Math 225
Review problems for midterm 1

Read your lecture notes and the textbook (Sections 1.1-1.5, 1.7, 2.1, 2.2) and solve as many exercises as possible from the suggested problems in homework 1 – 4. Below you can find some extra review problems:

1. Let A_1, A_2 be the following matrices

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 2 & 2 \end{bmatrix},$$

Solve the equations $A_1x = b$, $A_2x = b$ indicating the pivot positions and free variables, for
(a) $b = [1, 1, 3]^T$; (b) $b = [1, 1, 0]^T$, (c) $b = 0$.

2. Write down the solutions of the systems in problem 1 in a parametric vector form.

3. Explain the relation between the solution sets of systems of linear equations (SLEs) $Ax = b$ and $Ax = 0$, and give an example of your choice.

4. Let $x_1 = [1, 2, 3]^T$ and $x_2 = [1, 4, 5]^T$ be solutions of a SLE $Ax = b$. Find one more solution of the equation.

5. Compute the following matrices (a) A^{100}; (b) $A(B+C)+2C(B^T-A)$; (c) $AC-CA$, where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. (a) Give the definition of linearly independent vectors $v_1, ..., v_k \in R^n$,
(b) Explain how to find out whether given vectors $v_1, ..., v_k \in R^n$ are linearly independent, and give an example of your choice.

7. Determine whether the following vectors are linearly independent
(a) $v_1 = [1, 2, 3, 0, 1]^T$, $v_2 = [1, 2, 3, 1, 1]^T$, $v_3 = [3, 2, 3, 1, 1]^T$, $v_4 = [2, 4, 6, 1, 3]^T$;
(b) Columns of the matrices in the problems above;

8. Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix.
(a) Show that if the columns of B are linearly dependent then columns of the product AB are also linearly dependent (Hint: use the fact that $AB = [Ab_1 ... Ab_p]$ where $b_1, ..., b_p$ are the columns of the matrix B.)
(b) Show that if the columns of the product AB are linearly independent then columns of B are also linearly independent.
(c) Find an example showing that even if the columns of B are linearly independent the columns of the product AB may be linearly dependent.

9. (a) Give the definition of $\mathrm{Span}(v_1, v_2, ..., v_p)$ of vectors $v_1, v_2, ..., v_p \in R^n$
(b) Determine whether the vector v_4 in part a of problem 7 is in the span of the other three vectors v_1, v_2, v_3.

10. Let $v_1, v_2, , , , v_n \in R^m$. Explain how tho determine if $\mathrm{Span}(v_1, v_2, ..., v_n) = R^m$, and give an example of your choice.

11. Give the definiton of an inverse of a matrix and show that an inverse of an invertible matrix is unique.

12. Let A, B be invertible matrices.
(a) Show that A^{-1}, A^T are invertible and $(A^{-1})^{-1} = A, (A^T)^{-1} = (A^{-1})^T$
(b) Show that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

13. Find the inverse A^{-1} (if any) of the matrices:
(a) $A = \begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$; (b) $A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \\ 5 & 6 & 3 \end{bmatrix}$

14. Assume $A^5 = 0$ for an $n \times n$ matrix A. Calculate $B = (I - A)(I + A + A^2 + A^3 + A^4)$. Is $I - A$ an invertible matrix?