Isoperimetric inequalities for eigenvalues of triangles

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Outline

1. **Introduction**
   - Notation and classical results
   - New results
2. **Symmetrization techniques**
   - Steiner symmetrization
   - Continuous Steiner symmetrization
   - Polarization
3. **Proofs**
   - Freitas's lower bound
   - Improved lower bound
   - Isosceles triangles
   - Circular sectors
   - Monotonicity

Eigenvalues

Let D be an open set. Eigenvalues λ_i of the Dirichlet Laplacian on D will be called eigenvalues of D. They form a nondecreasing sequence such that $0 < \lambda_1 < \lambda_2$.

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Geometric quantities

- A - area of D
- R - inradius
- L - perimeter
- d - diameter
- $\lambda_D = \lambda_1$ - first eigenvalue of D

Triangles

- h - altitude perpendicular to the longest side
- γ - smallest angle

Classical isoperimetric inequality

Among all domains with fixed area A, the ball minimizes perimeter L.

$$L^2 \geq 4\pi A.$$

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Pólya's isoperimetric conjecture

Among all polygons $P(n)$ with n sides and fixed area A, the regular polygon $R(n)$ minimizes the first eigenvalue.

$$\lambda_{P(n)} A \geq \lambda_{R(n)} A.$$

(Proved for triangles and quadrilaterals.)

Known eigenvalues

- ball
- rectangles
- annuli
- circular sectors
- equilateral triangle
- right triangles with angles $\pi/4$ or $\pi/6$

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Methods of obtaining lower bounds

- Domain monotonicity (larger domain $\rightarrow$ smaller eigenvalue)
- Restricting to a subdomain (right isosceles triangle is a half of a square)
- Special analytical cases
- Symmetrization

Theorem (Freitas '06)

For arbitrary triangle T

$$\lambda_T \geq \pi^2 \left(\frac{4}{d^2} + \frac{d^2}{4A^2} \right).$$

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$$\lambda_T \geq \pi^2 \left(\frac{4}{d^2 + h^2} + \frac{d^2 + h^2}{4A^2} \right).$$

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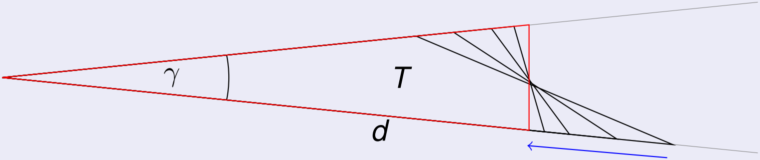
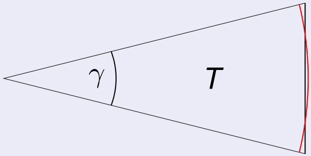
Theorem

Let T be a triangle with fixed area A and smallest angle γ. Then the eigenvalue λ_T is decreasing with diameter d.



Theorem

The eigenvalue λ_T of an isosceles triangle T with area A and smallest angle γ is bigger then the eigenvalue of a sector with the same area and angle.



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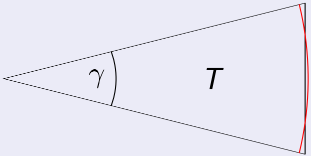

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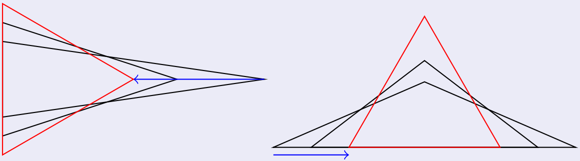
Given fixed area A, the eigenvalue of an isosceles triangle decreases when the smallest angle γ increases.



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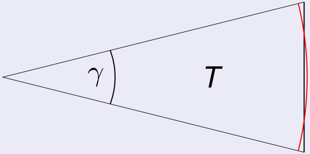

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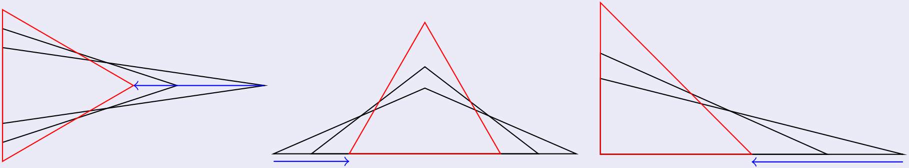
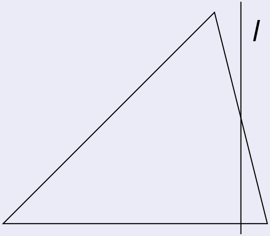


Definition of Steiner symmetrization

Fix a line l and domain D.

Action on triangles



Definition of Steiner symmetrization

Fix a line *l* and domain *D*. Consider cross-sections of a domain *D* perpendicular to *l*.

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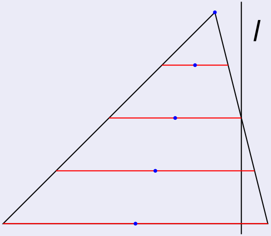

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Fix a line l and domain D. Consider cross-sections of a domain D perpendicular to l. The Steiner symmetrization of D is the domain D^* formed by the cross-sections centered around l.

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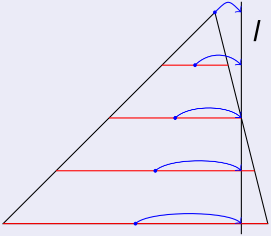

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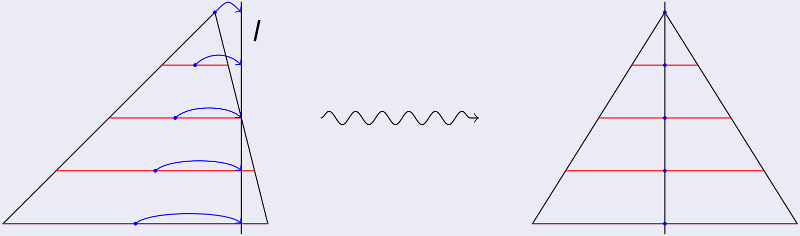

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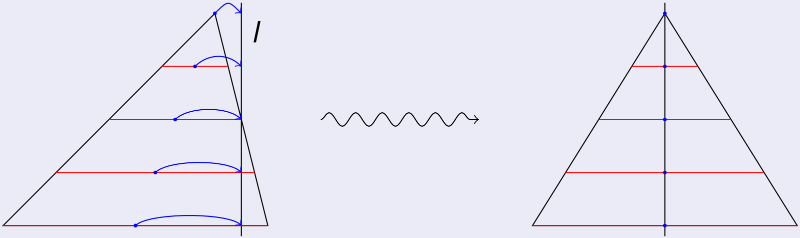
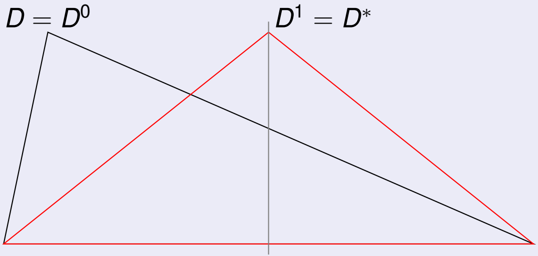


- preserves area
- decreases perimeter
- decreases the first eigenvalue

Definition of continuous Steiner symmetrization

Let $0 \leq t \leq 1$ be a time parameter. Let D^0 equal to initial domain D, and D^1 equal to the Steiner symmetrization D^*.

Action on triangles



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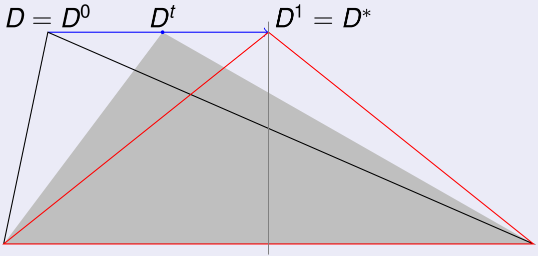

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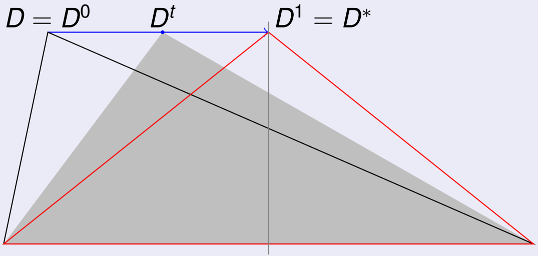


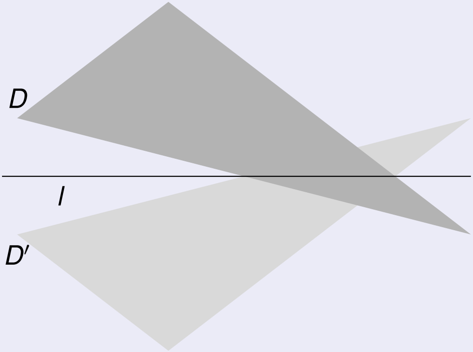
As t increases:

- area remains fixed
- perimeter decreases
- the first eigenvalue decreases

Definition of polarization

Fix a domain D and a line l. This line splits the space into two half-spaces H_1 and H_2. Let D' be a reflection of D about l. The polarization D^P of a set D consists of $D \cap H_1$, $D' \cap H_1$ and any point from $(D \cup D') \cap H_2$ such that its reflection is already included.

Action on triangles



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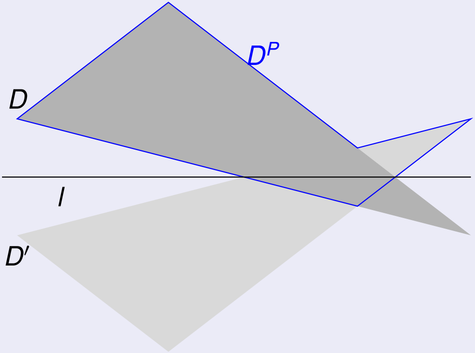

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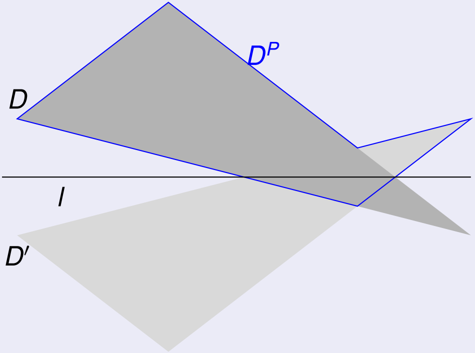
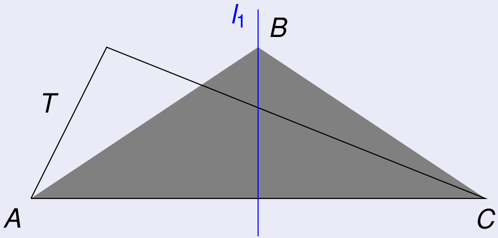


- preserves area
- decreases perimeter
- decreases the first eigenvalue

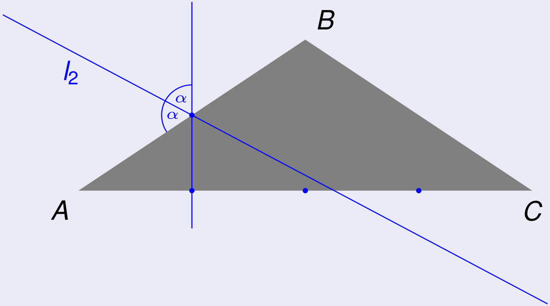
Alternative proof of Freitas's bound



Steps

1. Steiner symmetrization with respect to l_1.

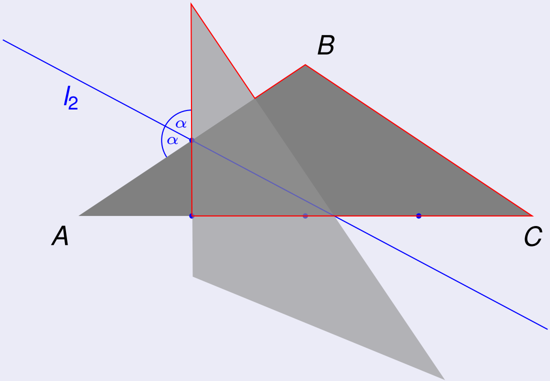
Alternative proof of Freitas's bound



Steps

2. Polarization with respect to l_2.

Alternative proof of Freitas's bound



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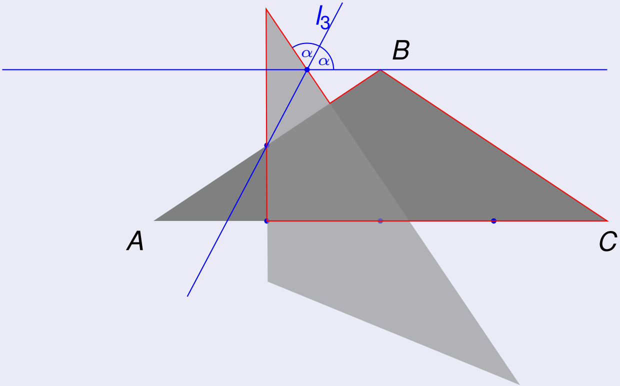
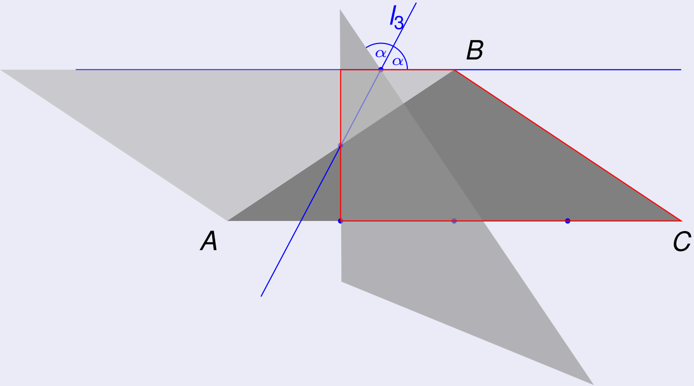


Steps

3. Polarization with respect to l_3.

Alternative proof of Freitas's bound



Steps

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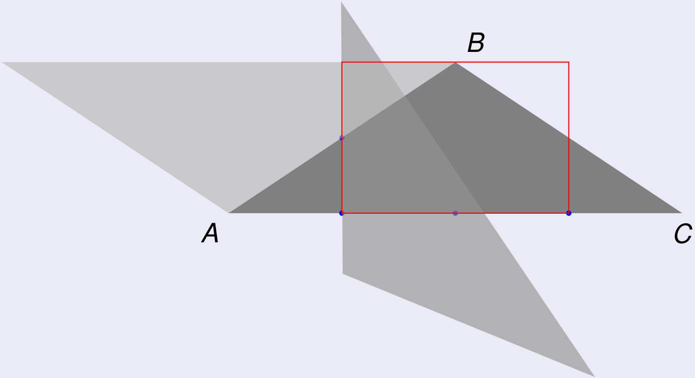
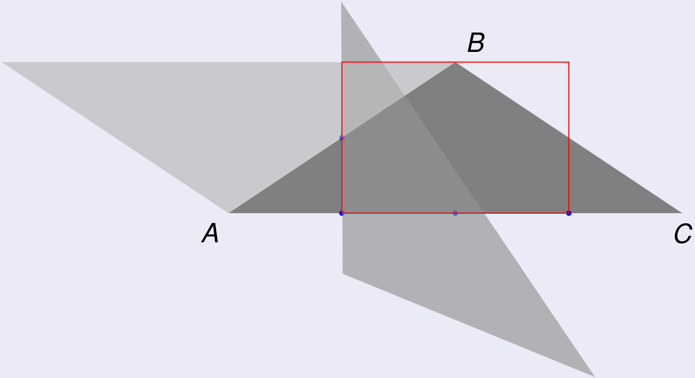


Steps

4. The same procedure on the other side

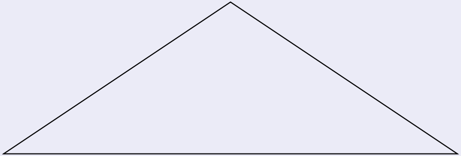
Alternative proof of Freitas's bound



Steps

5. $\lambda_T \geq \lambda_R = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) = \pi^2 \left(\frac{4}{d^2} + \frac{d^2}{4A^2} \right)$

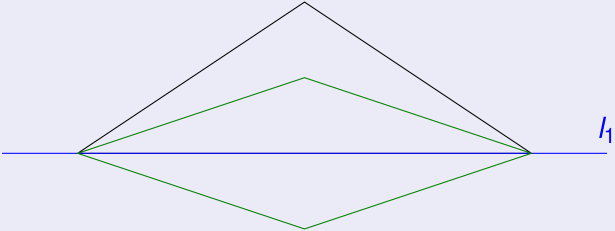
Improved lower bound using rectangles



Steps

- Start with an already symmetrized triangle.

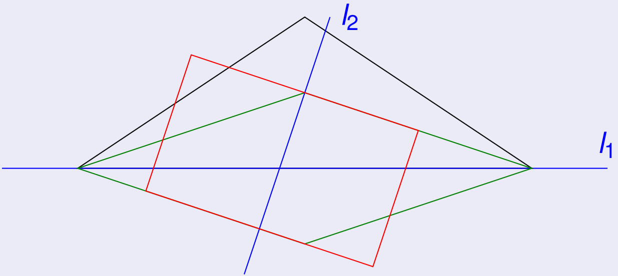
Improved lower bound using rectangles



Steps

- Start with an already symmetrized triangle.
- Steiner symmetrization with respect to the longest side.

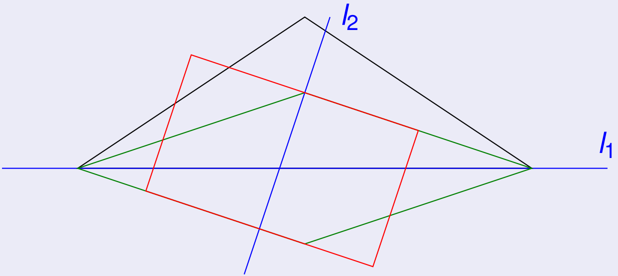
Improved lower bound using rectangles



Steps

- Start with an already symmetrized triangle.
- Steiner symmetrization with respect to the longest side.
- Steiner symmetrization with respect to the altitude.

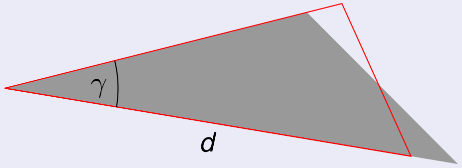
Improved lower bound using rectangles



Steps

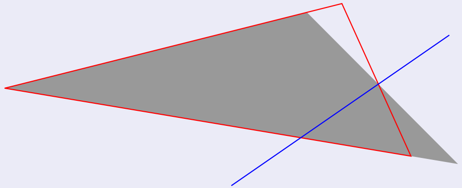
- Start with an already symmetrized triangle.
- Steiner symmetrization with respect to the longest side.
- Steiner symmetrization with respect to the altitude.
- $\lambda_T \geq \lambda_R = \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) = \pi^2 \left(\frac{4}{d^2+h^2} + \frac{d^2+h^2}{4A^2} \right)$

Symmetrization into isosceles triangles



We want to fit a gray triangle with area A inside a red one with area $A + \varepsilon$. Then by the scaling property we can take a limit $\varepsilon \to 0$. This shows that given area A and the smallest angle γ, the eigenvalue decreases with diameter.

Symmetrization into isosceles triangles



We need to define a sequence of polarizations with respect to certain bisectors.

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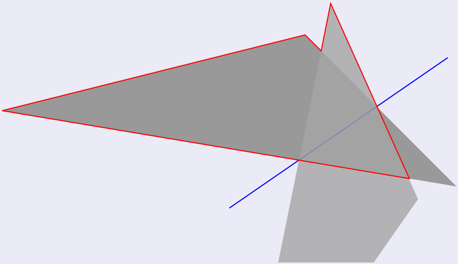

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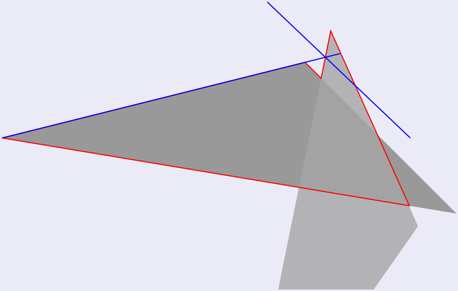

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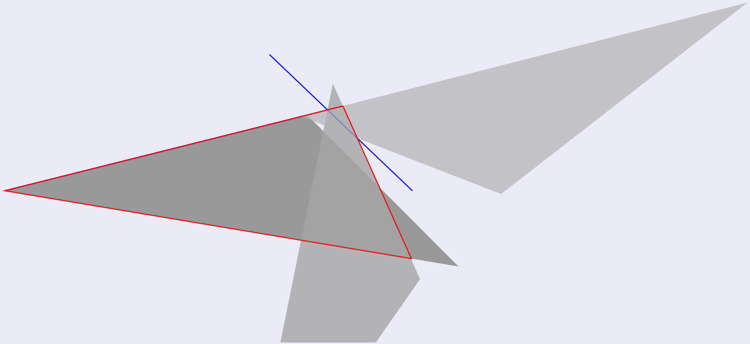

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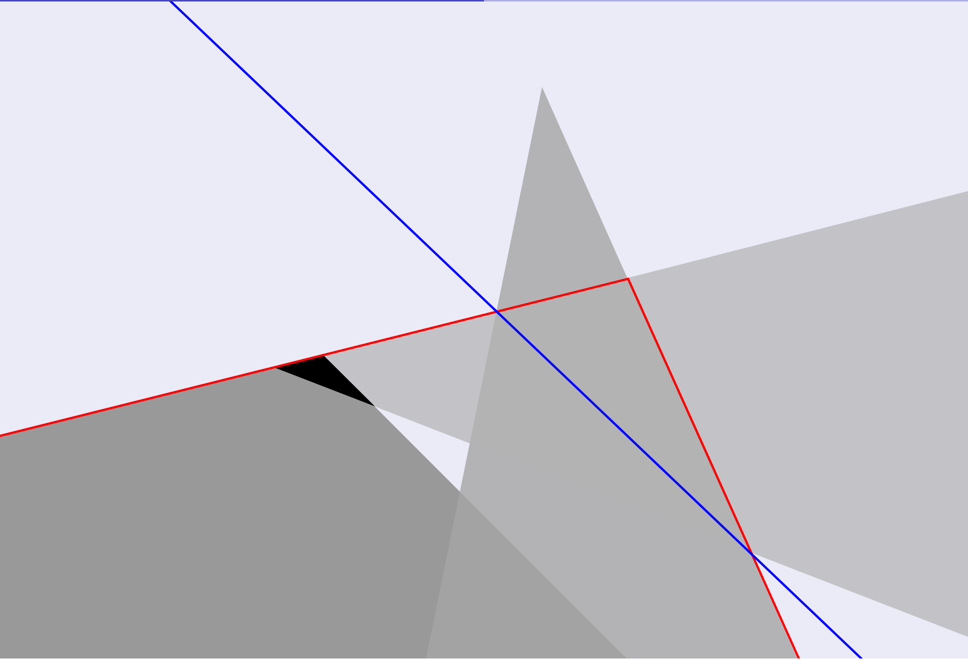

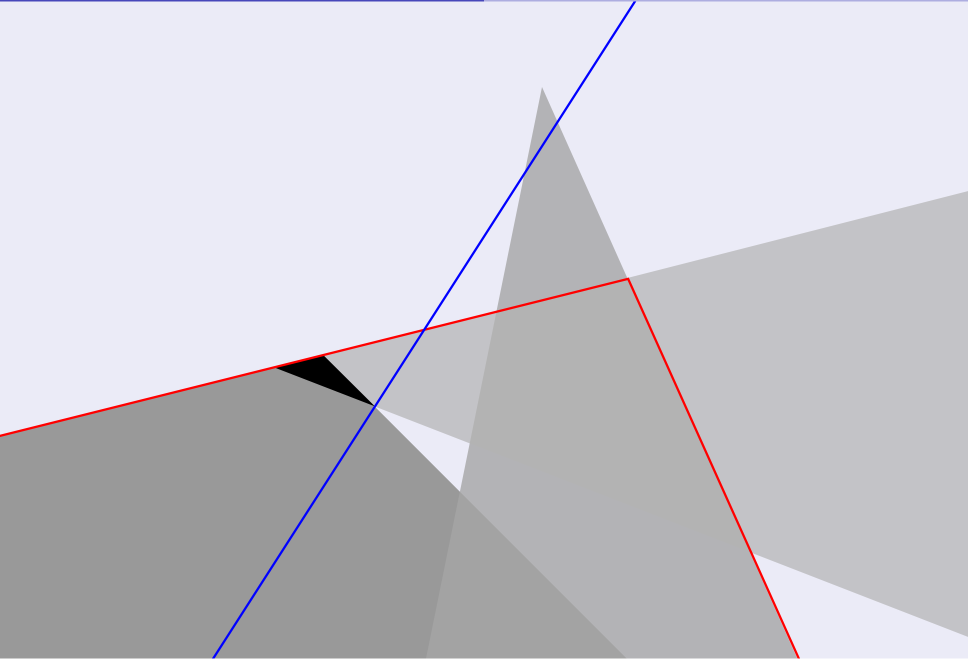

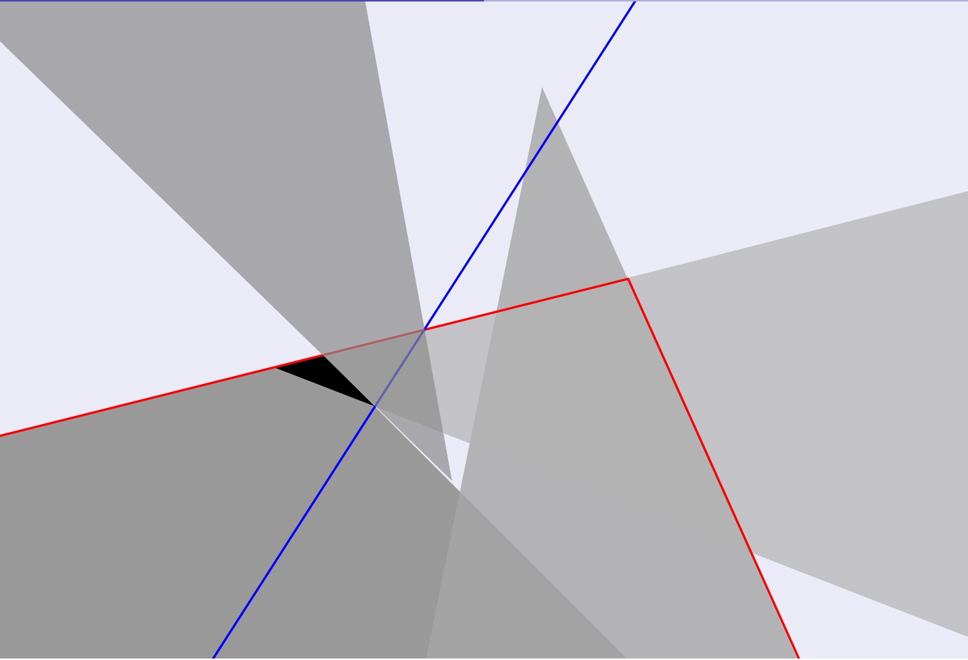

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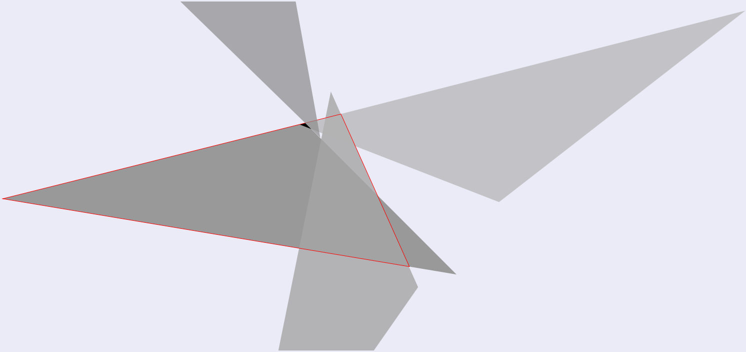
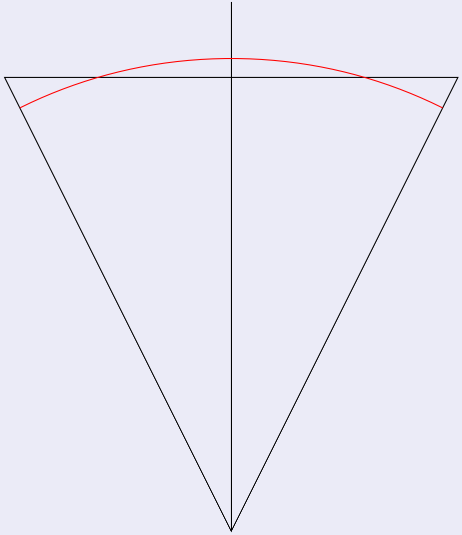


The reversed sequence of reflections gives a valid sequence of polarizations.

Symmetrization into circular sector



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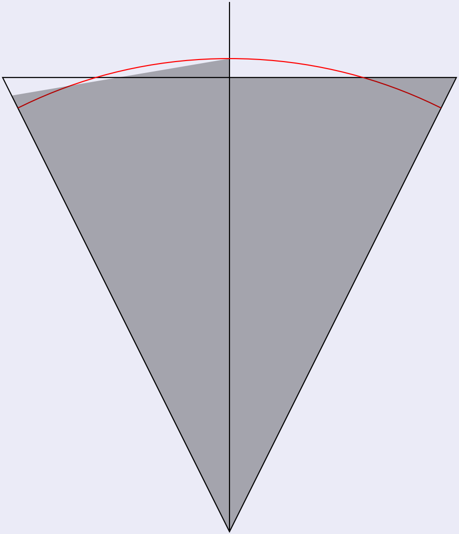

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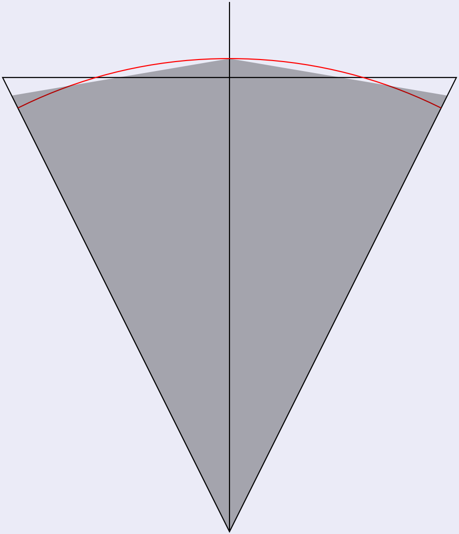

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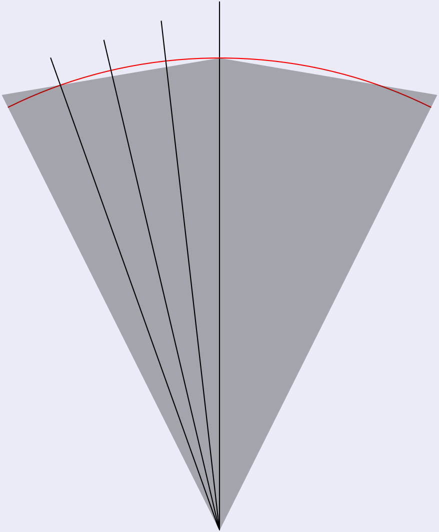

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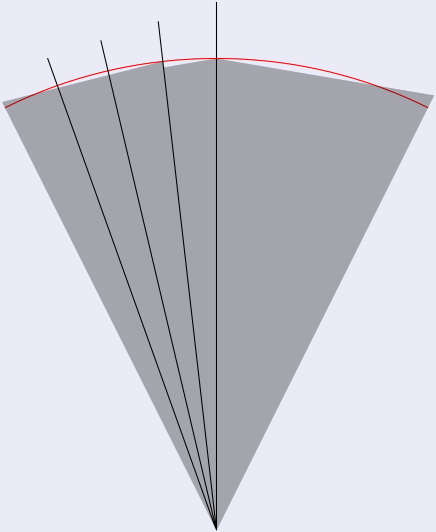
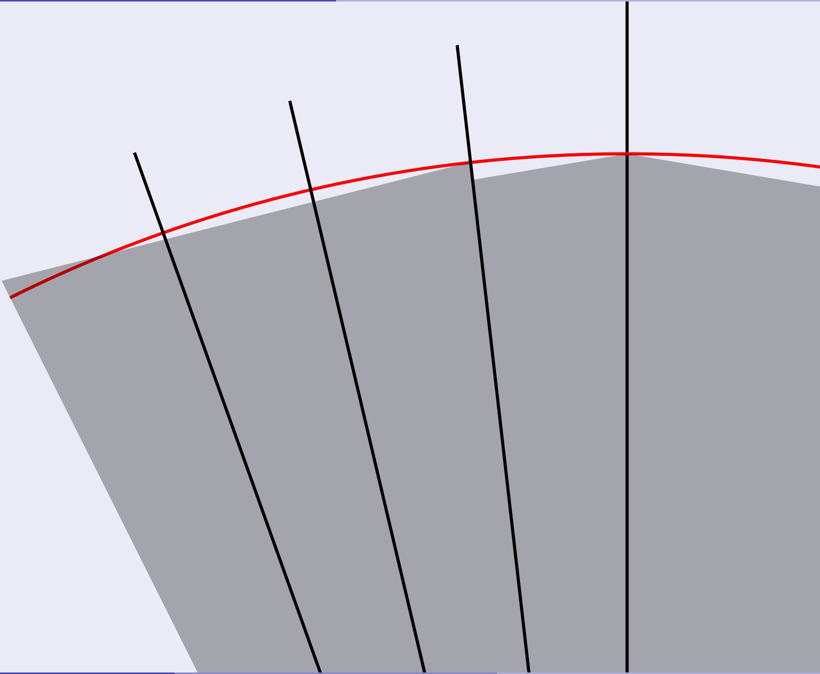
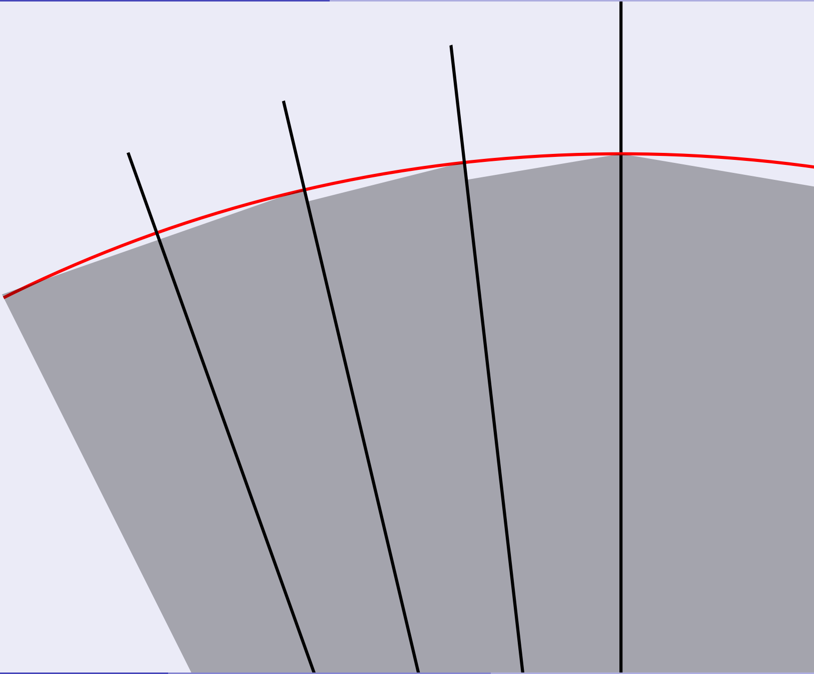
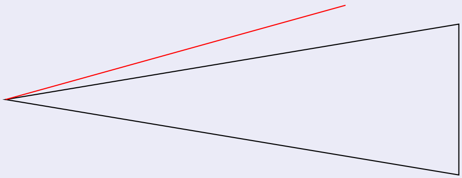






Monotonicity for isosceles triangles



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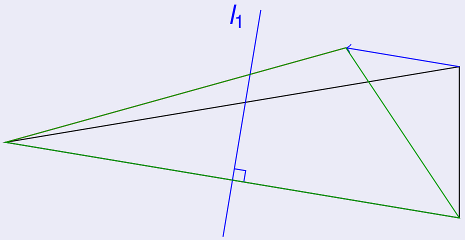
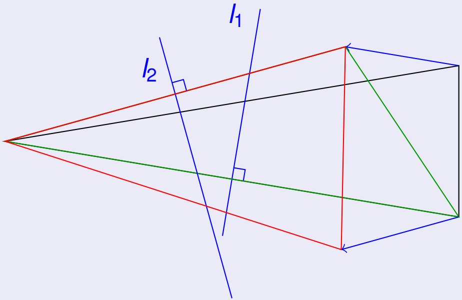


- Continuous Steiner symmetrization with respect to l_1.

Monotonicity for isosceles triangles



- Continuous Steiner symmetrization with respect to l_1.
- Continuous Steiner symmetrization with respect to l_2.