NAME:

Math 286 Spring 2009 Test 1

Total points: **75**. **Explain** all answers. No notes, books, or electronic devices.

1 (6 points). *Write down general solutions for the following equations. No explanation required.*

- $\frac{dP}{dt} = -4t$
  Solution:
  $$P(t) = -2t^2 + C.$$
- $\frac{dP}{dt} = -4P$
  Solution:
  $$P(t) = Ce^{-4t}.$$
- $\frac{d^2P}{dt^2} = -4P$
  Solution:
  $$P(t) = C\cos(2t) + D\sin(2t).$$

2 (4 points). *Find an example of a function f for which the directional field of $\frac{dy}{dx} = f(x, y)$ shows a vertically repeating pattern. Explain.*

Solution:

For example

$$f(x, y) = x \sin(y).$$

As long as $f(x, y)$ is periodic in y there will be a vertically repeating pattern. The function need not be periodic in x.

3 (4 points). *True/False and Explain: if the directional field of $\frac{dy}{dx} = f(x, y)$ shows a horizontally repeating pattern, then each solution $y(x)$ must be periodic.*

Solution:

Not true. Let

$$f(x, y) = 1 + \sin(x).$$

This function is periodic in x, hence we get horizontally repeating pattern. The solution

$$y(x) = x + \cos(x) + c$$

is clearly NOT periodic.

4 (20=14+2+4 points). *Write $x(t)$ for the height at time t of an object that is falling downward under the influence of gravity. Assume the object encounters air resistance proportional to the square of its velocity $v(t) = x'(t)$. Then by Newton law we get*

$$\frac{dv}{dt} = av^2 - g \qquad \text{(for some positive constants } a \text{ and } g\text{).}$$

- Solve for $v(t)$. Write $b = \sqrt{g/a}$ to simplify the calculations.

  Solution:

  This equation is separable.

$$\begin{aligned} \frac{dv}{a(v^2 - g/a)} &= dt \\ -\int \frac{dv}{b^2 - v^2} &= \int a dt \\ -\frac{1}{2b} \log\left|\frac{v+b}{v-b}\right| &= at + c \qquad \text{Any hard integrals will be written on the cover page.} \\ \frac{v+b}{v-b} &= \pm e^{-2bc} e^{2abt} = Ae^{-2abt} \\ v + b &= (v-b)Ae^{-2abt} \\ v(1 - Ae^{-2abt}) &= -b(1 + Ae^{-2abt}) \\ v &= -b\frac{1 + Ae^{-2abt}}{1 + Ae^{-2abt}} \end{aligned}$$

- Find the terminal velocity. Solution:

$$\lim_{t\to\infty} v(t) = -b\frac{1+0}{1-0} = -b.$$

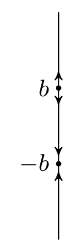
  The terminal velocity must be negative since the object is falling.

- Sketch the phase line for v. Does it agree with second part of the problem? Equilibrium points are the solutions of

$$av^2 - g = 0.$$

  That is $v = \pm b$.

  We check the sign of $av^2 - g$ to find the directions for the phase line. $av^2 - g$ is positive above b and below $-b$. Hence



5 (14 points). *Solve*

$$(\sec^2 y)y' + (\sec^2 x)\tan y = e^{-\tan x}.$$

Solution:

This equation is not linear, not separable, not Bernoulli and not homogeneous. So we need to try substitution. Let $v = \tan y$.

$$v' = (\sec^2 y)y'$$
$$v' + (\sec^2 x)v = e^{-\tan x}$$

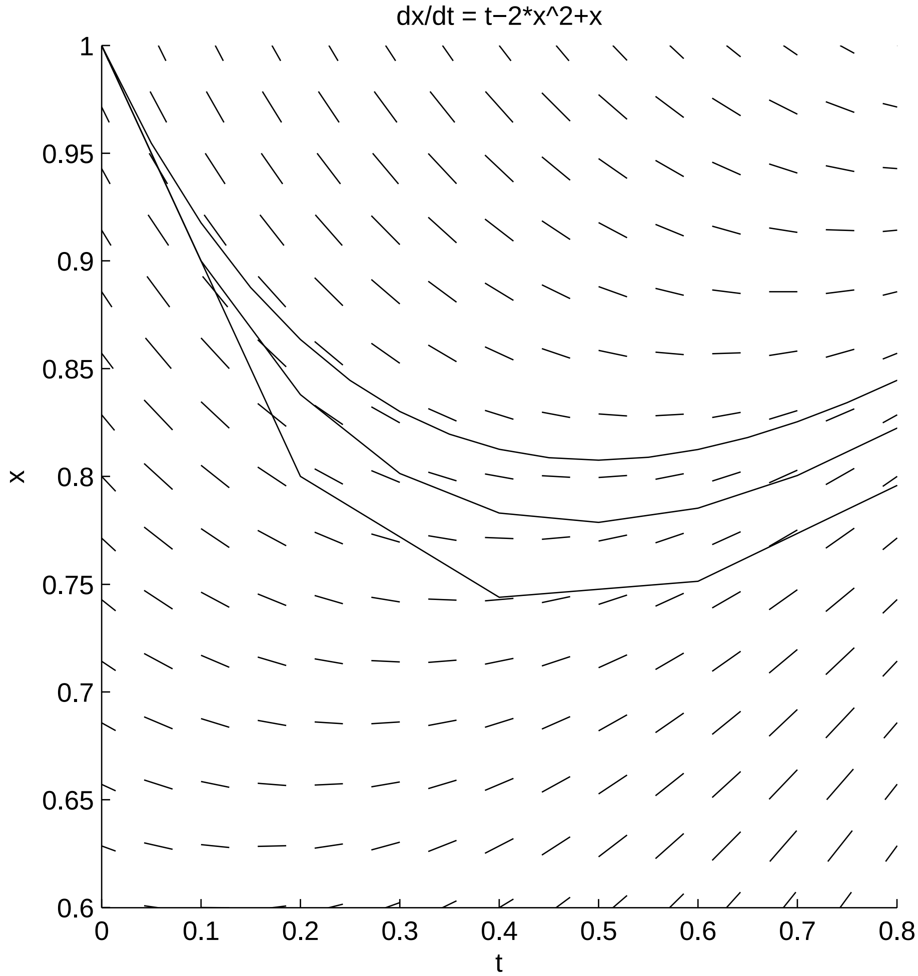
This equation is first order linear with integrating factor

$$e^{\int (\sec^2 x)dx} = e^{\tan x}.$$

We get

$$e^{\tan x}v' + e^{\tan x}(\sec^2 x)v = 1$$
$$(e^{\tan x}v)' = 1$$
$$e^{\tan x}v = x + c$$
$$c = (x + c)e^{-\tan x}$$
$$\tan y = (x + c)e^{-\tan x}$$
$$y = \arctan\left((x + c)e^{-\tan x}\right)$$

6 (15=9+2+4 points)**.** *Consider the first order differential equation* $\frac{dx}{dt} = t - 2x^2 + x$.



- *"Halving the step size halves the error in Euler's method." Illustrate the meaning of this principle with a suitable sketch above (take* $(t_0, x_0) = (0, 1)$*).*
- *State the Euler update formula. Not the Improved Euler update!*
  Solution:

$$x_{i+1} = x_i + hf(x_i, t_i).$$

- *Consider* $\frac{dx}{dt} = \sin(x - t^2)$*, with* $(t_0, x_0) = (0, 0)$ *and* $h = 0.1$*. Evaluate* t_1, x_1, t_2, x_2*.*
  Solution:

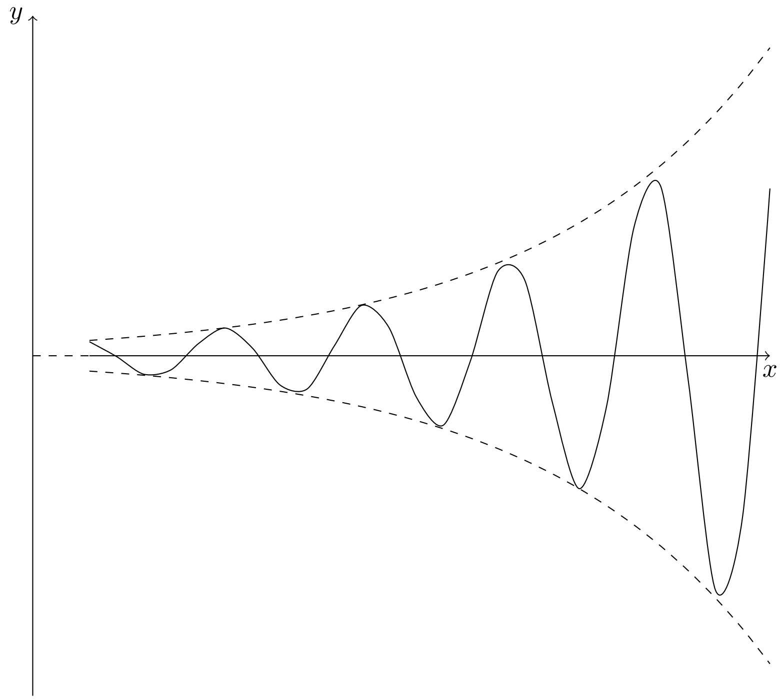
$$t_1 = t_0 + h = 0.1,$$
$$x_1 = x_0 + h\sin(t_0 - x_0^2) = 0 + 0.1\sin(0 - 0^2) = 0,$$
$$t_2 = t_1 + h = 0.2,$$
$$x_2 = x_1 + h\sin(t_1 - x_1^2) = 0 + 0.1\sin(0 - 0.1^2) = 0.1\sin(-0.01) \approx -0.001,$$

since $\sin x \approx x$ *for small* x*.*

7 (12=6+6 points)**.** *Suppose the characteristic equation of a constant coefficient, linear, homogeneous differential equation has roots* $-1, 2, 2, 3+4i, 3-4i$.

- *Write down the general solution of the equation (do not use complex exponents).*
  Solution:
  $$y(x) = c_1 e^{-x} + (c_2 + c_3 x)e^{2x} + e^{3x}(c_4 \cos 4x + c_5 \sin 4x).$$
- *Sketch the graph for large positive* x*-values. Carefully justify your reasoning. (Hint: Which term(s) are dominating.)*
  Solution:
  The highest exponent in the above solution is e^{3x}. *We pull it out and we get*
  $$\begin{aligned} y(x) &= e^{3x}[c_1 e^{-4x} + (c_2 + c_3 x)e^{-x} + (c_4 \cos 4x + c_5 \sin 4x)] \\ &\approx e^{3x}(c_4 \cos 4x + c_5 \sin 4x). \quad \textit{for very large } x \ (e^{-x} \approx 0). \end{aligned}$$

Therefore the solution should look like



The distance between every second zero of the solution should be $\pi/2$, *since* $\sin 4x$ *and* $\cos 4x$ *are dominant. The dashed line grows like* e^{3x}.