Sharp integrability condition of an exit time for for stable processes from unbounded domains

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Outline

Definition

Let $0 < \alpha < 2$ and $d \geq 2$. The symmetric α-stable process X_t on $\mathbb{R}^d$ is a strong Markov process with stationary increments and characteristic function

$$\mathbf{E}^x\left(\exp\left(i\xi(X_t - x)\right)\right) = \exp\left(-t|\xi|^\alpha\right).$$

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Transition densities

The process X_t has symmetric and continuous transition densities.

$$\mathbf{P}^x \left(X_t \in A \right) = \int_A p(t,x,y)dy,$$

$$p(t,x,y) \asymp \min \left\{ t^{-d/\alpha}, \frac{t}{|x-y|^{d+\alpha}} \right\}.$$

Lévy measure

The process X_t is a Lévy process with the Lévy measure

$$\nu(x) = c_{\alpha,d}|x|^{-d-\alpha}.$$

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Exit time and killed process

Let τ_D denote the first exit time from an open set D.
The transition density of the process killed while exiting D is defined by

$$p_D(t,x,y) = p(t,x,y) - \mathbf{E}^x\left(\tau_D < t, p(t-\tau_D, X(\tau_D), y)\right).$$

We have

$$\mathbf{P}^x(\tau_D > t) = \int_D p_D(t,x,y)dy.$$

Theorem (Burkholder '77)

Let D_γ be a cone with angle γ. For an arbitrary γ there exists $p(\gamma, d)$ such that for Brownian motion

$$\mathbf{E}^x((\tau_{D_\gamma})^p) < \infty \quad \iff \quad p < p(\gamma, d).$$

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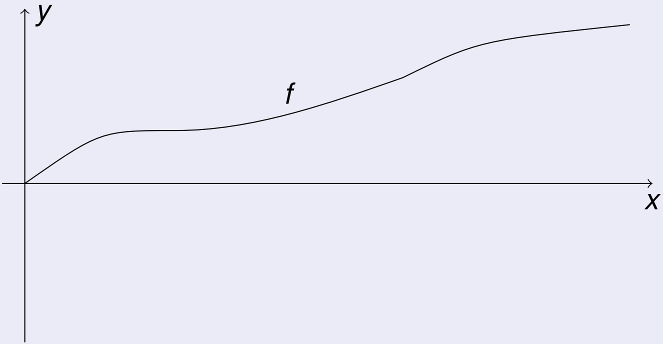
Theorem (DeBlassie, '90 (d=2); Méndez-Hernández, '02)

For the symmetric α-stable process X_t

$$\mathbf{E}^x\left(\tau_{D_\gamma}\right) = \infty.$$

Let $f : (0, \infty) \to (0, \infty)$ be a nondecreasing function.



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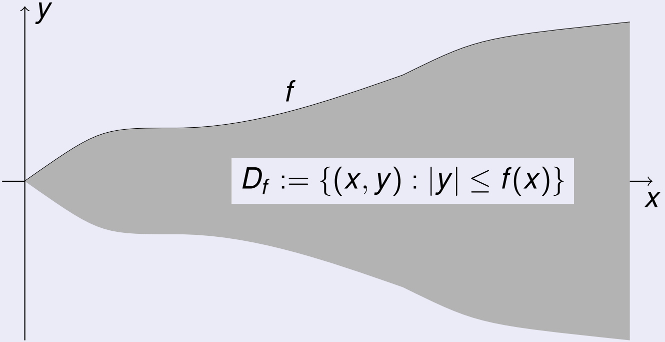

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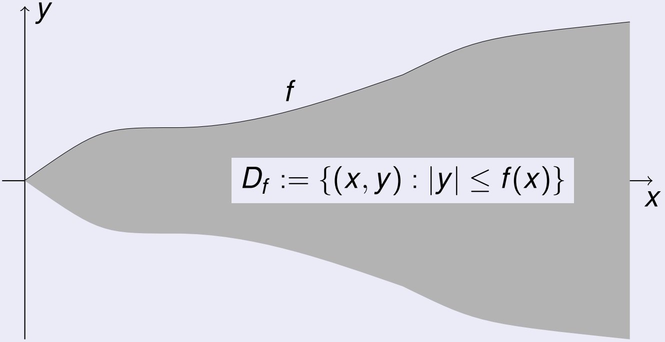

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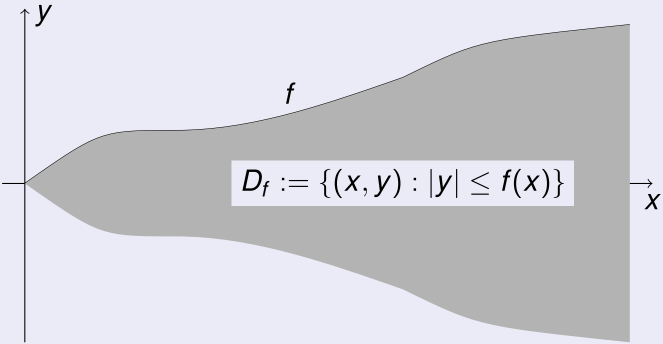

- If we set $f(t) = ct$, we get a family of cones with angles depending on c.
- Can we find a function f such that finiteness of the exit time for stable process from D_{cf} depends on c?

Theorem (Bañuelos, DeBlassie, Smits '01)

Let $d = 2$ and $f(t) = c\sqrt{t}$. For Brownian motion

$$\mathbf{E}^x\left(\exp(b\tau_{D_f}^a)\right)$$

is finite if $a < 1/3$ and infinite if $a > 1/3$.

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Theorem (Bañuelos, Bogdan '05)

Let $f(t) = t^\beta$ for $\beta < 1$. For the symmetric α-stable process X_t

$$\mathbf{E}^x(\tau_{D_f}^p) < \infty \iff p < \frac{(d-1)(1-\beta)+\alpha}{\alpha\beta}.$$

Let $M : (0, \infty) \to (0, \infty)$ be nondecreasing and $M'(t) \geq 1$ for $t > M_0$. Is it possible to get a finiteness condition for

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- $M(t) = t^p \ln^q(t)$ for arbitrary q.
- Is there $M(t)$ such that constant c in D_{cf} makes a difference?

Let

$$I_c(x) = \int_0^\infty M'(t) \exp\left(\frac{-ct}{f^\alpha(x)}\right) dt.$$

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Theorem

There exist constants $\lambda_0 > \lambda_1$ *such that for* X_t

$$\mathbf{E}^x\left(M(\tau_{D_f})\right) < \infty \Rightarrow \int_1^\infty \frac{f^{d-1}(x)}{x^{d+\alpha}} I_{\lambda_0}(x)dx < \infty,$$

$$\mathbf{E}^x\left(M(\tau_{D_f})\right) < \infty \Leftarrow \int_1^\infty \frac{f^{d-1}(x)}{x^{d+\alpha}} I_{\lambda_1}(x)dx < \infty,$$

Remark

If $M'(ct) \leq C(c)M'(t)$ then $I_c(x) \leq C_1(c)I_1(x)$. This means that the condition in the theorem becomes “if and only if”.

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Corollary

For $p \geq 1$ and $q \geq 0$ ($p > 1$ and q arbitrary) we have

$$\mathbf{E}^x\left(\tau_{D_f}^p \ln^q(\tau_{D_f})\right) < \infty \iff \int_1^\infty \frac{f^{d+\alpha p-1}(x)\ln^q(f(x))}{x^{d+\alpha}}dx < \infty.$$

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As a special case of this corollary we get all results for stable processes mentioned before.

Corollary

Let $M(t) = \exp\sqrt{t}$ *and* $f(t) = \beta\ln(t)$*. The expected exit time is now finite for* $\alpha < 1$ *and infinite for* $\alpha > 1$*. For* $\alpha = 1$*, there exists* β_0 *such that the expected exit time is finite if* $\beta < \beta_0$ *and infinite if* $\beta > \beta_0$*.*

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Remark

The most general result is valid for any domain such that the cross-sections are convex and nondecreasing. In this case the integral involves the area of the cross-section and its width.

Theorem (Ikeda, Watanabe '62)

For any open, bounded set D and any Borel set A such that $dist(A, D) > 0$ we have,

$$\mathbf{P}^x(\tau_D < \infty, X(\tau_D) \in A) = \int_D G_D(x,y) \int_A \nu(y-z)dzdy,$$

where $G_D(x,y) = \int_0^\infty p_D(t,x,y)dt$.

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where $G_D(x, y) = \int_0^\infty p_D(t, x, y) dt$.

Theorem (Kulczycki,S. '06)

For any open set D, any Borel set A such that $A \subset (\bar{D})^c$ and any Borel set B

$$\mathbf{P}^x(\tau_D < t, X(\tau_D) \in A, X(t) \in B)$$
$$= \int_D \int_0^t p_D(s, x, y) \int_A \nu(y - z) \mathbf{P}^z(X(t - s) \in B) ds dz dy.$$

Theorem (Kulczycki,S. '06)

Suppose that a set D has a uniform outer ball condition with radius ε. That is, for any point on the boundary, there exists a ball with radius ε such that its intersection with the domain is equal to this boundary point. Then for $t < T$ and $|x-y| > 1$

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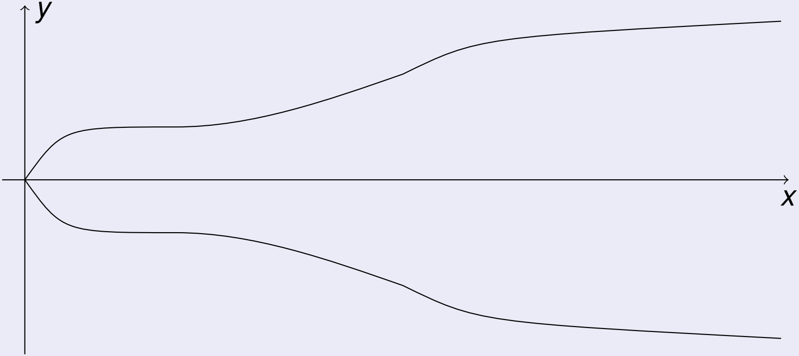
Theorem

Let D be a convex domain with width R. Then

$$p_D(t,x,y) \leq c \exp\left(\frac{-\lambda t}{R^\alpha}\right) p(t,x,y) \times \\ \times \min\left\{1, \frac{\delta_D^{\alpha/2}(x)}{\sqrt{t}}, \frac{\delta_D^{\alpha/2}(y)}{\sqrt{t}}, \frac{\delta_D^{\alpha/2}(x)\delta_D^{\alpha/2}(y)}{t}\right\}.$$

Partition of the domain



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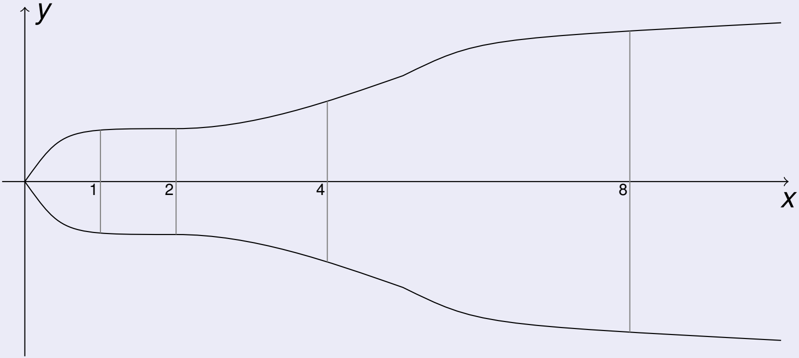

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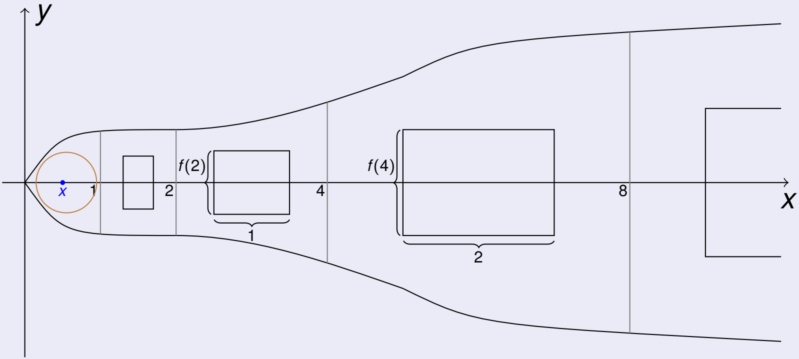

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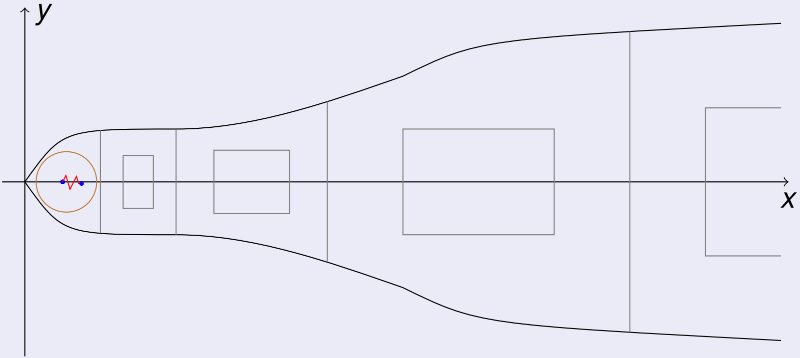

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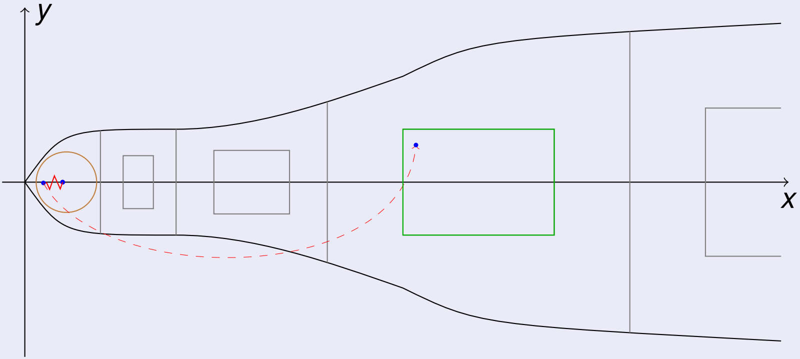
Event



$\mathbf{P}^x(\tau_{D_f} > t) \geq$? (for $t \geq 1$)

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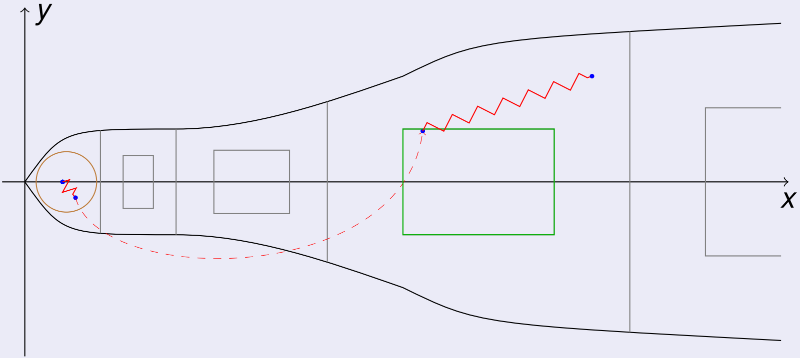
Event



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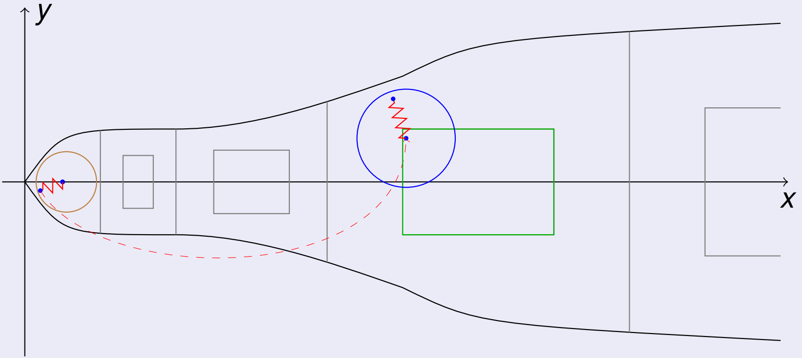

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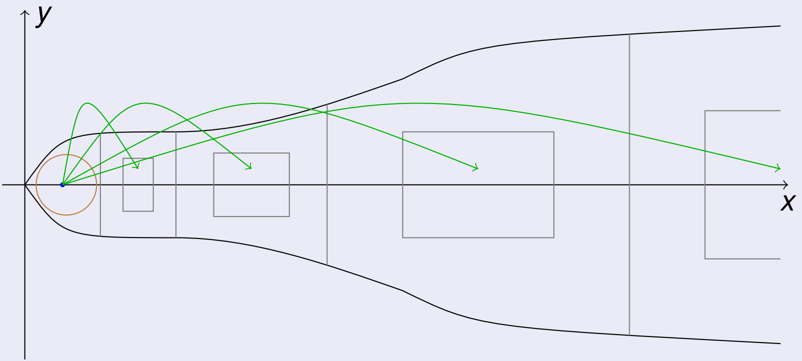
$$\mathbf{P}^x(E(3,t)) = \mathbf{P}^x(\tau_A < 1, X(\tau_A) \in C_3, \tau_{D_f} > t)$$
$$\geq \mathbf{E}^x(\tau_A < 1, X(\tau_A) \in C_3; \mathbf{P}^{X(\tau_A)}(\tau_{B_3} > t));$$

Event



$\mathbf{P}^x(\tau_{D_f} > t) \geq$? (for $t \geq 1$)

$$\mathbf{P}^x(E(n,t)) = \mathbf{P}^x(\tau_A < 1, X(\tau_A) \in C_n, \tau_{D_f} > t)$$
$$\geq \mathbf{E}^x(\tau_A < 1, X(\tau_A) \in C_n; \mathbf{P}^0(\tau_{B(0,f(2^n))} > t));$$

$$\mathbf{E}^x(M(\tau_{D_f})) = \int_0^\infty M'(t)\mathbf{P}^x(\tau_{D_f} > t)dt$$

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\mathbf{E}^x(M(\tau_{D_f})) &= \int_0^\infty M'(t)\mathbf{P}^x(\tau_{D_f} > t)dt \\
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&\geq \sum_{n=1}^\infty \mathbf{P}^x(\tau_A < 1, X(\tau_A) \in C_n) \int_1^\infty M'(t)\mathbf{P}^0(\tau_{B(0,f(2^n))} > t) \\
&= \sum_{n=1}^\infty \int_A \int_0^1 p_A(s,x,y) \int_{C_n} \frac{c}{|y-z|^{d+\alpha}} dzdsdy \times \\
&\quad \times \int_1^\infty M'(t) \exp\left(\frac{-\lambda t}{f^\alpha(2^n)}\right) dt
\end{aligned}$$

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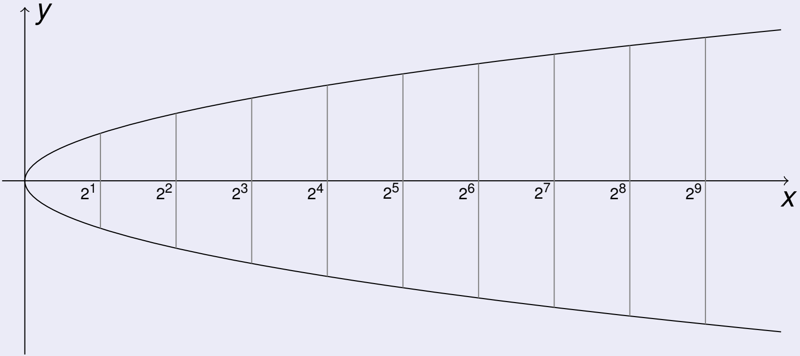
&\quad \times \int_1^\infty M'(t)\exp\left(\frac{-\lambda t}{f^\alpha(2^n)}\right)dt \geq c\sum_{n=1}^\infty p_n,
\end{aligned}
$$

where

$$
p_n = \frac{f^{d-1}(2^n)}{(2^n)^{d+\alpha-1}} I_\lambda(2^n).
$$

Compound event



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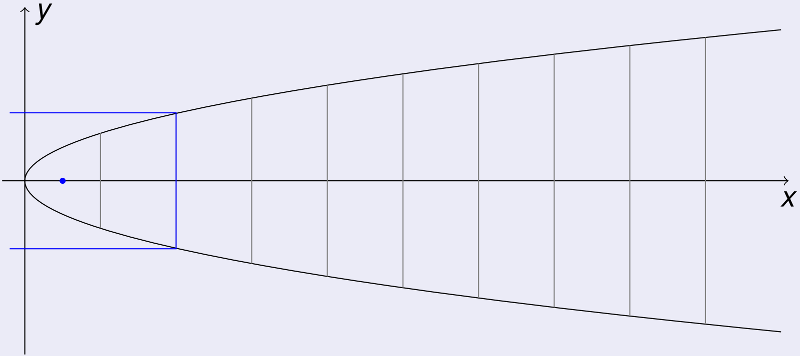

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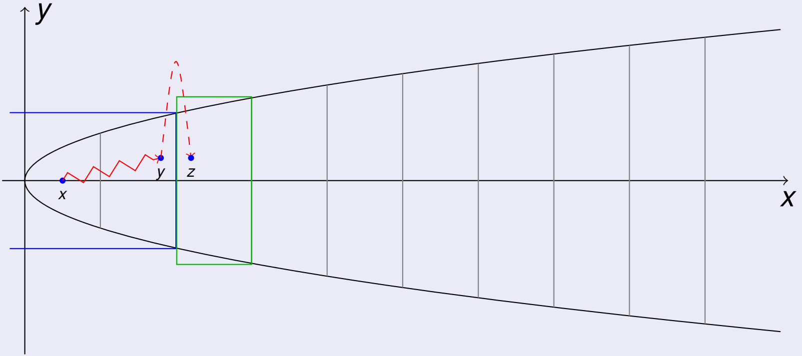
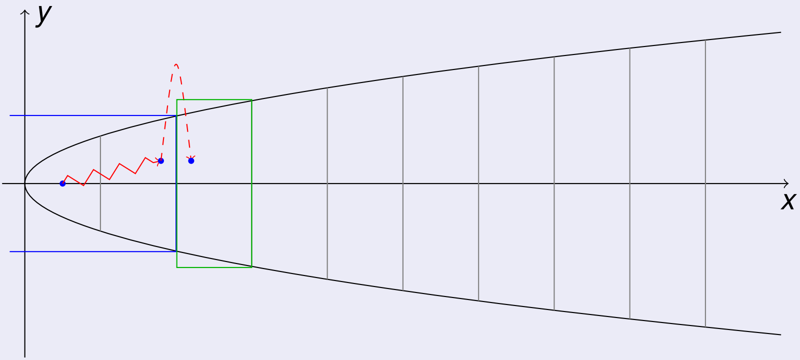


Ikeda-Watanabe formula:

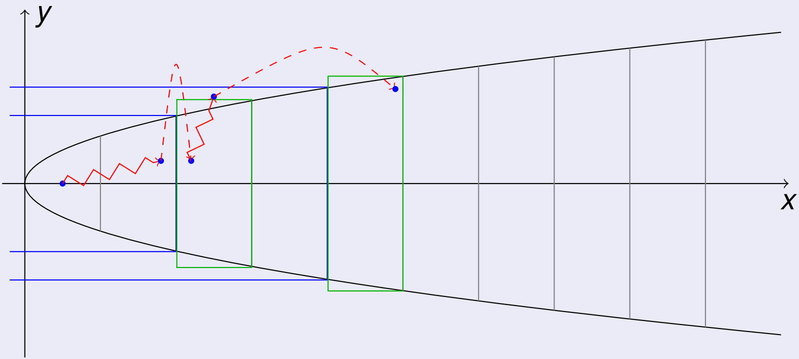
$$\int_A \int_0^t p_{A}(s,x,y) \int_{C} |y-z|^{-d-\alpha} dz ds dy, \text{ and } p_{A}(s,x,y) \asymp \frac{\delta_{A}{}^{\alpha/2}(y)}{|x-y|^{d+\alpha}}.$$

Compound event



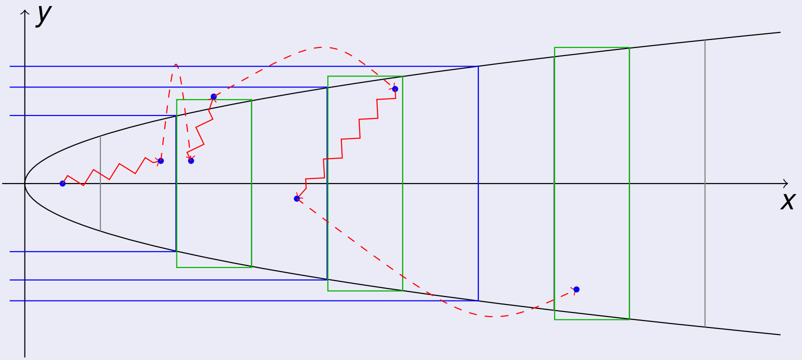
$$\mathbf{P}^x(E(3 \qquad))$$
$$= \mathbf{P}^x(X(\tau_{A_2}) \in C_3 \qquad)$$

Compound event



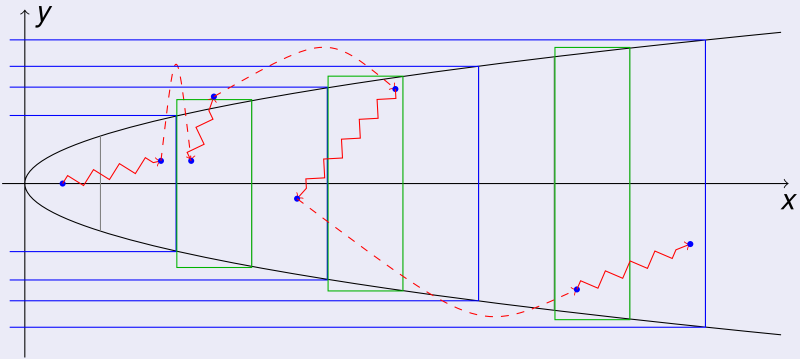
$$\mathbf{P}^x(E(3,5\qquad))$$
$$= \mathbf{P}^x(X(\tau_{A_2}) \in C_3, X(\tau_{A_4}) \in C_5 \qquad\qquad)$$

Compound event



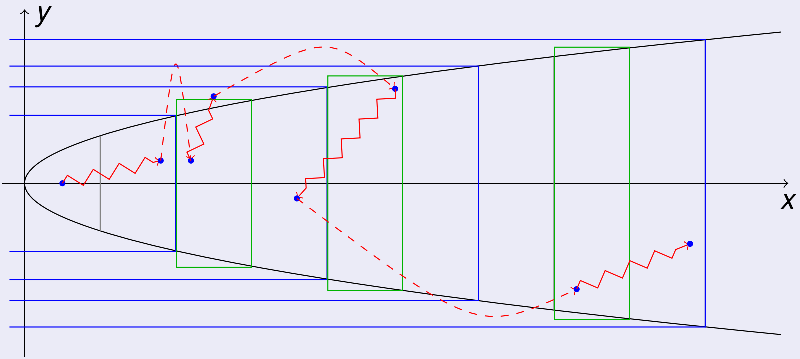
$$\mathbf{P}^x(E(3,5,8\quad))$$
$$= \mathbf{P}^x(X(\tau_{A_2}) \in C_3, X(\tau_{A_4}) \in C_5, X(\tau_{A_6}) \in C_8 \qquad\qquad)$$

Compound event



$$\mathbf{P}^x(E(3,5,8;t)) = \mathbf{P}^x(X(\tau_{A_2}) \in C_3, X(\tau_{A_4}) \in C_5, X(\tau_{A_6}) \in C_8, \tau_{A_6} < t < \tau_{A_9})$$

Compound event



$$
\begin{aligned}
&\mathbf{P}^x(E(3, 5, 8; t)) \\
&\quad = \mathbf{P}^x(X(\tau_{A_2}) \in C_3, X(\tau_{A_4}) \in C_5, X(\tau_{A_6}) \in C_8, \tau_{A_6} < t < \tau_{A_9}) \\
&\quad \leq (cp_3)(cp_5)A(8, t).
\end{aligned}
$$

$$\mathbf{E}^x(M(\tau_{D_f})) = \int_0^\infty M'(t)\mathbf{P}^x(\tau_{D_f} > t)dt$$

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$$\leq \sum_{k=1}^\infty \sum_{\{n_i+1<n_{i+1}\}_{i=1}^k} \int_0^\infty M'(t)\mathbf{P}^x(E(n_1, n_2, \cdots, n_k; t))dt$$

$$\mathbf{E}^x(M(\tau_{D_f})) = \int_0^\infty M'(t)\mathbf{P}^x(\tau_{D_f} > t)dt$$

$$\leq \sum_{k=1}^{\infty} \sum_{\{n_i+1<n_{i+1}\}_{i=1}^{k}} \int_0^\infty M'(t)\mathbf{P}^x(E(n_1, n_2, \cdots, n_k; t))dt$$

$$\leq \sum_{k=1}^{\infty} \sum_{\{n_i+1<n_{i+1}\}_{i=1}^{k}} \left(\prod_{i=1}^{k-1} cp_{n_i}\right) \int_0^\infty M'(t)A(n_k, t)dt$$

$$\mathbf{E}^x(M(\tau_{D_f})) = \int_0^\infty M'(t)\mathbf{P}^x(\tau_{D_f} > t)dt$$

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$$\leq \sum_{k=1}^\infty \sum_{\{n_i+1<n_{i+1}\}_{i=1}^k} \left(\prod_{i=1}^{k-1} cp_{n_i}\right) \int_0^\infty M'(t)A(n_k, t)dt$$

$$\leq \sum_{k=1}^\infty \sum_{\{n_i+1<n_{i+1}\}_{i=1}^k} \prod_{i=1}^{k} cp_{n_i}$$

$$
\begin{aligned}
\mathbf{E}^x(M(\tau_{D_f})) &= \int_0^\infty M'(t)\mathbf{P}^x(\tau_{D_f} > t)dt \\
&\le \sum_{k=1}^\infty \sum_{\{n_i+1<n_{i+1}\}_{i=1}^k} \int_0^\infty M'(t)\mathbf{P}^x(E(n_1, n_2, \cdots, n_k; t))dt \\
&\le \sum_{k=1}^\infty \sum_{\{n_i+1<n_{i+1}\}_{i=1}^k} \left(\prod_{i=1}^{k-1} cp_{n_i}\right) \int_0^\infty M'(t)A(n_k, t)dt \\
&\le \sum_{k=1}^\infty \sum_{\{n_i+1<n_{i+1}\}_{i=1}^k} \prod_{i=1}^{k} cp_{n_i} \le \prod_{n=1}^\infty (1 + cp_n)
\end{aligned}
$$

$$
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\mathbf{E}^x(M(\tau_{D_f})) &= \int_0^\infty M'(t)\mathbf{P}^x(\tau_{D_f} > t)dt \\
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&\le \sum_{k=1}^{\infty} \sum_{\{n_i+1<n_{i+1}\}_{i=1}^k} \left(\prod_{i=1}^{k-1} cp_{n_i}\right) \int_0^\infty M'(t)A(n_k, t)dt \\
&\le \sum_{k=1}^{\infty} \sum_{\{n_i+1<n_{i+1}\}_{i=1}^k} \prod_{i=1}^{k} cp_{n_i} \le \prod_{n=1}^{\infty}(1 + cp_n) \le \exp\left(c\sum_{n=1}^{\infty} p_n\right).
\end{aligned}
$$