On Eigenvalue Inequalities for the Klein-Gordon Operators

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Table of contents

1 Klein-Gordon operators

2 Results
   • An Upper Bound
   • A Ratio Bound

3 Weyl Asymptotics and a Berezin-Li-Yau type inequality
   • Weyl Asymptotics
   • A Counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau Inequality
   • An Improvement to the Berezin-Li-Yau type inequality

4 Sharp Bounds for Some Spectral Functions of $H_{0,\Omega}$
Table of Contents

1 Klein-Gordon operators

2 Results
   • An Upper Bound
   • A Ratio Bound

3 Weyl Asymptotics and a Berezin-Li-Yau type inequality
   • Weyl Asymptotics
   • A Counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau Inequality
   • An Improvement to the Berezin-Li-Yau type inequality

4 Sharp Bounds for Some Spectral Functions of $H_{0,\Omega}$
Klein-Gordon Equation

The Klein-Gordon equation can be written as

\[ i\hbar \frac{\partial}{\partial t} \psi = \sqrt{-\hbar^2 c^2 \nabla^2 + m^2 c^4} \psi. \]

We restrict \( \sqrt{|P|^2 + m^2} = \sqrt{-\Delta + m^2} \) to a bounded domain \( \Omega \) in \( \mathbb{R}^d \) and designate

\[ H_{m,\Omega} = \sqrt{|P|^2 + m^2}\bigg|_{\Omega}. \]
Klein-Gordon Operator: Definition

For $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\sqrt{-\Delta} + m^2 \varphi := \mathcal{F}^{-1} \sqrt{|\xi|^2 + m^2 \hat{\varphi}(\xi)}.$$

$H_{0,\Omega}$ coincides with the generator of the Cauchy stochastic process with a killing condition on $\partial \Omega$. (*Fractional Laplacian with power $\frac{1}{2}$*).
Consider the quadratic form on $C_c^\infty(\Omega)$ given by

$$\varphi \rightarrow \int_\Omega \overline{\varphi} \sqrt{-\Delta + m^2} \varphi.$$ 

(Here $\sqrt{-\Delta + m^2}$ is calculated for $\mathbb{R}^d$.) Since this quadratic form is positive and defined on a dense set, it extends to a unique minimal positive operator (the Friedrichs extension) on $L^2(\Omega)$, which we designate $H_{m,\Omega}$.
Table of Contents

1 Klein-Gordon operators

2 Results
   • An Upper Bound
   • A Ratio Bound

3 Weyl Asymptotics and a Berezin-Li-Yau type inequality
   • Weyl Asymptotics
   • A Counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau Inequality
   • An Improvement to the Berezin-Li-Yau type inequality

4 Sharp Bounds for Some Spectral Functions of $H_{0,\Omega}$
Trace formulae

Lemma (Harrell-Stubbe, 1997)

Let $H$ be a self-adjoint operator on $L^2(\Omega)$, $\Omega \in \mathbb{R}^d$, with discrete spectrum $\beta_1 \leq \beta_2 \leq \cdots$. Denoting the corresponding normalized eigenfunctions $\{u_j\}$, assume that for a Cartesian coordinate $x_\alpha$, the functions $x_\alpha u_j$ and $x_\alpha^2 u_j$ are in the domain of definition of $H$. Then

$$\sum_{j: \beta_j \leq z} (z - \beta_j) \langle u_j, [x_\alpha, [H, x_\alpha]] u_j \rangle - 2 \| [H, x_\alpha] u_j \|^2 \leq 0, \quad (1)$$

and

$$\sum_{j: \beta_j \leq z} (z - \beta_j)^2 \langle u_j, [x_\alpha, [H, x_\alpha]] u_j \rangle - 2(z - \beta_j) \| [H, x_\alpha] u_j \|^2 \leq 0. \quad (2)$$
Eigenvalue inequalities for Klein–Gordon operators

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For a real number $r$ and an integer $k > 0$, the normalized moment of the eigenvalues is

$$\beta_k^r := \frac{1}{k} \sum_{j=1}^{k} \beta_j^r.$$
An Upper Bound

Theorem

If \( d \geq 2 \), then for each \( n \), the eigenvalues \( \beta_n \) of \( H_{m,\Omega} \) satisfy

\[
\beta_{n+1} \leq \frac{1}{(d - 1)\beta_n^{-1}} \left( d + \sqrt{d^2 - (d^2 - 1)\beta_n \beta_n^{-1}} \right).
\]  

(3)

By using the Cauchy-Schwarz inequality \( 1 \leq \beta_n \beta_n^{-1} \),

A simple upper bound

\[
\beta_{n+1} \leq \frac{d + 1}{(d - 1)\beta_n^{-1}} \leq \frac{d + 1}{d - 1} \beta_n.
\]

Fundamental Ratio

\[
\frac{\beta_2}{\beta_1} \leq \frac{d + 1}{d - 1}.
\]
Sketch of the Proof of the Theorem

\[ [H_{m,\Omega}, x_\alpha] \varphi = -i \chi_{\Omega} F^{-1} \frac{\xi_\alpha}{\sqrt{|\xi|^2 + m^2}} \hat{\varphi}. \]

and

\[ [x_\alpha, [H_{m,\Omega}, x_\alpha]] \varphi = \chi_{\Omega} F^{-1} \left[ \left( \frac{1}{\sqrt{|\xi|^2 + m^2}} - \frac{\xi_\alpha^2}{(|\xi|^2 + m^2)^{3/2}} \right) \hat{\varphi} \right]. \]
Sketch of the Proof of the Theorem

After summing over $\alpha$ and simplifying expressions we obtain

Trace Inequality

\[(d - 1) \sum_{j=1}^{n} (z - \beta_j)^2 \langle u_j, H_{m,\Omega}^{-1} u_j \rangle - 2 \sum_j (z - \beta_j) \leq 0, \quad (4)\]

provided $z \in [\beta_n, \beta_{n+1}]$.

Or, equivalently,

\[(d - 1) \beta_n^{-1} z^2 - 2dz + (d + 1) \bar{\beta}_n \leq 0, \quad (5)\]

Since $\frac{1}{\beta_j} \leq \langle \hat{u}_j, H_{m,\Omega}^{-1} \hat{u}_j \rangle$. 
Setting $z = \beta_{n+1}$, we see that $\beta_{n+1}$ must be less than the larger root of (5).

\[
\beta_{n+1} \leq \frac{1}{(d - 1)\beta_n^{-1}} \left( d + \sqrt{d^2 - (d^2 - 1)\beta_n \beta_n^{-1}} \right).
\]
A Ratio Bound

With \( a_+ := \max(0, a) \), let

\[
R_{\sigma}(z) := \sum_{k} (z - \beta_k)^{\sigma}_+
\]

and

\[
U(z) := \sum_{k} \frac{(z - \beta_k)^2_+}{\beta_k},
\]

where \( z \) is a real variable.
The function $z^{-(d+1)}U(z)$ is nondecreasing in the variable $z$. Moreover, for $d \geq 2$ and any $n \geq 1$, we have

$$R_1(z) \geq \left( \frac{2n(d-1)^d}{(d+1)^{d+1}\beta_n^d} \right) z^{d+1}$$

for all $z \geq \left( \frac{d+1}{d-1} \right) \beta_n$. 
Sketch of the Proof of the Theorem

Trace inequality (4) can be rewritten as

\[(d + 1) \sum_{j=1}^{n} \frac{(z - \beta_j)^2}{\beta_j} - 2z \sum_{j=1}^{n} \frac{(z - \beta_j)}{\beta_j} \leq 0, \tag{6}\]

which implies

\[(d + 1) U(z) - zU'(z) \leq 0,\]

or, equivalently,

\[\frac{d}{dz} \left\{ \frac{U(z)}{z^{d+1}} \right\} \geq 0.\]
For $z \in [\beta_n, \beta_{n+1}]$, trace inequality also implies

$$R_1(z) \geq \frac{d - 1}{2} U(z).$$

Since $z^{-(d+1)} U(z)$ is nondecreasing, for $z \geq z_{n^*} \geq \beta_n$,

$$R_1(z) \geq \frac{d - 1}{2} \left( \frac{z}{z_{n^*}} \right)^{d+1} U(z_{n^*}).$$

Using the Cauchy-Schwarz inequality $1 \leq \beta_n \beta_{n-1}$ and then optimizing right side with respect to $z_{n^*}$ viz., $z_{n^*} = \frac{d + 1}{d - 1} \beta_n$, implies

$$R_1(z) \geq \left( \frac{2n(d - 1)^d}{(d + 1)^{d+1} \beta_n^d} \right) z^{d+1}$$
Sketch of Proof of the Theorem

For $k - 1 < w < k$, (Laptev-Weidl (2002), [Harrell-Hermi(2007)])

$$R_1^*(w) = (w - [w]) \beta_{[w]+1} + [w] \beta_{[w]},$$

(7)

where $[w]$ denotes the greatest integer $\leq w$. When $w \uparrow k$, $R_1^*(k) = k \beta_k$. Then

$$k \beta_k \leq \frac{d \beta_n}{2^{1/d} n^{1/d}(d - 1)} k^{\frac{d+1}{d}}.$$

Corollary

For $k > 2n$, we have

$$\frac{\beta_k}{\beta_n} \leq \frac{d}{2^{1/d}(d - 1)} \left( \frac{k}{n} \right)^{\frac{1}{d}}.$$
Improvement to $k > n$

By using an argument from the article [Harrell-Stubbe (2009)], the condition $k > 2n$ can be improved to $k > n$:

**Theorem**

For all $k > n$,

$$
\overline{\beta}_k \leq \frac{d^{1+1/d}(d - 1)^{-1/d}}{(d + 1)\overline{\beta}_n^{-1}} \left( \frac{d}{d - 1} + \sqrt{D_n} \right)^{1-1/d} \left( \frac{k}{n} \right)^{1/d} .
$$

(8)

where

$$
D_n := \left( \frac{d}{d - 1} \right)^2 - \left( \frac{d + 1}{d - 1} \right) \overline{\beta}_n^{-1} \overline{\beta}_n \leq \frac{1}{(d - 1)^2}.
$$
Improvement to $k > n$, simpler but weaker

Applying the Cauchy-Schwarz inequality (twice) we obtain:

**Corollary**

For all $k > n$,

$$\frac{\beta_k}{\beta_n} \leq \left(\frac{d}{d - 1}\right) \left(\frac{d}{d + 1}\right)^{1/d} \left(\frac{k}{n}\right)^{1/d}.$$  \quad (9)
Table of Contents

1 Klein-Gordon operators

2 Results
   • An Upper Bound
   • A Ratio Bound

3 Weyl Asymptotics and a Berezin-Li-Yau type inequality
   • Weyl Asymptotics
   • A Counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau Inequality
   • An Improvement to the Berezin-Li-Yau type inequality

4 Sharp Bounds for Some Spectral Functions of $H_{0,\Omega}$
Weyl Asymptotics

Define the partition function for $t > 0$ as

$$Z(t) := \sum e^{-\beta_j t},$$

$$Z(t) = \int e^{-\beta t} dN(\beta),$$

where $N(\beta) = \sum_{\beta_j \leq \beta} 1$ is the usual counting function.

$$Z(t) = \int_{\Omega} p_\Omega(x, x, t) dx.$$
As $\beta \to \infty$, 

$$N(\beta) \sim \frac{|\Omega|}{(4\pi)^{d/2}\Gamma(1 + d/2)}\beta^d.$$ 

Equivalently,

$$\beta_k \sim \sqrt{4\pi}\left(\frac{\Gamma(1 + d/2)k}{|\Omega|}\right)^{1/d}.$$ 

$$U(z) \sim \frac{2|\Omega|}{(4\pi)^{d/2}(d^2 - 1)\Gamma(1 + d/2)}z^{d+1}.$$
Karamata’s Tauberian Theorem

**Theorem 4.3.** If $\alpha(t)$ is non-decreasing and such that the integral

$$f(s) = \int_0^\infty e^{-st} \, d\alpha(t)$$

converges for $s > 0$, and if for some non-negative number $\gamma$

$$f(s) \sim \frac{A}{s^\gamma} \quad (s \to 0^+)$$

$$(s \to \infty),$$

then

$$\alpha(t) \sim \frac{At^\gamma}{\Gamma(\gamma + 1)} \quad (t \to \infty)$$

$$\quad (t \to 0^+).$$

(Book: Laplace Transform by Widder)
We have
\[ p_\Omega(x, y, t) < p_0(x - y, t) \] (10)
on \Omega, where \( p_\Omega \) is the integral kernel of the semigroup \( e^{-tH_{0, \Omega}} \) and
\[ p_0(t, x) = \frac{c_d t}{(t^2 + |x|^2)^{(d+1)/2}}. \]

Define
\[ r_\Omega := p_0(x - y, t) - p_\Omega(x, y, t), \]
and let \( \delta_\Omega(x) := \text{dist}(x, \partial \Omega) \).
According to [Bañuelos-Kulczyzki(2004)],

\[ 0 \leq r_\Omega \leq \frac{t}{\delta_\Omega^{d+1}(x)} c_d \mathcal{P}^y(\tau_\Omega < t), \]

where \( \mathcal{P}^y(\tau_\Omega) \) is the probability that a path originating at \( y \) exits \( \Omega \) before time \( t \). From (10),

\[ \int_{\Omega} p_\Omega(x, x, t) dx \leq \int_{\Omega} p_0(0, t) dx = c_d \frac{|\Omega|}{t^d}, \]
\[ \int_{\Omega} p_{\Omega}(x, x, t) \, dx = \int_{\{x: \delta_{\Omega}(x) < \sqrt{t}\}} p_{\Omega}(x, x, t) \, dx \]
\[ + \int_{\{x: \delta_{\Omega}(x) > \sqrt{t}\}} (p_{0}(0, t) - r_{\Omega}(x, x, t)) \, dx \]
\[ \leq c_d t^{-d} |\{x: \delta_{\Omega}(x) < \sqrt{t}\}| \]
\[ + c_d t^{-d} |\{x: \delta_{\Omega}(x) > \sqrt{t}\}| - \frac{t}{t^{(d+1)/2} |\Omega|}. \]

We thus validate the condition allowing the application of Karamata’s Tauberian Theorem.
Thus, by taking $\gamma = d$ and $A = c_d |\Omega|$ in Karamata, we get

$$\lim_{\beta \to \infty} \beta^{-d} N(\beta) = \frac{c_d |\Omega|}{\Gamma(d + 1)}.$$

Since $c_d = \frac{d!}{(4\pi)^{d/2} \Gamma(1 + d/2)}$ and $\Gamma(d + 1) = d!$. 

Selma Yildirim Yolcu

On Eigenvalue Inequalities for the Klein-Gordon Operators
A Counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau Inequality

\[ \bar{\beta}_k \geq \frac{(d - 1)2^{1/d}\sqrt{4\pi}}{d + 1} \left( \frac{\Gamma(1 + d/2)k}{|\Omega|} \right)^{1/d}. \]  

(11)

Next, we improve the term $(d - 1)2^{1/d}$ in (11) to $d$ and obtain

**Theorem**

*For all $k = 1, \ldots, $ the eigenvalues $\beta_k$ of $H_{0,\Omega}$ satisfy*

\[ \bar{\beta}_k \geq \frac{\sqrt{4\pi d}}{d + 1} \left( \frac{\Gamma(1 + d/2)k}{|\Omega|} \right)^{1/d}. \]  

(12)
A Counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau Inequality

Lemma (Attributed in (Li-Yau) to Hörmander)

Let $f: \mathbb{R}^d \to \mathbb{R}$ satisfy $0 \leq f(\xi) \leq M_1$. Assume that the weight function $w$ is nonnegative and nondecreasing and that

$$\int_{\mathbb{R}^d} f(\xi)w(|\xi|)d\xi \leq M_2. \quad (13)$$

Define $R = R(M_1, M_2)$ by the condition that

$$\int_{B_R} w(|\xi|)d\xi = \omega_{d-1} \int_0^R w(r)r^{d-1}dr = \frac{M_2}{M_1}. \quad (14)$$
Lemma

Here $\omega_{d-1}$ denotes the volume of the $d$-dimensional unit ball $\mathbb{S}^{d-1}$, i.e.,

$$\omega_{d-1} := \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

Then

$$\int_{\mathbb{R}^d} f(\xi) d\xi \leq \frac{\pi^{d/2} M_1}{\Gamma(1 + d/2)} R^d. \quad (15)$$
Lemma

As a special case, if \( w(\xi) = |\xi|^p \), then \( R = \left[ \frac{M_2(d + p)}{M_1 w_{d-1}} \right]^{\frac{1}{d+p}} \), and so

\[
\int_{\mathbb{R}^d} f(\xi) d\xi \leq \frac{1}{d} \left( (d + p)M_2 \right)^{\frac{d}{d+p}} (w_{d-1}M_1)^{-\frac{p}{d+p}}
\]

\[
= \left( \frac{d + p}{d} M_2 \right)^{\frac{d}{d+p}} \left( \frac{\pi^{d/2} M_1}{\Gamma(1 + d/2)} \right)^{\frac{p}{d+p}}.
\]
An Improvement to the BLY-type inequality

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AN IMPROVEMENT TO A BEREZIN-LI-YAU TYPE INEQUALITY

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This paper is dedicated to Professor Evans M. Harrell.

ABSTRACT. In this article we improve a lower bound for \( \sum_{j=1}^{k} \beta_j \) (a Berezin-Li-Yau type inequality) in \([8]\). Here \( \beta_j \) denotes the \( j \)th eigenvalue of the Klein Gordon Hamiltonian \( H_{0,\Omega} = |p| \) when restricted to a bounded set \( \Omega \subset \mathbb{R}^n \).

\( H_{0,\Omega} \) can also be described as the generator of the Cauchy stochastic process with a killing condition on \( \partial \Omega \). (cf. [2], [3].) To do this, we adapt the proof of Melas ([14]), who improved the estimate for the bound of \( \sum_{j=1}^{k} \lambda_j \), where \( \lambda_j \) denotes the \( j \)th eigenvalue of the Dirichlet Laplacian on a bounded domain in \( \mathbb{R}^n \).
An Improvement to the BLY-type inequality

Theorem

For \( k \geq 1 \) and the bounded set \( \Omega \),

\[
\sum_{j=1}^{k} \beta_j \geq \frac{d \tilde{C}_d}{d+1} |\Omega|^{-1/d} k^{1+1/d} + \tilde{M}_d \frac{|\Omega|^{1+1/d}}{I(\Omega)} k^{1-1/d},
\]

(16)

where \( \tilde{C}_d = \sqrt{4\pi} \Gamma(1 + d/2)^{1/d} \) and the constant \( \tilde{M}_d \) depends only on the dimension \( d \).

Here \( I(\Omega) \) denotes the moment of inertia, i.e.,

\[
I(\Omega) = \min_{u \in \mathbb{R}^d} \int_{\Omega} |x - u|^2 dx.
\]
# Table of Contents

1. Klein-Gordon operators

2. Results
   - An Upper Bound
   - A Ratio Bound

3. Weyl Asymptotics and a Berezin-Li-Yau type inequality
   - Weyl Asymptotics
   - A Counterpart for $H_{0,\Omega}$ to the Berezin-Li-Yau Inequality
   - An Improvement to the Berezin-Li-Yau type inequality

4. Sharp Bounds for Some Spectral Functions of $H_{0,\Omega}$
Joint work with L. Hermi

Define the function

$$U_\sigma(z) := \sum_j \frac{(z - \beta_j)^\sigma_{+}}{\beta_j}.$$ 

With this notation, we have

$$U_2(z) \leq \frac{2}{d + 1} z U_1(z).$$  \hfill (17)
Sharp Bounds for Spectral Functions

**Theorem**

For $\sigma > 0$, we have

$$U_\sigma(z) \leq C_\sigma z U_{\sigma - 1}(z).$$  \hspace{1cm} (18)

where

$$C_\sigma = \begin{cases} \frac{\sigma}{d+1}, & \text{when } \sigma \geq 2 \\ \frac{2}{d+1}, & \text{when } \sigma \leq 2 \end{cases}$$
Proof of the case when $\sigma > 2$

Trace inequality for $t \leq z$, becomes

$$\sum_j \frac{(z - \beta_j - t)^2}{\beta_j} \leq \frac{2}{d+1} z \sum_j \frac{(z - \beta_j - t)_+}{\beta_j}. \quad (19)$$

Multiplying (19) by $t^{\sigma-3}$ and integrating from $t = 0$ to $t = \infty$ yield

$$\int_0^\infty \sum_j \frac{(z - \beta_j - t)^2}{\beta_j} t^{\sigma-3} dt \leq \frac{2}{d+1} z \int_0^\infty \sum_j \frac{(z - \beta_j - t)_+}{\beta_j} t^{\sigma-3} dt. \quad (20)$$
Proof of the case when $\sigma > 2$

Next, set $t = (z - \beta_j)_+\tau$. We exploit the fact that

$$
\int_0^\infty (1 - t)^{p-1} t^{q-1} dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)},
$$

to obtain (after cancelations and simplifications)

$$
\sum_j \frac{(z - \beta_j)^\sigma_+}{\beta_j} \leq \frac{\sigma}{d + 1} z \sum_j \frac{(z - \beta_j)^{\sigma-1}_+}{\beta_j} U_\sigma(z) \leq C_\sigma z U_{\sigma-1}(z),
$$

where $C_\sigma = \frac{\sigma}{d + 1}$ for $\sigma > 2$. 
Proof of the case when $\sigma \leq 2$

Lemma

For all $\sigma \leq \sigma'$, \[ \frac{U_\sigma(z)}{U_{\sigma-1}(z)} \leq \frac{U_{\sigma'}(z)}{U_{\sigma'-1}(z)} \quad (21) \]
Proof of the case when $\sigma \leq 2$

The key point in the proof of Lemma 14 is the reverse Chebyshev’s inequality from [Hardy-Littlewood-Polya]:

Lemma

Let $\{a_j\}$ and $\{b_j\}$ be two real sequences, one nonincreasing and the other nondecreasing, and let $\{w_j\}$ be a sequence of nonnegative weights. Then

\[
\sum_{j=1}^{n} w_j \sum_{j=1}^{n} w_j a_j b_j \leq \sum_{j=1}^{n} w_j a_j \sum_{j=1}^{n} w_j b_j \tag{22}
\]
Proof of the case when $\sigma \leq 2$

Choose

$$w_j = \frac{(z - \beta_j)^\sigma}{\beta_j}, \quad a_j = (z - \beta_j)^{\sigma' - \sigma} \quad \text{and} \quad b_j = (z - \beta_j)^{-1}$$

in (22). Since

$$0 \leq \beta_1 < \beta_2 < \cdots$$

and $\sigma < \sigma'$, the sequence $\{a_j\}$ is decreasing and the sequence $\{\beta_j\}$ is increasing. After simplifications we get

$$U_\sigma(z)U_{\sigma'-1}(z) \leq U_{\sigma'}(z)U_{\sigma-1}(z),$$

for $\sigma < \sigma'$. 
Proof of the case when $\sigma \leq 2$

Choose $\sigma' = 2$ in (21) to obtain

$$\frac{U_\sigma(z)}{U_{\sigma-1}(z)} \leq \frac{U_2(z)}{U_1(z)},$$

where $\sigma < 2$. Recall that

$$\frac{U_2(z)}{U_1(z)} \leq \frac{2z}{d+1}.$$

After combining the two inequalities above we get

$$U_\sigma(z) \leq C_\sigma z U_{\sigma-1}(z),$$

where $C_\sigma = \frac{2}{d+1}$ for $\sigma < 2$. 

Selma Yildirim Yolcu
A bound in terms of $|\Omega|$ for the function $U_{\sigma}$

Lemma (Frank, Loss, Weidl)

Let $\rho > \sigma \geq 0$ and $t > z$. Then for all $\beta_j \geq 0$,

$$(z - \beta_j)^\sigma \leq C(\sigma, \rho)(t - z)^{\sigma - \rho}(t - \beta_j)^{\rho},$$

with

$$C(\sigma, \rho) := \begin{cases} 1 & \text{if } \sigma = 0 \\ \rho^{-\rho}\sigma^\sigma(\rho - \sigma)^{\rho - \sigma} & \text{if } \rho > \sigma > 0. \end{cases}$$

We choose $\rho = 2$ in the lemma and optimize the right side with respect to $t$ to obtain the following upper bound for $U_{\sigma}(z)$:
A bound in terms of $|\Omega|$ for the function $U_\sigma$

**Theorem**

For $0 \leq \sigma < 2$ and $t > z$, 

$$U_\sigma(z) \leq \frac{\sigma^\sigma |\Omega|(d + 1)^d}{2(d - 1)(4\pi)^{d/2}\rho \left(1 + \frac{d}{2}\right) (\sigma + d - 1)^{\sigma + d - 1}}.$$
Sharp Bounds for the Function $\sum_j \frac{e^{-\beta_j t}}{\beta_j}$

We utilize the Laplace transform to obtain some bounds for $t^{d-1} \tilde{Z}(t)$.

Theorem

The function $t^{d-1} \tilde{Z}(t)$ is nonincreasing.
Sharp Bounds for the Function $\sum_j \frac{e^{-\beta_j t}}{\beta_j}$

Observe that the Laplace transform of $(z - \beta_j)^\sigma_+$ is

$$\mathcal{L}((z - \beta_j)^\sigma_+) = \frac{\Gamma(\sigma + 1)e^{-\beta_j t}}{t^{\sigma+1}}.$$

Then,

$$\mathcal{L}(U_\sigma(z)) = \frac{\Gamma(\sigma + 1)}{t^{\sigma+1}} \sum_j \frac{e^{-\beta_j t}}{\beta_j} = \frac{\Gamma(\sigma + 1)}{t^{\sigma+1}} \tilde{Z}(t).$$
Sharp Bounds for the Function $\sum_j \frac{e^{-\beta_j t}}{\beta_j}$

An application of the Laplace transform to both sides yields

$$\frac{\Gamma(3)}{t^3} \ddot{Z}(t) \leq \frac{2}{d+1} \left( \frac{\Gamma(3)}{t^3} \ddot{Z}(t) - \frac{\Gamma(2)}{t^2} \dot{Z}'(t) \right)$$

$$ (d - 1) \ddot{Z}(t) + t \dot{Z}'(t) \leq 0$$

$$ \frac{d}{dt} \left( t^{d-1} \ddot{Z}(t) \right) \leq 0.$$
Sharp Bounds for the Function \[
\sum_j \frac{e^{-\beta_j t}}{\beta_j}
\]

Recall that the Weyl asymptotic formula implies

\[
U_2(z) \leq \frac{|\Omega|}{(4\pi)^{d/2}(d^2 - 1)\Gamma\left(1 + \frac{d}{2}\right)} z^{d+1}.
\]

An application of the Laplace transform to both sides and by using

\[
\frac{\Gamma(d+1)}{(4\pi)^{d/2}\Gamma\left(1 + \frac{d}{2}\right)} = \pi^{-\left(\frac{d+1}{2}\right)}\Gamma\left(\frac{d+1}{2}\right)
\]

we obtain the theorem:

**Theorem**

For \(d > 1\),

\[
t^{d-1}\tilde{Z}(t) \leq \frac{\Gamma\left(\frac{d+1}{2}\right)}{2(d - 1)\pi^{\frac{d+1}{2}}} |\Omega|.
\]
Sharp Bounds for the Function $\sum_j \frac{e^{-\beta_j t}}{\beta_j}$

Observe that the last two theorems imply that as $t \to 0^+$,

$$t^{d-1} \tilde{Z}(t) \to \frac{\Gamma \left( \frac{d+1}{2} \right)}{2(d-1)\pi^{\frac{d+1}{2}}} |\Omega|.$$

This resembles what Kac obtained in [Kac] for the Laplacian.

**Theorem**

$$\frac{\Gamma \left( \frac{d+1}{2} \right)}{2(d-1)(d+1)!\pi^{\frac{d+1}{2}}} |\Omega| \geq \frac{U_2(z_0)}{z_0^{d+1}}. \quad (25)$$
Sharp Bounds for the Function $\sum_j \frac{e^{-\beta_j t}}{\beta_j}$

$z^{-(d+1)} U_2(z)$ is nondecreasing in the variable $z$. Then for $\tau = z - z_0 > 0$,

$$U_2(z_0 + \tau) \geq U_2(z_0) \left( \frac{\tau + z_0}{z_0} \right)^{d+1}.$$  \hspace{1cm} (26)

After an application of the Laplace transform, we obtain

$$\mathcal{L}((\tau + z_0 - \beta_j)^2) = e^{(z_0 - \beta_j) t} \left( \frac{\Gamma(3)}{t^3} - \int_0^{(z_0 - \beta_j) + t} e^{-tu} u^2 du \right).$$  \hspace{1cm} (27)
\[ \mathcal{L}(\tau + z_0)^{d+1} = e^{z_0 t} \left( \frac{\Gamma(d + 2)}{t^{d+2}} - \int_0^{z_0 t} e^{-tu} u^{d+1} du \right). \quad (28) \]

Inserting these expressions in (26) we have that

\[ \sum_j \frac{e^{(z_0 - \beta_j) + t}}{\beta_j} \left( \frac{2}{t^3} - \int_0^{(z_0 - \beta_j) + t} e^{-tu} u^2 du \right) \]

\[ \geq \frac{U_2(z_0)}{z_0^{d+1}} e^{z_0 t} \left( \frac{\Gamma(d + 2)}{t^{d+2}} - \int_0^{z_0 t} e^{-tu} u^{d+1} du \right). \quad (29) \]
Observe that
\[
\sum_j e^{(z_0 - \beta_j)t} \geq e^{z_0 t} \tilde{Z}(t).
\]

With the aid of this observation and after some algebra,
\[
\frac{t^{d-1}}{(d+1)!} \tilde{Z}(t) \geq \frac{U_2(z_0)}{z_0^{d+1}} + R(t),
\]
where the remainder term is
\[
R(t) = \frac{t^{d-1} e^{-z_0 t}}{(d+1)!} \sum_j \frac{e^{(z_0 - \beta_j)t}}{\beta_j} \gamma(3, (z_0 - \beta_j) + t^2)
- \frac{U_2(z_0)}{z_0^{d+1}(d+1)!} \gamma(d + 2, z_0 t^2).
\]

Here the incomplete gamma function is defined as
\[
\gamma(a, z) = \int_0^z e^{-\tau} \tau^{a-1} d\tau.
\]
Note that $R(t) \to 0$ as $t \to 0^+$. Since
\[ t^{d-1} \tilde{Z}(t) \to \frac{\Gamma \left( \frac{d+1}{2} \right)}{2(d - 1)\pi^{d+1} |\Omega|}, \]
as $t \to 0^+$, we have
\[ \frac{\Gamma \left( \frac{d+1}{2} \right)}{2(d - 1)(d + 1)!\pi^{d+1} |\Omega|} \geq \frac{U_2(z_0)}{z_0^{d+1}}. \]
Thank You!