of a compact set. Thus by Lemma 3.2, $\widetilde{D}(M)$ is finitely connected. So $M$ is finitely connected as well. Now Lemma 3.1 applies, and implies that $\chi(M)$ is bounded below. By the formulas for the Euler characteristic of a finitely connected surface, $M$ may assume only finitely many topological types. □

We need only one more observation:

**Lemma 3.4.** *Let $\Gamma$ be a finite collection of closed $C^2$ curves immersed in a Riemannian manifold $N$, and $M$ be any $C^1$ surface in $N$ bounded by $\Gamma$. Then*

$$\int_\Gamma \kappa_M \geq -\int_\Gamma \kappa,$$

*where $\kappa_M$ and $\kappa$ denote, respectively, the geodesic curvatures of $\Gamma$ in $M$ and $N$.*

*Proof.* Let $\gamma\colon [0,L] \to \overline{M}$ be a unit speed parametrization for $\Gamma$, where $L$ denotes the length of $\Gamma$, and $v\colon [0,L] \to \mathbf{R}^3$ be the unit normal vector field along $\gamma$ which is tangent to $M$ and points inside $M$ ($v$ is the "conormal" of $\Gamma$ with respect to $M$). By the Cauchy-Schwartz inequality,

$$\int_\Gamma \kappa_g := \int_0^L \langle \overline{\nabla}_{\gamma'(t)}\gamma', v(t)\rangle\, dt \geq -\int_0^L \|\overline{\nabla}_{\gamma'(t)}\gamma'\|\, dt =: -\int_\Gamma \overline{\kappa}_g\, dt.$$

□

The last two results immediately yield Theorem 1.3.

## 4. Compact Higher-dimensional Fillings

This section considers topological finiteness of compact fillings of dimension $n > 2$. Our finiteness theorems will be proved by placing them in the context of Alexandrov spaces of curvature bounded below.

4.1. **Alexandrov spaces and finiteness.** By a *geodesic*, we mean a distance-realizing path parametrized by arclength. All our spanning hypersurfaces are *length metric spaces* in their intrinsic metrics, that is, every two points are joined by a geodesic.

Recall that an *Alexandrov space of curvature* $\geq K$ is a length metric space in which geodesic triangles are *wider* than comparison triangles in the simply connected space form $S_K$ of constant curvature $K$. Specifically, for any geodesic triangle $\triangle$ of perimeter $< 2\pi/\sqrt{K}$, the distance from a vertex to a point on the opposite side is at least the distance between corresponding points on a geodesic triangle with the same sidelengths in $S_K$. (Here the perimeter bound ensures that the model triangle is confined to an open hemisphere, and hence has a distinguished interior.) It follows that angles between geodesics may be defined, and the angle at each vertex of $\triangle$ is no less than the corresponding angle of the model triangle. (See [7] for a discussion.)

A complete Riemannian manifold $M$ with boundary is an Alexandrov space of curvature bounded below if and only if its interior sectional curvatures are bounded below and its boundary is locally convex. Any nonconvexity of the boundary produces branching of geodesics of the metric space $M$, and hence produces negatively infinite curvature in the form of triangles that do not satisfy the wideness condition for any $K$.

There is a deep relation between Alexandrov spaces of curvature bounded below and topological finiteness. Here are the fundamental theorems:

**Theorem 4.1** (Gromov Compactness)**.** *Let* $\mathcal{M} = \mathcal{M}(K, n, V, D)$ = *Alexandrov spaces of curvature* $\geq K$*, dim* $= n$*, vol* $\geq V$*, and diam* $\leq D$*, where* $\mathcal{M}$ *carries the Gromov-Hausdorff metric. Then* $\mathcal{M}$ *is compact.*

**Theorem 4.2** (Perelman Stability)**.** *If* $X \in \mathcal{M}$*, then any* $Y \in \mathcal{M}$ *sufficiently close to* $X$ *is homeomorphic to* $X$*.*

For a discussion of Gromov Compactness, one may consult [7, §10.7]. A simplified proof of Perelman's Stability Theorem [22] is given in detail by Kapovitch [19]. As an immediate consequence of Theorems 4.1 and 4.2, $\mathcal{M}$ contains only finitely many topological types.

4.2. **Proof of Theorem 1.4.** We are going to exploit the regularity of the boundary submanifold $\Gamma$ in order to access this finiteness theory. The connection is via Wong's warped product collar construction (Lemma 4.3) and Kosovskii's gluing theorem (Lemma 4.4). Lemma 4.3 allows the construction of a Riemannian collar $C = \Gamma\times[0,T]$ such that $\Gamma\times\{T\}$ is totally geodesic, and $\Gamma\times\{0\}$ has a degree of convexity in $C$ that exceeds the concavity of $\Gamma$ in any $M$ in the filling family. Lemma 4.4 then allows all $M$ in the family to be extended to Alexandrov spaces with a uniform lower curvature bound.

**Lemma 4.3** ([34, 35])**.** *Suppose $\Gamma$ is any manifold (without boundary) having sectional curvature bounded below by $K_1$. Then for any $T > 0$ and any $\lambda > 0$, there exists a smooth metric on $C = \Gamma \times [0,T]$ such that $\mathrm{II}_{\Gamma\times\{0\}} - \lambda\,\mathrm{I}$ is positive definite, where $\mathrm{II}_{\Gamma\times\{0\}}$ is the second fundamental form of $\Gamma \times \{0\}$ in $C$; $\Gamma\times\{0\}$ is isometric to $\Gamma$; $\Gamma\times[T-\epsilon,T]$ is isometric to the direct product of $c\,\Gamma$ for some $0 < c < 1$ with the interval $[T-\epsilon,T]$; and the sectional curvature of $\Gamma\times[0,T]$ is bounded below by a constant $K(K_1,\lambda,T)$.*

Typically, the constant $K = K(K_1,\lambda,T)$ will be strictly less than $K_1$.

**Lemma 4.4** ([20])**.** *Let $M_1$ and $M_2$ be two Riemannian manifolds-with-boundary, each having sectional curvatures bounded below by $K$, and whose boundaries are isometric and have respective second fundamental forms the sum of which is positive semidefinite. Then the space obtained by isometrically gluing $M_1$ to $M_2$ along their common boundary is an Alexandrov space of curvature bounded below by $K$.*

Now we are ready to prove topological finiteness for compact, smooth fillings with uniform bounds below on curvature and above on intrinsic diameter, of a compact immersed submanifold $\Gamma$ of arbitrary codimension in an arbitrary given Riemannian manifold $N$. Let $\mathcal{F}_\Gamma$ denote such a family of filling submanifolds with boundary $\Gamma$.

*Proof of Theorem 1.4.* Since $\Gamma$ is $\mathcal{C}^2$ smooth, the second fundamental form of $\Gamma$ in $N$ or in any $M \in \mathcal{F}_\Gamma$ is well-defined. Denote these forms by $\mathrm{II}^N$ and $\mathrm{II}^M$ respectively. Recall that $\mathrm{II}^N$ assigns a bilinear form to each element of the unit normal bundle of $\Gamma$ in $N$, which we denote by $\mathrm{II}^N_{(p,u)}$, and similarly for $M$. Further note that for any $p \in M$, we have $\mathrm{II}^M_{(p,v)} = \mathrm{II}^N_{(p,v)}$, where $v$ is a normal vector in $M$ to $\Gamma$ at $p$. Let $|\,\mathrm{II}^M\,|$ and $|\,\mathrm{II}^N\,|$ be, respectively, the maximum of the norms $|\,\mathrm{II}^M_{(p,v)}\,|$ and $|\,\mathrm{II}^N_{(p,u)}\,|$ (i.e. the largest of the absolute values of the eigenvalues of the associated quadratic forms). Then, for any $M \in \mathcal{F}_\Gamma$, $|\,\mathrm{II}^M\,| \le |\,\mathrm{II}^N\,|$.

Now we apply Lemma 4.3, with $\lambda := |\,\mathrm{II}^N\,|$, to obtain the collar $C = \Gamma \times [0,T]$. By construction, $C$ carries a metric such that $\Gamma\times\{0\}$ is isometric to $\Gamma$, and $\Gamma\times\{0\}$ has a degree of convexity in $C$ that exceeds the concavity of $\Gamma$ in any $M \in \mathcal{F}_\Gamma$, i.e., the sum of the respective second fundamental forms is positive definite. Since all $M \in \mathcal{F}_\Gamma$ have the common boundary $\Gamma$, the same collar $C$ can be glued to each $M$. Let $K_1$ be a lower bound for the intrinsic sectional curvature of $\Gamma$. By isometrically gluing $C$ to $M$ along $\Gamma\times\{0\}$, using Lemma 4.4, one obtains an Alexandrov space $M \cup_\Gamma C$ of curvature bounded below by a uniform constant $K(K_1,\lambda,T)$.

By construction of $C$, the diameter of any $M \cup_\Gamma C$ is bounded above by the sum of the intrinsic diameter of $M$ and twice the collar width, $2T$. Since we are assuming the diameters of the $M \in \mathcal{F}_\Gamma$ to be uniformly bounded, so are the diameters of the glued spaces $M \cup_\Gamma C$. Clearly, the volume of $M \cup_\Gamma C$ is uniformly bounded below by the volume of $C$.

Thus the family $\{M \cup_\Gamma C : M \in \mathcal{F}_\Gamma\}$ has uniform lower curvature bound, upper diameter bound and lower volume bound, and hence lies in a class $\mathcal{M}$ that contains only finitely many topological types. Since $M \cup_\Gamma C$ is homeomorphic to $M$, the class $\mathcal{F}_\Gamma$ itself admits only finitely many topological types. □

**Remark 4.5.** As the preceding proof makes clear, Lemmas 4.3 and 4.4 imply the following *intrinsic* finiteness theorem for manifolds with boundary:

**Theorem 4.6** ([34, 35]). *There are only finitely many topological types of compact Riemannian n-manifolds with boundary having uniform bounds below on curvature and volume, and above on diameter and the norm of the second fundamental form of their boundaries.*

4.3. **Proof of Theorem 1.6.** In this section, we consider a compact, immersed submanifold $\Gamma$ of codimension 2 in $\mathbf{R}^{n+1}$. Let $\mathcal{F}_\Gamma$ denote the family of all compact, nonnegatively curved, immersed hypersurfaces having $\Gamma$ as boundary. Since $\mathbf{R}^{n+1}$ is projectively equivalent to a hemisphere in $\mathbf{S}^{n+1}$, we may regard the spaces $M \in \mathcal{F}_\Gamma$ as hypersurfaces of sectional curvature $\geq 1$ in $\mathbf{S}^{n+1}$. Indeed, since projective maps preserve semidefiniteness of the second fundamental forms of hypersurfaces, this claim is a consequence of the Gauss Equation (see [11]). From now on in this subsection, we use the Riemannian metrics induced from $\mathbf{S}^{n+1}$.

Denote the components of $\Gamma$ by $\Gamma_i$, $1 \leq i \leq l$. The following lemma bounds the intrinsic diameter of any $M \in \mathcal{F}_\Gamma$ in terms of the intrinsic diameters $\operatorname{diam}(\Gamma_i)$ of the $\Gamma_i$.

**Lemma 4.7.** *The intrinsic diameter of $M \in \mathcal{F}_\Gamma$ is uniformly bounded above by*

$$(l+1)\pi + \Sigma_{i=1}^{l} \operatorname{diam}(\Gamma_i).$$

*Proof.* First note that the distance of any point in $M \in \mathcal{F}_\Gamma$ to $\Gamma$ is $\leq \pi$; for, otherwise, the distance from some $p \in M$ to $\Gamma$ would be realized by an $M$-geodesic $\gamma$ of length $> \pi$, and $\gamma$ would have a subsegment from $p$ of length $> \pi$ that does not contact $\Gamma$. Since $M$ has sectional curvature $\geq 1$, such a subsegment would contain an internal pair of conjugate points by the Rauch comparison theorem. Therefore the subsegment could be shortened by a variation with endpoints fixed in an arbitrarily small neighborhood of itself (see, for example, [29, p.316]), yielding a contradiction.

Now let $\gamma$ be a geodesic in $M$. The distance from the initial point to the first intersection with the boundary, say with $\Gamma_1$, is $\leq \pi$. The distance from the first intersection point to the last intersection point of $\gamma$ with $\Gamma_1$ is at most $\operatorname{diam}(\Gamma_1)$. Now reiterate this argument, starting from the last intersection point of $\gamma$ with $\Gamma_1$. □

With this preparation, the proof of Theorem 1.6 is immediate:

*Proof of Theorem 1.6.* The proof proceeds just as in the proof of Theorem 1.4, except that here we have not assumed an *a priori* bound on the diameters of the $M \in \mathcal{F}_\Gamma$, but rather, may invoke Lemma 4.7. □

**Remark 4.8.** As we have just shown, Theorem 1.6 also holds in $\mathbf{S}^{n+1}$ for spanning hypersurfaces with curvature $\geq 1$. Similarly, by projective transformation, Theorem 1.6 holds in $\mathbf{H}^{n+1}$ for compact hypersurfaces with curvature $\geq -1$. The latter case is interesting for being a finiteness theorem where there is not a uniform bound below on total negative curvature.

### 4.4. **Proof of Theorem 1.5.**

## 5. Complete Higher-dimensional Fillings

5.1. **Proof of Theorem 1.7.** Finally we turn to the case of noncompact complete fillings. In this section, we again assume $n > 2$, since we already have established Theorem 1.3 for surfaces.

Our procedure is first to bound the number of ends of any positively curved hypersurface spanning $\Gamma$. Then we "clip off" the ends, and reduce the proof of Theorem 1.7 to that of the compact case already considered in Section 4. The new elements we draw on here for the noncompact case include a theorem of Cai that bounds the number of ends of a Riemannian manifold whose Ricci curvature is nonnegative off a compact set and everywhere bounded below [10]; a theorem of Perelman on intrinsic gluing and smoothing under a lower Ricci curvature bound [23]; a theorem of Alexander and Currier on the existence of convex end representatives in Euclidean hypersurfaces that are positively curved off compact sets [1]; and an extrinsic smoothing theorem of Ghomi for convex hypersurfaces [12].

By an *end representative* in a complete Riemannian manifold $M$, we mean an unbounded component of the complement of a compact subset of strictly$M$. An *end* is an equivalence class of nested decreasing sequences of end representatives that eventually lie outside every compact subset of $M$, where two sequences are equivalent if any member of either contains a truncation of the other.

**Lemma 5.1** ([10])**.** *In a complete $n$-dimensional Riemannian manifold without boundary whose Ricci curvature satisfies ric $\geq 0$ off a metric ball of radius $a$ and ric $> \Lambda$ everywhere, there is a universal bound $c(n, a, \Lambda)$ on the number of ends.*

This control on the number of ends only requires lower Ricci bounds, and these behave well under gluing and smoothing, as the following theorem shows.

**Lemma 5.2** ([23])**.** *Let $N_1$ and $N_2$ be two Riemannian manifolds with isometric compact boundaries and $\mathrm{ric}(N_i) > \Lambda$. Suppose the sum of the second fundamental forms is positive definite at each point on the identified boundaries. Then the induced metric on the gluing $N_1 \cup N_2$ can be smoothed in an arbitrarily small tubular neighborhood of the identified boundaries to yield a complete $\mathcal{C}^2$ manifold with* $\mathrm{ric} > \Lambda$.

**Remark 5.3.** This theorem was originally stated for positive Ricci curvature, but the proof holds for any lower bound. The proof described in [23] is written out in detail in an appendix of Wang's thesis [32].

**Lemma 5.4.** *Given a compact submanifold* $\Gamma$ *of codimension* 2 *in* $\mathbf{R}^{n+1}$*, consider all complete, immersed, nonnegatively curved spanning hypersurfaces* $M$ *of* $\Gamma$*. Then the number of ends of* $M$ *is uniformly bounded.*

*Proof.* Let $\{\Gamma_1, \ldots, \Gamma_l\}$ be the components of $\Gamma$. For any spanning hypersurface $M$, we consider the glued manifold $M \cup_\Gamma C$, where $C = \cup_{i=1}^{l} \Gamma_i \times [0,T]$ is a (fixed) collar which may be constructed, with the aid of Lemma 4.3, similarly to the description in the proof of Theorem 1.4. In particular, setting $\lambda := |\,\mathrm{II}^{\mathbf{R}^{n+1}}\,|$, i.e. the norm of the second fundamental form of $\Gamma$ in $\mathbf{R}^{n+1}$, we can make sure that the sum of the second fundamental forms of $M$ and $C$ at the corresponding points of gluing across $\Gamma$ is positive definite.

Since $C$ is compact and the spanning hypersurfaces $M$ are nonnegatively curved, there is a uniform lower bound on the sectional curvatures of $C$ and the manifolds $M$, and consequently a uniform Ricci bound, $\mathrm{ric} > \Lambda$. Therefore, by Lemma 5.2, the metric of each $M \cup_\Gamma C$ may be smoothed arbitrarily close to $\Gamma$ while retaining $\mathrm{ric} > \Lambda$. Denote the resulting manifold by $\widetilde{M}$. Since, by Lemma 4.3, each $\widetilde{M}$ is cylindrical near its boundary $\Gamma \times \{T\}$, doubling across the boundary yields a complete, smooth Riemannian manifold $D(\widetilde{M})$ still satisfying $\mathrm{ric} > \Lambda$.

Now note that removing the two copies of $\mathrm{int}(M)$ in $D(\widetilde{M})$ leaves $l$ components $D(C_i)$, where $C_i$ is a component of $C$ (namely, $\Gamma_i \times [0,T]$ carrying the restriction of the metric of $\widetilde{M}$) and $D(C_i)$ is the doubling of $C_i$ across $\Gamma_i \times \{T\}$. We may join $D(C_i)$ to $D(C_{i+1})$, for $1 \leq i \leq k-1$, by a surgery procedure inside $D(C)$, i.e., we remove from the interiors of $D(C_i)$ and $D(C_{i+1})$ a pair of small open balls, replace them by an annulus, and use a partition of unity to endow the gluing with a smooth metric which coincides with the metric on $D(C)$ near its boundaries. Thus these gluings transform $D(\widetilde{M})$ and $D(C)$ to smooth connected Riemannian manifolds $D(\widetilde{M})^*$ and $D(C)^*$ respectively, such that, for all $M \in \mathcal{F}_\Gamma$, $D(\widetilde{M})^* - D(C)^* = D(\widetilde{M}) - D(C)$.

Since $D(C)^*$ is compact, it has a lower Ricci bound $\Lambda'$ which does not depend on $M$. Further, since $D(C)^*$ is connected, its diameter $\Delta$ is well defined and yields a uniform upper bound for diameter of $D(C)^*$ in $D(\widetilde{M})^*$. Thus there exists a point $p$ of $D(C)^*$ such the metric ball $B$ in each $D(\widetilde{M})^*$ centered at $p$ and of uniform radius $\Delta + \epsilon$ contains an open neighborhood of $D(C)^*$. Outside of this neighborhood, we may assume $D(\widetilde{M})^*$ to have nonnegative curvature since, as described earlier in this proof when we invoked Lemma 5.2, $\widetilde{M}$ coincides with $M$ outside of a uniformly small neighborhood of $C$. Thus $D(\widetilde{M})^* - B$ has nonnegative Ricci curvature, and $B$ has a uniform lower Ricci bound given by $\min\{\Lambda, \Lambda'\}$.

By Lemma 5.1, it then follows that the number of ends of $D(\widetilde{M})^*$ is uniformly bounded. Therefore there is a uniform bound for the number of ends of $\widetilde{M}$, and hence of $M \in \mathcal{F}_\Gamma$, since $M$ and $\widetilde{M}$ are homeomorphic. □

By a *convex body* in a Euclidean space we mean a convex subset with interior points; its boundary is a *convex hypersurface*. A *convex cap* is the intersection of a convex hypersurface with an open halfspace.

**Lemma 5.5** ([1])**.** *Let $M$ be a complete, smoothly immersed hypersurface without boundary of $\mathbf{R}^{n+1}$, $n > 2$. Suppose $M$ is positively curved off a compact set. Then each end of $M$ has a* convex representative*, i.e., a representative that is embedded onto a convex cap.*

The above lemma yields that:

**Corollary 5.6.** *Let $M$ be as in Lemma 5.5. Then each end of $M$ has a representative whose closure is homeomorphic to $\mathbf{S}^{n-1} \times [0, 1)$.*

*Proof.* Let $E$ be an end representative of $M$ which is a convex cap as described in Lemma 5.5, i.e., $E = \partial K \cap \operatorname{int} H^+$, where $H \subset \mathbf{R}^{n+1}$ is a hyperplane, which determines a closed halfspace $H^+$, and $K \subset H^+$ is an unbounded convex body in $\mathbf{R}^{n+1}$. We may assume that $K \cap H$ has interior points in $H$, after moving $H$ parallel to itself into $\operatorname{int} H^+$. Thus $\partial E := \overline{E} - E = \partial(K \cap H)$, where $\overline{E}$ denotes the closure of $E$, is a convex hypersurface of $H$. Further note that $\partial E$ is compact, since, by definition of an end representative, $E$ is a complement of a convex subset $C$ of $M$, and $\partial E = \overline{E} \cap C$, i.e., $\partial E$ is a closed subset of a compact set. So $\partial E$ is homeomorphic to $\mathbf{S}^{n-1}$. Consequently, interior of $K \cap H$ is homeomorphic to an open ball $B^n$. So, since $\overline{E} = \partial K - \operatorname{int}(K \cap H)$, it follows that $\overline{E}$ is homeomorphic to $\partial K$ minus an open ball. On the other hand $\partial K$ is a noncompact complete convex hypersurface of $\mathbf{R}^{n+1}$ with a strictly convex point (since by assumption $M$ is positively curved ouside a compact subset). So, by a theorem of van Heijenoort [33], $\partial K$ is homeompohic to $\mathbf{R}^n$, which implies that $\overline{E}$ is homeomorphic to $\mathbf{S}^{n-1} \times [0, 1)$. □

The next lemma shows that we may "clip off" the ends of a smooth positively hypersurface to make it compact without compromising either local convexity or smoothness.

**Lemma 5.7.** *Let $M$ be a $\mathcal{C}^{k \geq 2}$, complete hypersurface with boundary in $\mathbf{R}^{n+1}$. Suppose there exists a hyperplane $H \subset \mathbf{R}^{n+1}$ such that $X := H \cap M$ is compact and lies in the interior of $M$, and $M$ has positive curvature on $X$. Further, when $n = 2$ suppose that each component of $X$ is embedded. Let $H^+$ be one of the closed halfspaces determined by $H$ and set $M^+ := M \cap H^+$. Then there exists a $\mathcal{C}^k$ complete hypersurface $\overline{M}^+$ such that $\overline{M}^+ \supset M^+$, $\partial \overline{M}^+ = \partial M^+ - X$, and $\overline{M}^+ - M^+$ has positive curvature.*

*Proof.* Since $X$ is compact, it has finitely many connected components $X_i$. Further, since $M$ has positive curvature on $X$ and $X$ lies in the interior of $M$, $X_i$ is a closed hypersurface of $H$ with positive curvature in $H$. In particular, by Hadamard's theorem (which we may apply when $n > 2$), $X_i$ is embedded in $H$. So it follows that there exists an open neighborhood $U_i$ of $X_i$ in $M$ such that the closure of $U_i$ is a compact "strictly convex hypersurface" of $\mathbf{R}^{n+1}$ as defined in [12]. Therefore, by the main result of [12], there exists a closed $\mathcal{C}^k$ hypersurface $O_i$ of positive curvature such that $O_i \supset U_i$. By the Jordan-Brouwer separation theorem $X_i$ separates $O_i$ into a pair of subsets, bounded by $X_i$. Gluing one of these subsets to $M^+$ along each $X_i$ yields the desired surface $\overline{M}^+$. □

Now we are ready to prove our finiteness theorem for noncompact, complete, positively curved fillings of a given compact submanifold $\Gamma$ of codimension 2 in $\mathbf{R}^{n+1}$, $n > 2$. Let $\mathcal{F}_\Gamma$ denote all complete, smoothly immersed spanning hypersurfaces of $\Gamma$ having positive sectional curvature. We must show that $\mathcal{F}_\Gamma$ contains only finitely many topological types. When $M \in \mathcal{F}_\Gamma$ is noncompact, its projective image in $\mathbf{S}^{n+1}$ need not be complete, so we do not begin with a projective transformation as in Section 4, but rather with the original positively curved metric induced from $\mathbf{R}^{n+1}$ on each $M \in \mathcal{F}_\Gamma$.

*Proof of Theorem 1.7.* By Lemma 5.1, the number of ends of is uniformly bounded. Further, by Lemma 5.5, each end of $M$ has a convex representative which is positively curved. Thus for each end of $M$, a convex representative may be "clipped off" by Lemma 5.7. Therefore after clipping, the $M \in \mathcal{F}_\Gamma$ become compact, smooth locally convex hypersurfaces.

Now the proof given in Section 4 for the compact case shows that after we clip and smooth the $M \in \mathcal{F}_\Gamma$, only finitely many homeomorphism classes remain. Moreover, by Lemma 5.1, the number of clipped ends is uniformly bounded, and by Corollary 5.6 the closure of each clipped end representative is homeomorphic to $\mathbf{S}^{n-1} \times [0, 1)$. So it follows that $\mathcal{F}_\Gamma$ contains only finitely many homeomorphism classes. Here we have used the fact that any gluing of the end representative to the rest of the manifold is determined by a homeomorphism of $\mathbf{S}^{n-1}$ to itself, which is isotopic to the identity; thus, topologically speaking, there is only one way to glue the end representative back to the rest of $M$, which in turn yields the topological finiteness that we seek. □

5.2. **Comments and Questions.** It is interesting to ask to what extent Theorem 1.7 might reflect *intrinsic* finiteness phenomena. Recall that a complete, open, positively curved Riemannian manifold $M$ without boundary is diffeomorphic to $\mathbf{R}^n$, by a theorem of Gromoll and Meyer [15]. Perelman proved the same statement if $M$ is merely nonnegatively curved with a point of strictly positive curvature [24].

**Question 5.8.** Does topological finiteness extend to the class of complete open Riemannian manifolds that are positively curved off sets with uniform diameter and lower curvature bounds?

One can also ask to what extent the positive curvature hypothesis in Theorem 1.7 can be relaxed. [DO WE WANT TO INCLUDE THIS PARAGRAPH?] In particular, we conjecture that finiteness still holds for the class of nonnegatively curved spanning hypersurfaces $M$ for which the second fundamental form has nullity at most 1 everywhere. In this case, it follows from [1] that each end of $M$ has a representative that is embedded onto the union of two convex caps and is still homeomorphic to $\mathbf{S}^{n-1} \times [0, 1)$. Controlled clipping is still possible but does not lend itself to established smoothing techniques. Thus there are difficulties to be overcome in order to prove this conjecture, to which we plan to return since it is a finiteness statement that requires no upper size bounds and no points of strictly positive curvature.

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Department of Mathematics, University of Illinois at Urbana-Champaign, IL 61801
*E-mail address*: sba@math.uiuc.edu
*URL*: www.math.uiuc.edu/~sba

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332
*E-mail address*: ghomi@math.gatech.edu
*URL*: www.math.gatech.edu/∼ghomi

Department of Mathematics, University of Toronto, Toronto, ON, M5S 2E4
*E-mail address*: jawong1@math.toronto.edu
*URL*: www.math.toronto.edu/∼jawong1