Math 124 C1 Homework #4
Ch. 6.2 #11, 12, 32
Ch. 6.3 #6, 10, 12, 14, 24, 26, 34

Chapter 6.2

#11

We are rolling two fair six-sided dice, and we want the probability of not rolling a sum of seven. Note that implicitly we are using the equally-likely probability function here, so we will be counting outcomes. We could choose to find the number of outcomes with sum 2, 3, 4, and so on, but the best way to do this problem is to apply the complementation rule.

Let E be the event of rolling a sum of seven. Noting that the underlying sample space S consists of ordered pairs of numbers one through six, we see that E has six outcomes: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1). Using that the $n(S) = 6^2 = 36$,

$$\Pr(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

The actual quantity we are asked to find is the probability that we do *not* roll a sum of 7! Thus, we are really interested in finding $\Pr(E')$. We use the rule for complementation:

$$\Pr(E') = 1 - \Pr(E) = 1 - \frac{1}{6} = \frac{5}{6}$$

#12

We are drawing three cards from a standard deck and asking the probability that at least one of them is a spade. We again exploit the complementation rule; instead of finding the probability of at least one spade, we find the probability of getting no spades. We first count the sample space, S, by noting that it consists of 3-permutations of the set with 52 cards. Therefore, $n(S) = P(52, 3)$. Next, we count how many 3-permutations there are in which no card is a spade; the set of such 3-permutations will be called E'. We can think of this as finding the number of 3-permutations from the set of cards that are not spades; there are 39 such cards. Thus $n(E') = P(39, 3)$. Again, using the equally likely probability function, we see that:

$$\Pr(E') = \frac{n(E')}{n(S)} = \frac{P(39, 3)}{P(52, 3)}$$

Of course, E' is not the even whose probability we want; rather, we want to find $\Pr(E)$. Hence

$$\Pr(E) = 1 - \Pr(E') = 1 - \frac{P(39, 3)}{P(52, 3)}$$

#32

Let F be the event that the salesman sells a full-sized car and I the event that he sells an intermediate-sized car. We seek to find the quantity $\Pr(F \cap I)$. We are given that $\Pr(F) = 0.14$ and $\Pr(I) = 0.0.37$. We are given that the probability that he sells at least one car is 0.45. We may interpret this event as $F \cup I$. Using inclusion-exclusion, we know that

$$\begin{aligned} \Pr(F \cup I) &= \Pr(F) + \Pr(I) - \Pr(F \cap I) \\ 0.45 &= 0.14 + 0.37 - \Pr(F \cap I) \\ 0.06 &= \Pr(F \cap I) \end{aligned}$$

Thus the probability that he sells both is 0.06.

6.3

#6

Let F be the event that the card is a face card, and K be the event that it is a king. The probability that the card is a king given that it is a face card can be expressed as $\Pr(K|F)$. Since there are 12 face cards, and there are 4 face cards that are kings,

$$\begin{aligned} \Pr(K|F) &= \frac{\Pr(K \cap F)}{\Pr(F)} = \frac{n(K \cap F)}{n(F)} \\ &= \frac{4}{12} = \frac{1}{3} \end{aligned}$$

#10a

We are calculating $\Pr(E|F) = \frac{n(E \cap F)}{n(F)}$, so first we find out how many outcomes have at least one one. There are six events with the first die a 1 and six with the second die a 1. We notice that rolling two ones is in both events, so there are $n(F) = 6 + 6 - 1 = 11$ outcomes in which at least one die is a 1.

Now, of these 11 events, how many also show a sum of 7? If at least one die is a 1, in order for the sum to be 7, the other die must be a six. Hence, $n(E \cap F) = 2$. Therefore, the probability of E given F is $\frac{2}{11}$.

#10b

We begin by noting there are 6 outcomes in which the sum of the two dice is seven: $(1, 6), (2, 5), (3, 4), (5, 2), (6, 1)$. We see that only two of these have a 1 showing on either die. Therefore, $\Pr(F|E) = \frac{2}{6} = \frac{1}{3}$.

#10c

We first find the number of outcomes in which the sum of the two dice is odd. We recall that when adding two numbers together, the result is odd if and only if one of the numbers is even and the other odd. Using our usual counting tricks, if the first die is odd, there are 3 choices for its value. The second die must then be one of 3 even numbers, for a total of 9. But we see that there are also 9 outcomes in which the first die is even and the second odd, so there are 18 such outcomes overall. We now need to find the size of the intersection of E and G. If the sum of the dice is 7, then the sum is also odd, so every event in E is an event in G, so $n(E \cap G) = n(E)$. As usual, there are 6 ways to roll a sum of 7 with two dice. Therefore, $\Pr(E|G) = \frac{6}{18} = \frac{1}{3}$.

#10d

We seek the probability that the sum of the dice is odd, given that the sum of the dice is 7. Since we already know the sum is 7, which is an odd number, the sum of the dice must automatically be odd. Thus $\Pr(G|E) = 1$.

#12a

We think of the sample space as 2-tuples, listing the value of the red die first and the white die second. For $\Pr(E|F)$, we need $n(E \cap F)$ and $n(F)$. Note that if the red die shows a 1, the white die can be any of the six possible values; thus $n(F) = 6$. Now if the red die shows 1 and the sum is 7, the white die must be a 6. There is exactly one way for this to occur, so the probability $\Pr(E|F) = \frac{1}{6}$.

#12b

As above, we know that $n(E \cap F) = 1$. We need to find $n(E)$, which is 6, from a quick count of the die outcomes that total 7. Thus $\Pr(F|E) = \frac{1}{6}$.

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#14

Let F be the event that the plane is not shot down and E the probability that the target is hit. We use the total probability formula.

$$\Pr(F) = \Pr(E)\Pr(F|E) + \Pr(E')\Pr(F|E')$$

We notice that if the plane is shot down, the target cannot be hit, so $\Pr(F|E') = 0$. Hence,

$$\Pr(F) = \Pr(E)\Pr(F|E) = (1 - .38)(0.84) = 0.5208$$

#24

We are finding the probability of a drawn ball being white. Let W be the event of drawing a white ball, U_1 the event of drawing from urn 1, and U_2 the event of drawing from urn 2. We use the total probability equation, since every outcome involves drawing from one of the two urns. We further assume that we choose between either of the urns with equal probability.

$$\begin{aligned}
\Pr(W) &= \Pr(U_1)\Pr(W|U_1) + \Pr(U_2)\Pr(W|U_2) \\
&= \frac{1}{2} \cdot \frac{9}{13} + \frac{1}{2} \cdot \frac{2}{10} \\
&= \frac{9}{26} + \frac{1}{10} \\
&= \frac{90}{260} + \frac{26}{260} \\
&= \frac{116}{260}
\end{aligned}$$

#26

Let R be the event that the drawn ball is red, U_i the ith urn is chosen, and B_i the event that the ith box is chosen. Note that we can only choose urns 1 and 2 if we choose box 1, and we can only choose urns 3, 4, and 5 if we choose box 2.

$$\begin{aligned}
\Pr(R) &= \Pr(B_1)\Pr(R|B_1) + \Pr(B_2)\Pr(R|B_2) \\
&= \Pr(B_1)[\Pr(U_1)\Pr(R|U_1) + \Pr(U_2)\Pr(R|U_2)] + \Pr(B_2)(\Pr(U_3)\Pr(R|U_3) \\
&\quad + \Pr(U_4)\Pr(R|U_4) + \Pr(U_5)\Pr(R|U_5)) \\
&= \frac{1}{2}(\frac{1}{2} \cdot \frac{3}{6} + \frac{1}{2} \cdot \frac{4}{6}) + \frac{1}{2}(\frac{1}{3} \cdot \frac{5}{8} + \frac{1}{3} \cdot \frac{2}{9} + \frac{1}{3} \cdot \frac{4}{7}) \\
&= \frac{3}{24} + \frac{4}{24} + \frac{5}{48} + \frac{2}{54} + \frac{4}{42} \\
&= 0.528
\end{aligned}$$

#34a

We will assume that player A goes first. For this part, there are 3 balls in the box, and so no game is longer than three rounds. We can write out a game by listing the order in which the balls are drawn.

First, we find the probability that A wins on round 1. To win, he must draw the white ball, of which there is one, from the pool of 3 balls total. Thus A has a $\frac{1}{3}$ chance of winning on the first round.

A cannot win on the second round, because A does not draw the ball in the second round.

A can win on the third round, only if he did not win on the first round and B did not win on the second round. A has a $\frac{2}{3}$ chance of not drawing the white ball in round 1, and so B has a $\frac{1}{2}$ chance of not drawing the white ball in round 2. In round 3, one ball is left, the white one, so A will draw it with probability 1. Therefore, A has a $\frac{2}{3} \times \frac{1}{2}$ chance of winning on round 3.

The probability of A winning is therefore $\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}$.

Since one of A and B must win, B then has a $1 - \frac{2}{3} = \frac{1}{3}$ chance of winning.

34b

We again calculate the probability that A wins, and then use complementation to find B's chances.

Since there are six balls, there are six rounds. We again break A's chances of winning into the chances that he wins on any round in which he draws a ball– the 1st, 3rd, and 5th rounds.

Since 2 of the 6 balls are white, A has a $\frac{1}{3}$ chance of winning on round 1.

In order to win on round 3, all balls drawn on previous rounds must be black. A has a $\frac{4}{6}\frac{3}{5}\frac{2}{4} = \frac{1}{5}$ chance of winning on round 3.

In order to win on round 5, all balls drawn on the previous 4 rounds must be black. Thus the probability is $\frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \cdot 1 = \frac{1}{15}$. The chance of drawing a white ball on round 5 is, of course, 1, because there are only 4 black balls in the box, all of which must have been drawn in the previous four rounds.

The total probability of A winning is therefore $\frac{1}{3} + \frac{1}{5} + \frac{1}{15} = \frac{9}{15}$.