NAME:

Math 285 Spring 2003 — Test 2

Total points: **100**. Do all questions. Explain all answers. No notes, books, calculators or computers.

1. [6 points] For the following differential equation, write down the form of the complementary solution y_c, and of the particular solution y_p (by the method of undetermined coefficients). You do *not* have to evaluate any coefficients.

$y'' + 9y = x \cos 3x$.

$y_c =$

$y_p =$

2. [25=20+5 points]
(a) For each value of $\omega_0 > 0$, solve the forced undamped oscillator equation

$$x''(t) + \omega_0^2 x(t) = \sin(2t),$$

by Undetermined Coefficients.

(b) Roughly sketch a typical solution $x(t)$ for large t-values, for $\omega_0 = 2, 3$.

3. [8 points] What does "resonance" mean? Use problem #2 as an example.

4. [10=7+3 points]

(a) Solve the boundary value problem

$$(\frac{d}{dx} - 2)(\frac{d}{dx} - 3)y = 0, \quad y(0) = 0, \quad y(1) = -4.$$

(b) Does $y(x)$ grow or decay as $x \to \infty$? (Circle one answer.) Explain in detail.

5. [12 points] A certain shock absorber is described by

$$mx''(t) + x'(t) + x(t) = 0.$$

Find all m-values such that the amplitude of oscillation gets reduced by at least 80%, during each unit of time. (You may assume $m > 1/4$.)

6. [14=6+8 points] Consider the damped, forced oscillator equation

$$mx''(t) + cx'(t) + kx(t) = F_0 \cos(\omega t),$$

where m, c, k are positive constants.

(a) Write down the form of the steady periodic response to the forcing. (You do *not* have to evaluate any of the coefficients.)

(b) Take $x(0) = 1000, x'(0) = 2000$ and $\omega = 2$. Roughly sketch the shape of the solution $x(t)$ for large t, as best you can. Explain.

7. [25=20+5 points] Consider the eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < \pi,$$

under the Neumann boundary conditions $X'(0) = 0, X'(\pi) = 0$. Show that the eigenvalues are $\lambda_n = n^2$ for $n = 0, 1, 2, 3, \ldots$, with corresponding eigenfunctions $X_0(x) = 1$ and $X_n(x) = \cos(nx)$ for $n = 1, 2, 3, \ldots$.

Hint. Consider cases $\lambda = 0, \lambda < 0, \lambda > 0$.

(b) If $n, m \geq 0$ and $n \neq m$ then $\int_0^\pi \cos(nx)\cos(mx)\,dx = 0$. Explain why this is true.

Formulas

Here are some formulas you might be able to use on the test:

$$
\begin{aligned}
y &= y_c + y_p \\
\omega_0 &= \sqrt{\frac{k}{m}}, \quad p = \frac{c}{2m}, \quad \omega_1 = \sqrt{\omega_0^2 - p^2} \\
e^{(a \pm ib)x} &= e^{ax}(\cos bx \pm i \sin bx) \\
y &= -y_1 \int \frac{y_2 f}{W}\, dx + y_2 \int \frac{y_1 f}{W}\, dx \\
W &= y_1 y_2' - y_1' y_2
\end{aligned}
$$