On a sharp rearrangement inequality for the Coulomb energy

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Symmetrization.

If $A \subset \mathbb{R}^n$ has finite measure, its

- **symmetric rearrangement** $A^*$

is the centered ball of the same measure.

If $f : \mathbb{R}^n \to \mathbb{R}_+$ vanishes at infinity, its

- **symmetric decreasing rearrangement** $f^*$

is the radially decreasing function equimeasurable with $f$. 
Rearrangement inequalities.

\[(A \cap B)^* \subset A^* \cap B^*\]
order-preserving

\[\text{Per}(A) \geq \text{Per}(A^*)\]
isoperimetry

\[(A + B)^* \supset A^* + B^*\]
Brunn-Minkowski

\[\int f g \leq \int f^* g^*\]
Hardy-Littlewood

\[\|\nabla f\|_p \geq \|\nabla f^*\|_p\]
Pólya-Szegő

\[\int f(g * h) \leq \int f^*(g^* * h^*)\]
Riesz

\[
\begin{cases}
-\Delta u = f \\
-\Delta v = f^*
\end{cases}
\implies u^* \leq v
\]
Talenti
We consider Riesz’ inequality for the Coulomb energy

\[ Q(f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|} \, dxdy. \]

- positive definite quadratic form;
- \( Q(f) \leq \text{Const.} \|f\|_{6/5}^2 \);
- by Riesz’ inequality \( Q(f) \leq Q(f^*), \) with equality only if \( f \circ \tau = f^* \) for some translation \( \tau. \)

Is it true that \( Q(f^*) - Q(f) \geq \inf_{\tau} Q(f^* - f \circ \tau) \)?
Coulomb energy:

\[ Q(f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)f(y)}{|x-y|} \, dx \, dy. \]

Is it true that

\[ Q(f^*) - Q(f) \geq C \cdot \inf_{\tau} Q(f^* - f \circ \tau) \]

for some \( C > 0 \) ?
Motivation: Dynamical stability of a gaseous star.

**Euler-Poisson System:** density \( \rho(x,t) \geq 0 \), velocity field \( u(x,t) \in \mathbb{R}^3 \), pressure \( P = \rho^\gamma \), gravitational potential \( V \),

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\rho \partial_t u + \rho (u \cdot \nabla) u &= -\nabla P - \rho \nabla V, \\
\Delta V &= 4\pi \rho.
\end{align*}
\]

Conserved energy:

\[
E = \frac{1}{2} \int |u|^2 \rho \, dx + \frac{1}{\gamma - 1} \rho^\gamma \, dx - \frac{1}{2} Q(\rho).
\]

Minimize \( E(\rho, 0) \) under the mass constraint \( \int u = M \) to obtain radial steady states.

Are these steady-states dynamically stable?
Dynamical stability (continued).

Suppose $\rho_{\text{min}}$ is a radially decreasing function so that $(\rho_{\text{min}}, 0)$ achieves the minimal energy $E_{\text{min}}$ for its mass.

Let $(\rho_0, u_0)$ be a perturbation of $(\rho_{\text{min}}, 0)$ that has the same mass and almost the same energy $E(\rho_0, u_0) \approx E_{\text{min}}$.

- By conservation of mass and energy,
  
  \[ E(\rho_t, u_t) = E(\rho_0, u_0) \approx E_{\text{min}}, \]

  and hence $u_t \approx 0$ and $Q(\rho_t) \approx Q(\rho_t^*)$ for all $t > 0$.

- YG’s conjecture $\Rightarrow \rho_t \approx \rho_t^* \circ \tau_t$ for some translation $\tau_t$.

- Stability under radial perturbations implies $\rho_t^* \approx \rho_{\text{min}}$.

So $\rho_t \circ \tau_t$ must be close to $\rho_{\text{min}}$!

(The argument reduces the stability problem to radial perturbations, which is much easier.)
THEOREM (with Yan Guo, 2005).

Assume $g$ is symmetric decreasing, $0 < Q(g) < \infty$. Consider a sequence $\{f_k\}_{k \geq 0}$.

If

$$Q(f_k^* - g) \to 0, \quad \text{and} \quad Q(f_k) \to Q(g),$$

then there exists translations $\{\tau_k\}_{k \geq 0}$ such that

$$Q(g - f_k \circ \tau_k) \to 0.$$

(This is a weak version of the conjecture, which still implies dynamical stability. It is proved by a compactness argument.)
Quantitative isoperimetric inequality.

Assume $A \subset \mathbb{R}^n$ has finite measure. Define its

- **asymmetry** $\alpha(A) = \inf_\tau \frac{\text{Vol}(A^* \Delta \tau A)}{\text{Vol}(A)}$;

- **isoperimetric deficit** $\delta(A) = \text{Per}(A) - \text{Per}(A^*)$.

**THEOREM** (Hall 1992; Fusco, Maggi, Pratelli 2008)

There exist constants $C(n)$ such that for every $A \subset \mathbb{R}^n$,

$$\alpha(A) \leq C(n)(\delta(A))^{1/2}.$$  

(The exponent $1/2$ is largest-possible.
Similar results exist for Sobolev inequalities.)
Inequality for uniform charge distributions.

Assume $A \subset \mathbb{R}^n$ has finite measure. Set
\[ Q(A) = \int_A \int_A |x - y|^{-(n-2)} \, dx \, dy , \]
and define its

- **asymmetry** $\alpha(A) = \inf_{\tau} \frac{\text{Vol}(A^* \Delta \tau A)}{\text{Vol}(A)}$;

- **Coulomb deficit** $\delta(A) = \frac{Q(A^*) - Q(A)}{Q(A^*)}$.

**PROPOSITION** (with G. Chambers, 2009)

There exist constants $C(n)$ such that for every $A \subset \mathbb{R}^n$,
\[ \alpha(A) \leq C(n)(\delta(A))^{\frac{1}{n+2}} . \]

(The exponent is not largest-possible. YG’s conjecture would correspond to an exponent $\frac{n}{n+2}$.)
PROOF.

Consider the potential $\phi_A(x) = \int_A |x - y|^{-1} dy$.

KEY STEPS: Assume that $A$ has measure $\omega_n$.

- A simple geometric argument shows that
  $$\phi_A^*(0) - \phi_A(x) \geq c_1 \cdot (\alpha(A))^2 \quad \text{for all } x \in \mathbb{R}^n.$$ 

- Talenti’s comparison principle says that
  $$\phi_A^*(x) \geq (\phi_A)^*(x) \quad \text{for all } x \in \mathbb{R}^n.$$ 

We now estimate $Q(A) = \int_A \phi_A$ as follows:
PROOF (conclusion).

\[
\delta(A) = Q(A^*) - Q(A) \\
= \int_{A^*} \phi_{A^*}(x) \, dx - \int_A \phi_A(x) \, dx \\
\geq \int_{A^*} \phi_{A^*}(x) - (\phi_A)^*(x) \, dx \\
\geq \int_{A^*} \left[ \phi_A(x) - \phi_{A^*}(0) + c_1 (\alpha(A))^2 \right] + dx \\
\geq c_2 (\alpha(A))^{n+2}.
\]
Final remarks.

- YG’s conjecture is open, even for sets.

- It is not hard to see that a small Coulomb deficit $\delta(f)$ forces all level sets of $f$ to be close to balls. But it is less obvious why these balls should be almost concentric.

- Few stability results are known for convolutions and other multiple integrals. In some cases, not even the equality cases have been fully characterized.

- Such stability could be useful for taking continuum limits of discrete systems.