OPTIMAL RETIREMENT AGE

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In this short note we discuss the optimal retirement age for a single individual under certain assumptions. The final results are interesting because they quantify intuitive ideas about the relationship between savings rate, the rate of interest earned on savings and how soon a person can retire while maintaining their standard of living.

1. Assumptions and Conclusions

The assumptions are:

1. **There are no taxes.** This is not too problematic because we can simply adjust the income to account for it. The difference in tax rate between capital gains and income does muddy the waters a bit but doesn't seriously affect the conclusions.
2. **The rate of increase of the individual's income is known and constant.** This of course does not hold for recent graduates or other early career people, but is a reasonable assumption for mid to late career professionals.
3. **The saving rate of the individual is known and constant.** This assumption will be invalid at times of high personal expenses, for example when kids go to college or a new house is bought. We will, however, ignore such variations. Our model can be easily modified to account for such expenses.
4. **The interest rate earned on savings is known and constant.** We ignore the short-term variations in returns.
5. **Upon retirement, the annual expenses will be fixed at those during the last year of employment.** This satisfies the assumption about maintaining the standard of living upon retirement.

(6) **There is no inflation.** Inflation can be accounted for by reducing the rate of increase in income by the expected inflation rate. In other words, we assume that all calculations are in present day dollars.

Now let's assume that the current income is I, the saving rate is s, the long-term rate of return on investments is r and the rate of growth of employment income is g. The number of years the individual must work before retirement is n. The final result, derived in the next section is that

$$\boxed{n = \frac{1-s}{s}\left(1 + \frac{1-g}{r}\right)}$$

We can immediately observe several facts from this formula.

(1) The number of years of work necessary is directly proportional to the ratio of expenses to savings for the individual. On one end of the spectrum, if s is close to 0 and there are almost no savings, n is huge, pushing the retirement age up. On the other (rather unlikely) extreme, if there are almost no expenses, so that almost all the income is saved, the person can retire at any time.

(2) The current annual income I does not enter the equation. Maintaining the same standard of living after retirement negates the benefits of higher life-long earnings.

(3) A higher rate of growth of income (g) promises a quick retirement. This is a bit unexpected in light of the fact that the absolute level of income I is not important. The rate at which income grows is, however, somewhat important because higher income earlier in the career grows for a long time before it is needed during retirement.

(4) Higher returns on investments lead to quicker retirement.

To see whether our formula matches the real world, let us compute n for typical values of the inputs. Assuming a 20% savings rate ($s = 0.20$), a 10% annual raise ($g = 0.10$) and a return of 10% ($r = 0.10$), we get

$$n = \frac{1-0.20}{0.20}\left(1 + \frac{1-0.10}{0.10}\right) = 40$$

which means that a person who starts working in her mid-twenties can expect to retire in her mid-sixties.

2. Derivation of the formula

We have avoided the question of life expectancy up to this point. We will, in fact, assume an infinite life span. This might seem too generous an assumption but is justified for three reasons. *First*, life expectancy a few decades from now is hard to predict. *Second*, catastrophic medical expenses toward the end can be easily accounted for this way. *Third*, most people wish to leave some of their wealth for family or charity.

A minor fourth reason is that, as we will see later, if we are assuming 15 to 20 years of life after retirement, extending that to infinity doesn't change much. This is because the present value of expenses so far into the future is rather small.

We will calculate the value of all savings and expenses *at the time of retirement*. We begin with the savings. We are assuming that all savings are invested at the end of each year. The savings i years from now will be $s(1+g)^{i-1}I$ and in the next $n-i-1$ years these will grow to $s(1+g)^{i-1}(1+r)^{n-i-1}I$. Adding these up for years 1 through n gives us

$$\sum_{i=1}^{n} s(1+g)^{i-1}(1+r)^{n-i-1}I = sI\left[\frac{(1+r)^n-(1+g)^n}{r-g}\right]$$

The expenses for the year just after retirement will be $(1-s)(1+g)^{n-1}I$ and will remain constant thereafter. The value of the expenses for the i^{th} year after retirement is $(1-s)(1+g)^{n-1}I\cdot(1+r)^{-i+1}$ at the time of retirement. Adding these up for, let's say, N years of retirement gives

$$(1-s)(1+g)^{n-1}I\left[1+\frac{1}{1+r}+\frac{1}{(1+r)^2}+\ldots+\frac{1}{(1+r)^{N-1}}\right]$$

If N is in the 15 to 20 range, the sum of the series is not increased by much if we assume that it continues to infinity. So we replace it with

$$(1-s)(1+g)^{n-1}I\left[1+\frac{1}{1+r}+\frac{1}{(1+r)^2}+...\right]$$

which adds up to

$$(1-s)(1+g)^{n-1}\frac{1+r}{r}I.$$

Since the savings at retirement must provide for all future expenses, we get

$$sI\left[\frac{(1+r)^n-(1+g)^n}{r-g}\right]=(1-s)(1+g)^{n-1}\frac{1+r}{r}I$$

which, after some simplification, becomes

$$s\left[\left(\frac{1+r}{1+g}\right)^n-1\right]=(1-s)\left(\frac{1+r}{1+g}\right)\left(1-\frac{g}{r}\right)$$

We use the approximations $(1+r)^n\approx 1+nr$, $(1+g)^n\approx 1+ng$ and $\frac{1+r}{1+g}\approx 1+r-g$ to get

$$sn(r-g)=\frac{(1-s)(1+r-g)(r-g)}{r}$$

which simlifies to

$$n=\frac{1-s}{s}\left(1+\frac{1-g}{r}\right),$$

as promised.