# Math 285 Spring 2003 — Test 2 Solutions

Total points: **100**. Do all questions. Explain all answers. No notes, books, calculators or computers.

1. [6 points] For the following differential equation, write down the form of the complementary solution $y_c$, and of the particular solution $y_p$ (by the method of undetermined coefficients). You do *not* have to evaluate any coefficients.

$$y'' + 9y = x\cos 3x.$$

$$\boxed{y_c = c_1 \cos 3x + c_2 \sin 3x}$$

Here $f(x) = x\cos(3x) = (\text{polynomial of degree } 1) \cdot \cos(3x)$. So we guess $y_p = (A + Bx)\cos 3x + (C + Dx)\sin 3x$ by Undetermined Coefficients Rule 1. But now there is duplication with $y_c$, and so we have to use Rule 2: multiply through by $x$ to get

$$\boxed{y_p = x(A + Bx)\cos 3x + x(C + Dx)\sin 3x}$$

*Aside.* The answer is

$$y_p = \frac{1}{36}x\cos(3x) + \frac{18x^2 - 1}{216}\sin(3x),$$

but you were not required to find this.

2. [25=20+5 points]
(a) For each value of $\omega_0 > 0$, solve the forced undamped oscillator equation

$$x''(t) + \omega_0^2 x(t) = \sin(2t),$$

by Undetermined Coefficients.

*Solution.* First we write down the complementary solution, solving $x''(t) + \omega_0^2 x(t) = 0$:

$$x_c(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

Then by Undetermined Coefficients we guess $x_p(t) = A \cos 2t + B \sin 2t$.

There is no duplication if $\boxed{\omega_0 \neq 2}$, and so in that case we can just substitute $x_p$ into the DE to get

$$[-4A \cos 2t - 4B \sin 2t] + \omega_0^2 [A \cos 2t + B \sin 2t] = \sin(2t).$$

Equating coefficients of $\cos 2t$ gives $-4A + \omega_0^2 A = 0$ and so $A = 0$. Equating coefficients of $\sin 2t$ gives $-4B + \omega_0^2 B = 1$ and so $B = 1/(\omega_0^2 - 4)$, so that

$$x(t) = x_c(t) + x_p(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{1}{\omega_0^2 - 4} \sin 2t.$$

But there is duplication if $\boxed{\omega_0 = 2}$, and so in that case we must multiply our guess for $x_p$ by $t$ to get $x_p(t) = t[A \cos 2t + B \sin 2t]$. Substituting this into the DE yields

$$\begin{aligned} t''[A \cos 2t + B \sin 2t] + 2t'[A \cos 2t + B \sin 2t]' + t[A \cos 2t + B \sin 2t]'' \\ + \omega_0^2 t [A \cos 2t + B \sin 2t] = \sin 2t, \end{aligned}$$

where we have used the product rule $(FG)'' = F''G + 2F'G' + FG''$. Taking the derivatives and using $\omega_0 = 2$ gives

$$2[-2A \sin 2t + 2B \cos 2t] + t[-4A \cos 2t - 4B \sin 2t] + 4t[A \cos 2t + B \sin 2t] = \sin 2t,$$

which simplifies to

$$-4A \sin 2t + 4B \cos 2t = \sin 2t,$$

so that $-4A = 1$ and $4B = 0$, or $A = -1/4$ and $B = 0$. That is,

$$x(t) = x_c(t) + x_p(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4} t \cos 2t.$$

(b) Roughly sketch a typical solution $x(t)$ for large $t$-values, for $\omega_0 = 2, 3$.

*Solution.*

3. [8 points] What does "resonance" mean? Use problem #2 as an example.

*Solution.* "Resonance" is the phenomenon where the **amplitude** of an oscillation **increases to infinity**, in other words $x(t)$ grows and oscillates. It occurs when an **undamped** system is **forced at its natural frequency**.

In problem #2, the natural frequency is $\omega_0$ and the forcing frequency is $\omega = 2$, so resonance occurs when $\omega_0 = 2$. The graph in #2(b) for $\omega_0 = 2$ indeed shows the amplitude getting bigger and bigger (in fact, the amplitude grows linearly as $t \to \infty$, because of the "$t$" in $x_p = -\frac{1}{4}t\cos 2t$).

4. [10=7+3 points]
(a) Solve the boundary value problem

$$(\frac{d}{dx} - 2)(\frac{d}{dx} - 3)y = 0, \quad y(0) = 0, \quad y(1) = -4.$$

*Solution.* The roots of the characteristic equation are $r = 2, 3$ and so $\boxed{y = c_1 e^{3x} + c_2 e^{2x}}$. The boundary condition $y(0) = 0$ gives $c_1 + c_2 = 0$ and so $c_2 = -c_1$. Then the boundary condition $y(1) = -4$ implies

$$c_1(e^3 - e^2) = -4$$

and so $c_1 = -4/(e^3 - e^2)$. That is,

$$y(x) = -\frac{4}{e^3 - e^2}(e^{3x} - e^{2x}).$$

*Aside.* The DE can be expanded to $(\frac{d^2}{dx^2} - 5\frac{d}{dx} + 6)y = 0$ or $y'' - 5y' + 6y = 0$, and from this you can find the characteristic equation and find the roots. But there is no point in doing so, since the roots $r = 2, 3$ are given already in the factored form of the DE.

(b) Does $y(x)$ $\boxed{\text{grow}}$ or decay as $x \to \infty$? (Circle one answer.) Explain in detail.

*Solution.* We need to evaluate the limit of $y(x)$ as $x \to \infty$. We do so by pulling out the highest exponential as a common factor:

$$\lim_{x\to\infty} y(x) = -\lim_{x\to\infty} \frac{4}{e^3 - e^2} e^{3x}(1 - e^{-x}) = -\frac{4}{e^3 - e^2} \cdot \infty \cdot 1 = -\infty.$$

So $y(x)$ **grows** (to $-\infty$) as $x \to \infty$.

5. [12 points] A certain shock absorber is described by

$$mx''(t) + x'(t) + x(t) = 0.$$

Find all $m$-values such that the amplitude of oscillation gets reduced by at least 80%, during each unit of time. (You may assume $m > 1/4$.)

*Solution.* The system is underdamped because $c^2 - 4km = 1^2 - 4m < 0$, using the assumption $m > 1/4$. So the solution has the form

$$x(t) = Ce^{-pt}\cos(\omega_1 t - \gamma),$$

where $p = c/2m = 1/2m$. The amplitude at time $t$ is $Ce^{-pt}$ and the amplitude at time $t+1$ is $Ce^{-p(t+1)}$. So we want

$$\begin{aligned}
Ce^{-p(t+1)} &\le (20\%)Ce^{-pt} \\
e^{-p} &\le \frac{1}{5} \\
e^{p} &\ge 5 \\
p &\ge \log 5 \\
\frac{1}{2m} &\ge \log 5 \\
\frac{1}{4} < m &\le \frac{1}{2\log 5} \approx 0.31
\end{aligned}$$

6. [14=6+8 points] Consider the damped, forced oscillator equation

$$mx''(t) + cx'(t) + kx(t) = F_0 \cos(\omega t),$$

where $m, c, k$ are positive constants.

(a) Write down the form of the steady periodic response to the forcing. (You do *not* have to evaluate any of the coefficients.)

*Solution.* $\boxed{x_p(t) = A\cos(\omega t) + B\sin(\omega t)}$ by Undetermined Coefficients. There is definitely no duplication with $x_c(t)$: damping is present ($c > 0$) and so the complementary solution involves a decaying exponential ($x_c$ is transient), whereas our guess for $x_p$ has no exponentials.

(b) Take $x(0) = 1000, x'(0) = 2000$ and $\omega = 2$. Roughly sketch the shape of the solution $x(t)$ for large $t$, as best you can. Explain.

*Solution.* $\boxed{x = x_c + x_p}$ and the complementary solution is transient due to the damping, $x_c(t) \to 0$ as $t \to \infty$. So $x_c$ is negligible for large $t$. Hence the initial conditions are irrelevant for large $t$ (since they only affect $x_c$ and not $x_p$).

What matters is just the frequency and amplitude of the response $x_p$ to the forcing. The forcing frequency is $\boxed{\omega = 2}$, and the amplitude of $x_p$ (in other words $A$ and $B$) cannot be found by Undetermined Coefficients because we are not given values for $m, c, k, F_0$. So the best graph we can draw for large $t$ is a simple oscillation with frequency 2, period $\pi$:

*Warning.* $A$ and $B$ have nothing to do with the initial conditions 1000 and 2000. The values of $A$ and $B$ can only be determined by substituting $x_p$ into the original DE (Method of Undetermined Coefficients).

7. [25=20+5 points] Consider the eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < \pi,$$

under the Neumann boundary conditions $X'(0) = 0, X'(\pi) = 0$. Show that the eigenvalues are $\lambda_n = n^2$ for $n = 0, 1, 2, 3, \ldots$, with corresponding eigenfunctions $X_0(x) = 1$ and $X_n(x) = \cos(nx)$ for $n = 1, 2, 3, \ldots$.

*Hint.* Consider cases $\lambda = 0, \lambda < 0, \lambda > 0$.

*Solution.* First consider $\boxed{\lambda = 0}$, so that the equation is $X'' = 0$. This has general solution $X = Ax + B$. The boundary condition $X'(0) = 0$ implies $A = 0$, so that now $X = B$. Then the boundary condition $X'(\pi) = 0$ is true. So $\lambda = 0$ is an eigenvalue with corresponding eigenfunction $X = 1$ (where we take $B = 1$ for convenience).

Next consider $\boxed{\lambda < 0}$, so that $\lambda = -|\lambda|$. Then the equation is $X'' - |\lambda| X = 0$, which has general solution

$$X(x) = A\cosh(\sqrt{|\lambda|}x) + B\sinh(\sqrt{|\lambda|}x),$$

so that

$$X'(x) = A\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}x) + B\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}x).$$

(*Note.* In these eigenvalue problems, when one of the boundary conditions is at $x = 0$ it is always better to solve using cosh and sinh rather than in terms of exponentials.)

The boundary condition $X'(0) = 0$ implies $B = 0$, since $\cosh(0) = 1$ and $\sinh(0) = 0$ and $|\lambda| \neq 0$ by assumption. So now $X(x) = A\cosh(\sqrt{|\lambda|}x)$. Then the boundary condition $X'(\pi) = 0$ implies

$$A\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}\pi) = 0.$$

Now, $\sinh(u)$ is only equal to zero at $u = 0$ [just think about its graph, or the fact that $\sinh u = (e^u - e^{-u})/2$], and so $\sinh(\sqrt{|\lambda|}\pi) \neq 0$. Thus the only possibility is that $A = 0$. Then $X \equiv 0$, so that $\lambda < 0$ is not an eigenvalue.

Next consider $\boxed{\lambda > 0}$, so that the general solution of $X'' + \lambda X = 0$ is

$$X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x),$$

giving

$$X'(x) = -A\sqrt{\lambda}\sin(\sqrt{\lambda}x) + B\sqrt{\lambda}\cos(\sqrt{\lambda}x).$$

The boundary condition $X'(0) = 0$ implies $B = 0$, since $\cos(0) = 1$ and $\sin(0) = 0$ and $\lambda \neq 0$ by assumption. So now $X(x) = A\cos(\sqrt{\lambda}x)$. Then the boundary condition $X'(\pi) = 0$ implies

$$-A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) = 0.$$

Now, $\lambda \neq 0$ by assumption, and so either $A = 0$ or else $\sin(\sqrt{\lambda}\pi) = 0$. If $A = 0$ then $X \equiv 0$ again. So our only chance for $\lambda$ to be an eigenvalue is if $\sin(\sqrt{\lambda}\pi) = 0$. This happens whenever $\sqrt{\lambda}$ is a positive integer, or in other words:

$$\lambda = 1^2, 2^2, 3^2, \ldots.$$

So these $\lambda$-values (and no others) are the positive eigenvalues.

We saw above that the eigenfunction corresponding to eigenvalue $\lambda > 0$ is $X(x) = A\cos(\sqrt{\lambda}x)$. By taking $A = 1$ for convenience we get the following eigenfunctions corresponding to $\lambda = 1^2, 2^2, 3^2, \ldots$:

$$X(x) = \cos(1x), \cos(2x), \cos(3x), \ldots.$$

Remembering from above that $\lambda = 0$ is also an eigenvalue, with eigenfunction $X = 1$, we have shown that the complete collection of eigenvalues is $\boxed{\lambda_n = n^2}$ for integers $\boxed{n = 0, 1, 2, 3, \ldots}$, with corresponding eigenfunctions $\boxed{X_n(x) = \cos(nx)}$.

(*Check.* When $n = 0$, this reduces back to $\lambda_0 = 0$ and $X_0(x) = 1$.)

(b) If $n, m \geq 0$ and $n \neq m$ then $\int_0^\pi \cos(nx)\cos(mx)\,dx = 0$. Explain why this is true.

*Solution.* By part (a) we know $X_n(x) = \cos(nx)$ and $X_m(x) = \cos(mx)$ are eigenfunctions satisfying homogeneous Neumann boundary conditions at $x = 0, \pi$, with eigenvalues $\lambda_n = n^2$ and $\lambda_m = m^2$. If $n \neq m$ then $\lambda_n \neq \lambda_m$, and so $X_n$ and $X_m$ are orthogonal by the Orthogonality of Eigenfunctions Theorem. That is,

$$0 = \int_0^\pi X_n(x)X_m(x)\,dx = \int_0^\pi \cos(nx)\cos(mx)\,dx.$$

**Formulas**

Here are some formulas you might be able to use on the test:

$$
\begin{aligned}
y &= y_c + y_p \\
\omega_0 &= \sqrt{\frac{k}{m}}, \quad p = \frac{c}{2m}, \quad \omega_1 = \sqrt{\omega_0^2 - p^2} \\
e^{(a \pm ib)x} &= e^{ax}(\cos bx \pm i \sin bx) \\
y &= -y_1 \int \frac{y_2 f}{W}\, dx + y_2 \int \frac{y_1 f}{W}\, dx \\
W &= y_1 y_2' - y_1' y_2
\end{aligned}
$$