Arrangements and Computations III: \( \Lambda(\mathcal{V}) \) and BGG

\[
\begin{pmatrix}
\sum x_i & 0 & 0 & 0 & 0 \\
0 & \sum x_i & 0 & 0 & 0 \\
0 & 0 & \sum x_i & 0 & 0 \\
x_1 + x_4 + x_5 & 0 & 0 & x_0 + x_3 + x_4 & 0 \\
0 & x_2 + x_3 + x_5 & 0 & 0 & 0 \\
0 & 0 & x_0 + x_3 + x_4 & 0 & 0 \\
x_1 + x_4 + x_5 & 0 & 0 & x_0 + x_1 + x_2 & 0 \\
0 & x_2 + x_3 + x_5 & 0 & 0 & 0 \\
0 & 0 & x_1 & 0 & 0 \\
x_0 & x_1 & 0 & 0 & 0 \\
x_2 & x_0 & -x_2 & 0 & 0 \\
0 & x_1 & -x_5 & 0 & 0
\end{pmatrix}.
\]

4 10 15 20 25 \( \ldots \)

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Let $A$ be the Orlik-Solomon algebra of $\mathbb{C}^\ell \setminus \mathcal{A}$, with $|\mathcal{A}| = n$. For each $a = \sum a_ie_i \in A_1$, we consider the complex $(A, a)$.

The $i^{th}$ term is $A_i$, and differential is $\wedge a$:

$$(A, a): 0 \to A_0 \xrightarrow{a} A_1 \xrightarrow{a} A_2 \xrightarrow{a} \cdots \xrightarrow{a} A_\ell \to 0.$$ 

Arose in

- hypergeometric functions (Aomoto)
- cohomology with local system coefficients
  – Esnault, Schechtman, Viehweg
  – Schechtman, Terao, Varchenko

The resonance varieties of $\mathcal{A}$ are the loci of points $a = \sum_{i=1}^n a_ie_i \leftrightarrow (a_1 : \cdots : a_n) \in \mathbb{P}^{n-1}$ for which $(A, a)$ fails to be exact, that is:

**Definition 1** For each $k \geq 1$,

$$R^k(\mathcal{A}) = \{a \in \mathbb{P}^{n-1} \mid H^k(A, a) \neq 0\}.$$ 

Yuzvinsky: for generic $a$, $(A, a)$ is exact.
**Definition 2** \( \Pi \) partition of \( A \) is neighborly if 
\[ \forall Y \in L_2(A), \pi \text{ block of } \Pi, \]

\[ \mu(Y) \leq |Y \cap \pi| \rightarrow Y \subseteq \pi. \]

**Falk**: proved that components of \( R^1(A) \) arise from neighborly partitions, and conjectured that \( R^1(A) \) is a union of linear components.

This was proved by
- **Cohen–Suciu** and by
- **Libgober–Yuzvinsky** \( R^1(A) = \bigsqcup_i L_i^+ \)
- **Cohen–Orlik** also true for \( R^{\geq 2}(A) \)
- **Falk** can fail if characteristic \( \neq 0 \).

**Libgober–Yuzvinsky** connects \( R^1(A) \) to pencils/nets/webs; recent work in this area by:
- **Falk–Yuzvinsky**
- **Pereira–Yuzvinsky**

Recall conjectural connection to LCS ranks \( \phi_k \):

**Conjecture 3 (Suciu)** Under certain conditions,

\[ \prod_{k \geq 1} (1 - t^k) \phi_k = \prod_{L_i \in R^1(A)} (1 - (\dim(L_i) t)) \]
Example 4 Let $A = V(xy(x - y)z) \subseteq \mathbb{P}^2$, and $E = \Lambda(\mathbb{C}^4)$, with generators $e_1, \ldots, e_4$. The Orlik-Solomon algebra

$$A = E/\langle \partial(e_1e_2e_3), \partial(e_1e_2e_3e_4) \rangle,$$ with

$$\partial(e_1e_2e_3) = e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_3$$

To compute $R^1(A)$, we need only the first two differentials in the Aomoto complex. Use $e_{13}, e_{14}, e_{23}, e_{24}, e_{34}$ as a basis for $A_2$.

$$e_1 \mapsto e_1 \wedge (\sum_{i=1}^{4} a_i e_i) = a_2 e_{12} + a_3 e_{13} + a_4 e_{14}.$$ 

Since $e_{12} = e_{13} - e_{23}$, $a_2 e_{12} = a_2(e_{13} - e_{23})$,

giving $(a_2 + a_3)e_{13} + a_4 e_{14} - a_2 e_{23}$. compute!

$$
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a_2 + a_3 & -a_1 & -a_1 & 0 \\
0 & 0 & 0 & -a_1 \\
-a_2 & a_1 + a_3 & -a_2 & 0 \\
0 & a_4 & 0 & -a_2
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 0 & a_4 & -a_3
\end{pmatrix}
\rightarrow
\mathbb{C}^5$$
Letting $a = \sum_{i=1}^{n} a_i e_i$, we have

$$R^1(A) \iff H^1(A, \wedge a)$$
$$\iff \exists b \in E_1 \mid a \wedge b \text{ vanishes in } A_2$$
$$\iff \exists b \in E_1 \mid a \wedge b \in I_2$$
$$\iff \text{decomposable 2-tensors in } I_2$$
$$\iff \mathbb{P}(I_2) \cap \text{Gr}(2, E_1) \subseteq \mathbb{P}(\wedge^2 E_1)$$

$I_2$ is determined by the intersection lattice $L(A)$ in rank $\leq 2$, so to study $R^1(A)$, let $A \subseteq \mathbb{P}^2$. Grassmannian gives fastest computation of $R^1(A)$.

**Problem** Code up for $R^{\geq 2}(A)$ (Segre map).

Note interesting connection to syzygies. Since $a \wedge b \in I_2 \longrightarrow a \wedge b = \sum c_i f_i, c_i \in \mathbb{C}, f_i \in I_2$, the relations $a \wedge a \wedge b = 0 = b \wedge a \wedge b$ yield linear syzygies on $I_2$:

$$\sum ac_i f_i = 0 = \sum bc_i f_i.$$ 

That is,

$$R^1(A) \text{ is related to } Tor_{2}^{F}(A, \mathbb{C})_3$$
Example 5 For $A = V(xy(x - y)z) \subseteq \mathbb{P}^2$, the Orlik-Solomon algebra is just

$$A = E/\partial(e_1e_2e_3),$$

since the relation $\partial(e_1e_2e_3e_4)$ is redundant:

$$\partial(e_1e_2e_3e_4) = e_1 \wedge \partial(e_1e_2e_3) - e_4 \partial(e_1e_2e_3)$$

Observe that

$$e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_3 = (e_1 - e_2) \wedge (e_2 - e_3)$$

This means that the line

$$s(e_1 - e_2) + t(e_2 - e_3) \subseteq R^1(A) \subseteq \mathbb{P}(E_1)$$

Parametrically, this may be written

$$(s : t - s : -t : 0) = V(a_4, a_1 + a_2 + a_3)$$

Such components of $R^1(A)$ are called local.

(Compute) the corresponding linear syzygies.
WHO CARES? Conjecturally, $R^1(A)$ is (sometimes) connected to the LCS ranks. But it is always connected to the Chen ranks! Introduced by K.T. Chen, these are the LCS ranks of the maximal metabelian quotient of $G$:

$$\theta_k(G') := \phi_k(G/G'')$$


**Conjecture 6 (Suciu)** Let $G = G(A)$ be an arrangement group, and let $h_r$ be the number of components of $R^1(A)$ of dimension $r$. Then, for $k \gg 0$:

$$\theta_k(G) = (k - 1) \sum_{r \geq 1} h_r \binom{r + k - 1}{k}.$$ 

For Example 3, $R^1(A) \simeq \mathbb{P}^1$ and thus

$$\theta_k(G) = (k - 1).$$
How to determine the Chen ranks? The Alexander invariant $G'/G''$ is a module over $\mathbb{Z}[G/G']$. For arrangements, $\mathbb{Z}[G/G'] = \text{Laurent polynomials in } n\text{-variables.}$

Massey: $\sum_{k \geq 0} \theta_{k+2} t^k = HS(\text{gr } G'/G'' \otimes \mathbb{Q}, t)$

Easier to work with is the linearized Alexander invariant $B$ of Cohen-Suciu

$$(A_2 \oplus E_3) \otimes S \xrightarrow{\Delta} E_2 \otimes S \to B \to 0,$$ where $\Delta$ is built from Koszul diff. and $(E_2 \to A_2)^t$.

**Theorem 7 (Cohen-Suciu)**

$V(\text{ann } B) = R^1(\mathcal{A})$

**Theorem 8 (Papadima-Suciu)** For $k \geq 2$,

$$\sum_{k \geq 2} \theta_k t^k = HS(B, t).$$

In particular, the Chen ranks are combinatorially determined, and depend only on $L(\mathcal{A})$ in rank $\leq 2$. 
**Example 9** Recall the matroid for $A_3$ is:

![Diagram of a matroid]

For $A_3$, $B$ is the cokernel of the matrix on the first slide. (compute) $R^1(A_3) =$

\[
V(x_1 + x_4 + x_5, x_0, x_2, x_3) II \\
V(x_2 + x_3 + x_5, x_0, x_1, x_4) II \\
V(x_0 + x_3 + x_4, x_1, x_2, x_4) II \\
V(x_0 + x_1 + x_2, x_3, x_4, x_5) II \\
V(x_0 + x_1 + x_2, x_0 - x_5, x_1 - x_3, x_2 - x_4).
\]

and (compute) the Hilbert Series of $B$:

\[
\frac{(4t^2 + 2t^3 - t^4)}{(1-t)^2} = 4t^2 + 10t^3 + 15t^4 + 20t^5 + \cdots
\]

Magic Trick! (compute) $Tor^E_i(A_3, \mathbb{C})_{i+1}$

Magic Trick! (compute) free resolution of the cokernel of last map in the Aomoto complex.
Theorem 10 (Eisenbud-Popescu-Yuzvinsky)

For an arrangement $\mathcal{A}$, the Aomoto complex is exact, as a sequence of $S$-modules:

$$0 \to A_0 \otimes S^{\cdot a} A_1 \otimes S^{\cdot a} \cdots \otimes S^{\cdot a} A_\ell \otimes S \to F(A) \to 0.$$

Theorem 11 (–, Suciu) The linearized Alexander invariant $B$ is functorially determined by the Orlik-Solomon algebra:

$$B \cong \text{Ext}^{\ell-1}_S(F(A), S).$$

Use this, localization, and the result of Libgober-Yuzvinsky that $R^1(\mathcal{A}) = \bigoplus L_i$ to obtain:

Theorem 12 (–, Suciu) For $k \gg 0$,

$$\theta_k(G) \geq (k - 1) \sum_{L_i \in R^1(\mathcal{A})} \binom{\dim L_i + k - 1}{k}.$$

Problem Prove the remaining inequality! Note: $\theta_k(G)$ is polynomial in $k$, of degree $= \dim R^1(\mathcal{A})$. 
WHAT MAKES ALL THIS WORK IS BGG: the Bernstein-Gelfand-Gelfand correspondence.

Let $S = \text{Sym}(V^*)$ and $E = \wedge(V)$. BGG is an isomorphism between derived categories of

- bounded complexes of coherent sheaves on $\mathbb{P}(V^*)$.
- bounded complexes of f.gen’d, graded $E$–modules.

From this, can extract functors

$R$: f.gen’d, graded $S$-modules $\rightarrow$ linear free $E$-complexes.

$L$: f.gen’d, graded $E$-modules $\rightarrow$ linear free $S$-complexes.

Point: can translate problems to possibly simpler setting. For example, we’ll see this gives a fast way to compute sheaf cohomology, using Tate resolutions.
$P$ a f’gend, graded $E$-module, then $L(P)$ is the complex

\[
\cdots \rightarrow S \otimes P_{i+1} \overset{a}{\rightarrow} S \otimes P_i \overset{a}{\rightarrow} S \otimes P_{i-1} \overset{a}{\rightarrow} \cdots ,
\]

where $a = \sum_{i=1}^{n} x_i \otimes e_i$, so that $1 \otimes p \mapsto \sum x_i \otimes e_i \wedge p$

Note: elts of $V^*$ deg $= 1$, elts of $V$ deg $= -1$.

**Example 13** $P = E = \wedge \mathbb{C}^3$. Then we have

\[
0 \rightarrow S \otimes E_0 \rightarrow S \otimes E_1 \rightarrow S \otimes E_2 \rightarrow S \otimes E_3 \rightarrow 0.
\]

Clearly $1 \mapsto \sum_1^3 x_i \otimes e_i$. For $d_1$

- $e_1 \mapsto -x_2 e_{12} - x_3 e_{13}$
- $e_2 \mapsto x_1 e_{12} - x_3 e_{23}$
- $e_3 \mapsto x_1 e_{13} + x_2 e_{23}$

$d_2: e_{12} \mapsto x_3 e_{123}$, $e_{13} \mapsto -x_2 e_{123}$ $e_{23} \mapsto x_1 e_{123}$

Thus, $L(E)$ is

\[
S^1 \xrightarrow{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} S^3 \xrightarrow{\begin{bmatrix} -x_2 & x_1 & 0 \\ -x_3 & 0 & x_1 \\ 0 & -x_3 & x_2 \end{bmatrix}} S^3 \xrightarrow{\begin{bmatrix} x_3 & -x_2 & x_1 \end{bmatrix}} S^1
\]

The Koszul complex!
$M$ a f’gend, graded $S$-module, then $R(M)$ is the complex

$$
\cdots \longrightarrow \hat{E} \otimes M_{i-1} \overset{a}{\longrightarrow} \hat{E} \otimes M_i \overset{a}{\longrightarrow} \hat{E} \otimes M_{i+1} \overset{a}{\longrightarrow} \cdots,
$$

where $a = \sum_{i=1}^{n} e_i \otimes x_i$, so $1 \otimes m \mapsto \sum_{i=1}^{n} e_i \otimes x_i \cdot m$, and $\hat{E}$ is the $C$-dual of $E$:

$$\hat{E} \simeq E(n) = \text{Hom}_C(E, C).$$

Just as $L(P) = S \otimes_C P$, $R(M) = \text{Hom}_C(E, M)$.

**Example 14** $M = \mathbb{C} [x_0, x_1] / \langle x_0 x_1, x_0^2 \rangle$. Then

$$0 \longrightarrow E \otimes M_0 \longrightarrow E \otimes M_1 \longrightarrow E \otimes M_2 \longrightarrow E \otimes M_3 \longrightarrow \cdots$$

$$1 \mapsto e_0 \otimes x_0 + e_1 \otimes x_1$$

$$x_0 \mapsto e_0 \otimes x_0^2 + e_1 \otimes x_0 x_1$$

$$x_1 \mapsto e_0 \otimes x_0 x_1 + e_1 \otimes x_1^2$$

$$x_1^n \mapsto e_0 \otimes x_0 x_1^n + e_1 \otimes x_1^{n+1}$$

Thus, $R(M)$ is

$$
\begin{pmatrix}
e_0 \\
e_1
\end{pmatrix}
\xrightarrow{E(2)}
\begin{pmatrix}
0 & e_1
\end{pmatrix}
\xrightarrow{E(3)}
\begin{pmatrix}
e_1
\end{pmatrix}
\xrightarrow{E(4)}
\begin{pmatrix}
e_1
\end{pmatrix}
\xrightarrow{E(5)}
\cdots
$$
This complex is exact, except at the second step. Obviously the kernel of

\[
\begin{bmatrix}
0 & e_1 \\
\end{bmatrix}
\]

is generated by \( \alpha = [1, 0] \) and \( \beta = [0, e_1] \), with relations \( \text{im}(d_1) = \beta + e_0 \alpha = 0, e_1 \beta = 0 \), so that

\[
H^1(\mathbb{R}(M)) \cong E(3)/e_0 \wedge e_1
\]

Compute this, and compute the free resolution of \( M \). This illustrates

**Theorem 15 (Eisenbud-Fløystad-Schreyer)**

\[
H^j(\mathbb{R}(M))_{i+j} = \text{Tor}_i^S(M, \mathbb{C})_{i+j}.
\]

**Corollary 16** The Castelnuovo-Mumford regularity of \( M \) is \( \leq d \) iff \( H^i(\mathbb{R}(M)) = 0 \) for all \( i > d \).
What can be said about higher resonance varieties? **Cohen–Orlik** proved that for \( k \geq 2 \),

\[
R^k(A) = \bigcup L_i \text{ linear.}
\]

**Suciu** showed union need not be disjoint.

**Theorem 17 (Eisenbud-Popescu-Yuzvinsky)**

*Resonance persists:* \( p \in R^k(A) \longrightarrow p \in R^{k+1}(A) \).

The key observation is \( a \in R^k(A) \subseteq P(E) \) means

\[
H^k(A, a) \neq 0 \iff \text{Tor}^S_{\ell-k}(F(A), S/I(p)) \neq 0.
\]

The result follows from interpreting this in terms of Koszul cohomology.

**Theorem 18 (Denham, –)** As for \( R^1(A) \), higher resonance may be interpreted via \( Ext \):

\[
R^k(A) = \bigcup_{k' \leq k} V(\text{ann Ext}^{\ell-k'}(F(A), S)).
\]

Differentials in free resolution can be analyzed using BGG and Grothendieck spectral sequence (work in progress, Denham, –).
For a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^d$, there is a f'gend, graded $S$-module $M$ whose sheafification is $\mathcal{F}$. If $\mathcal{F}$ has Castelnuovo-Mumford regularity $r$, then the **Tate resolution** of $\mathcal{F}$ is obtained by splicing the complex $\mathbb{R}(M_{\geq r})$:

$$
\cdots \rightarrow \hat{E} \otimes M_r \xrightarrow{d^r} \hat{E} \otimes M_{r+1} \rightarrow \hat{E} \otimes M_{r+2} \rightarrow \cdots,
$$

with a free resolution $P_\bullet$ for the kernel of $d^r$:

$$
\cdots \rightarrow P_1 \rightarrow P_0 \xrightarrow{\text{ker}(d^r)} \hat{E} \otimes M_r \rightarrow \hat{E} \otimes M_{r+1} \rightarrow \cdots
$$

By Corollary 16, $\mathbb{R}(M_{\geq r})$ is exact except at the first step, so this yields an exact complex of free $E$-modules.

**Example 19** Since $M = S$ has regularity zero, we obtain Cartan resolutions in both directions, with splice map $E \rightarrow \hat{E} = E(d+1)$ multiplication by $e_0 \wedge e_1 \wedge \cdots \wedge e_d = \ker \left[ e_0, \ldots, e_d \right]^t$. 

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Theorem 20 (Eisenbud-Fløystad-Schreyer)

The $i^{th}$ free module $T^i$ in a Tate resolution for $F$ satisfies

$$
T^i = \bigoplus_j \widehat{E} \otimes H^j(F(i - j)).
$$

Example 21  Twisted cubic $I \subseteq S = \mathbb{C}[x, y, z, w]$

$$
0 \rightarrow S(-3)^2 \xrightarrow{\begin{bmatrix} -z & w \\ y & -z \\ -x & y \end{bmatrix}} S(-2)^3 \xrightarrow{\begin{bmatrix} y^2-xz & yz-xw & z^2-yw \end{bmatrix}} S \rightarrow S/I
$$

Display as a betti table:

$$
\begin{array}{c|ccc}
\text{total} & 1 & 3 & 2 \\
0 & 1 & - & - \\
1 & - & 3 & 2 \\
\end{array}
$$

This has regularity one, so now we can (compute) the Tate resolution:
Plugging these numbers into Theorem 20, we see that

\[

table

\begin{array}{|c|c|c|c|c|c|c|}
\hline
i  & -3 & -2 & -1 & 0 & 1 & 2 \\
\hline
h^1(F(i)) & 8 & 5 & 2 & 0 & 0 & 0 \\
\hline
h^0(F(i)) & 0 & 0 & 0 & 1 & 4 & 7 \\
\hline
\end{array}
\]

Does this make sense?

\[
\mathcal{F} = \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}(3)
\]

so

\[
h^1(\mathcal{F}(i)) = h^1(\mathcal{O}_{\mathbb{P}^1}(3i)) = h^0(\mathcal{O}_{\mathbb{P}^1}(-3i - 2))
\]

and

\[
h^0(\mathcal{F}(i)) = h^0(\mathcal{O}_{\mathbb{P}^1}(3i)) = 3i + 1, \ i \geq 0
\]

THE END! THANK YOU!


