THE INJECTIVE ENVELOPE AS THE SPACE OF EXTREMAL FUNCTIONS

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ABSTRACT

In this thesis, we study the injective envelope of metric spaces by viewing it as the space of extremal functions as defined by Isbell. Extremal functions are also Katětov functions, which satisfy two inequalities derived from the triangle inequality. One of these inequalities, along with a minimality requirement, is used to define the extremal functions. We compare the extremal functions to other classes of functions defined similarly using one of the two inequalities from the definition of Katětov functions. We also consider separability of the space of extremal functions. We give a general method for generating uncountably many extremal functions from one extremal function satisfying certain inequalities on a sequence of ordered pairs. Then we prove non-separability of the space of extremal functions over some metric subspaces of finite dimensional real Banach spaces and some bounded metric spaces by constructing such an extremal function. Lastly, we discuss some connections with Melleray's work on separability of the space of Katětov functions.

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CHAPTER 1

INTRODUCTION

A metric space Y is *injective* if every mapping that increases no distance from a subspace of any metric space X to Y can be extended, increasing no distance, over X. Isbell [1] proved that every metric space has a smallest injective space containing it, called its *injective envelope*. Isbell constructed the injective envelope of any metric space X as the space of *extremal functions* over X, denoted by $\varepsilon(X)$ (see Definition 1.1). Cohen [2] constructed the injective (linear) envelope for any real or complex Banach space. Isbell [3] proved that his injective envelopes are equal to Cohen's injective (linear) envelopes for real Banach spaces.

In this paper, we try to understand the injective envelope by studying the extremal functions. It turns out that extremal functions are also *Katětov functions*. The space of Katětov functions over X, denoted by $E(X)$ (see Definition 1.2), was defined by Katětov [4] to construct the Urysohn space. Katětov functions satisfy two inequalities: the sum of the function's values at any two points is at least the distance between the points, and the difference of the function's values at any two points is at most the distance between the points. Each of these two inequalities show up in definitions that are of interest to us.

First, extremal functions are defined as functions that are point-wise minimal such that the sum of the function's values at any two points is at least the distance between the points. Second, *finitely supported* functions (see Definition

2.1) are defined as Katětov functions satisfying the following condition: for some finite set of points, for any point x in the space, there is a point y from the finite set such that the difference of the function's values at x and y is equal to the distance between x and y. The finitely supported Katětov functions are crucial in Katětov's [4] construction of the Urysohn space and in Melleray's [5] characterization of Polish metric spaces with separable spaces of Katětov functions.

In Chapter 2, we consider functions arising from the two inequalities in the definition of Katětov functions. Using the extremal functions and the finitely supported functions as patterns, we define three new classes of functions: the *finitely witnessed functions*, the *min functions*, and the *max functions*. The finitely witnessed functions will be extremal and are defined symmetrically to the finitely supported functions. The min and max functions are generalizations of the finitely supported functions and finitely witnessed functions, respectively. We study how these classes relate to each other as well as to the classes of Katětov functions and extremal functions. Propositions 2.5 and 2.6 summarize these relationships.

In Chapter 3, we focus on the question of separability of the space of extremal functions. The question is difficult to answer for metric spaces in general, however, we do have results for some specific classes of metric spaces. From Isbell [1], we know that compact spaces have compact injective envelopes, so their spaces of extremal functions are separable. Also, metric spaces embed into their injective envelopes, thus any non-separable metric space has non-separable injective envelope. This leaves us with the separable non-compact metric spaces. We will focus on two particular classes of metric spaces, the proper spaces and the bounded metric spaces. Notice that the intersection of these two classes, the proper and bounded metric spaces, are exactly the compact spaces.

In the class of proper spaces, we consider metric subspaces of finite dimensional Banach spaces. Theorem 3.11 in Section 3.2 states that separability of $\varepsilon(\mathbb{R}^n, N)$ and $\varepsilon(\mathbb{Z}^n, N)$ is determined by the number of extreme points of the unit sphere given by the norm N. Lacey and Cohen [6] characterized the injective envelopes of real and complex Banach spaces in terms of the unit sphere in the dual space. Their result implies Theorem 3.11 for $\mathbb{R}^n$, but their method cannot be applied to $\mathbb{Z}^n$ as they use Cohen's construction of the (linear) injective envelope in [2], which is only for Banach spaces. Our method of proof for Theorem 3.11 is as follows: In the case that the injective envelope is separable, we construct an isometry between the space of extremal functions and $(\mathbb{R}^k, \|\cdot\|_\infty)$, for some k, which is determined by the shape of the unit sphere. In the case that the injective envelope is non-separable, we construct an uncountable set of extremal functions witnessing non-separability. This construction of extremal functions does not rely on any Banach-specific properties of the space, and can be carried out both in $\mathbb{R}^n$ and $\mathbb{Z}^n$, proving non-separability of the injective envelope of both spaces.

Section 3.3 deals with separability of the injective envelope of bounded metric spaces. We find that the existence of a sequence of ordered pairs whose distances satisfy certain properties, which we call *slack* (see Definition 3.31), is sufficient for the injective envelope to be non-separable. We also find that for a particular class of bounded metric spaces, those whose points can be arranged into an *approximately constant* (see Definition 3.32) sequence of n-tuples for some n, the existence of a slack sequence of ordered pairs is also necessary for the injective envelope to be non-separable. Theorem 3.33 states these results.

Proofs in Sections 3.2 and 3.3 use some general facts about extremal functions collected in Section 3.1. Here, we first show an extension property of extremal functions, and then prove three propositions regarding preservation of separability of the space of extremal functions. Proposition 3.2 states that the space of extremal functions over any metric space is isomorphic (as metric spaces) to the space of extremal functions over any dense subspace of it. Proposition 3.3 states that if X is any metric space with separable $\varepsilon(X)$, then $\varepsilon(X \cup K, d)$ is also separable for any compact space K and any metric d extending the metrics on X and K. Proposition 3.7 states that if $(d_i)_i$ is a sequence of metrics on X *converging uniformly* (see Definition 3.6) to a metric d on X, and $\varepsilon(X, d_i)$ is separable for all i, then $\varepsilon(X, d)$ is separable.

Following Melleray's work in [5], where he proves that the space of Katětov functions is separable if and only if the space of finitely supported functions is dense in it, we attempt to find a subspace of the extremal functions with the same property, i.e. a canonical separable subspace such that $\varepsilon(X)$ is separable if and only if this subspace is dense in it. We consider several natural subspaces of $\varepsilon(X)$. Unfortunately, none of them is as desired. The discussion and counterexamples for the subspaces we consider are in Section 3.4.

The two main definitions for this paper:

Definition 1.1. A function $f : X \to \mathbb{R}$ on a metric space (X, d) is called *extremal* if

$$d(x, y) \leq f(x) + f(y) \tag{1.1}$$

for all $x, y \in X$ and whenever $g : X \to \mathbb{R}$ such that $d(x, y) \leq g(f) + g(y)$ for all

$x, y \in X$ and $g(x) \leq f(x)$ for all $x \in X$, then $f = g$. Denote by $\varepsilon(X)$ the space of extremal functions on X.

Definition 1.2. A function $f : X \to \mathbb{R}$ on a metric space (X, d) is called *Katětov* if for all $x, y \in X$ it satisfies the inequalities

$$f(x) - f(y) \leq d(x, y) \leq f(x) + f(y). \tag{1.2}$$

Denote by $E(X)$ the space of Katětov functions on X.

We equip both $E(X)$ and $\varepsilon(X)$ with the sup metric:

$$d_E(f, g) = \sup \{|f(x) - g(x)| : x \in X\}.$$

It can be checked that for any $f, g \in E(X)$ and any $x_0 \in X$, $|f(x) - g(x)| \leq f(x_0) + g(x_0)$ for all $x \in X$. Hence, the above sup is finite and the metric is well-defined. Although not immediate from the definition, it is not hard to show, cf. Isbell [1] or see the argument at the beginning of Chapter 2, that extremal functions are also Katětov; i.e., $\varepsilon(X) \subseteq E(X)$.

CHAPTER 2

FINITELY SUPPORTED KATĚTOV FUNCTIONS AND EXTREMAL FUNCTIONS

Recall that inequality (1.2) from Definition 1.2 of Katětov functions has two parts:

$$f(x) - f(y) \leq d(x, y) \tag{2.1}$$

$$d(x, y) \leq f(x) + f(y). \tag{2.2}$$

Notice that (1.1) from Definition 1.1 of extremal functions is exactly (2.2). It turns out that extremal functions also satisfy (2.1), making them Katětov functions. Isbell showed this in [1], but we include the following short argument to illustrate the interaction between these two inequalities.

Notice that $d(x, y) \leq f(x) + f(y)$ can also be written as $f(x) \geq d(x, y) - f(y)$. This leads us to the following equivalent definition for $\varepsilon(X)$:

$$f \in \varepsilon(X) \Leftrightarrow \forall x \in X \ f(x) = \sup_{y \in X} d(x, y) - f(y). \tag{2.3}$$

We will use this equivalent definition to show that extremal functions are Katětov. Inequality (2.2) for all $x, y \in X$ is obviously implied by the sup, so it remains to show that

$$\forall x \in X \ f(x) = \sup_{y \in X} d(x, y) - f(y) \Rightarrow \forall x, y \in X \ f(x) - f(y) \leq d(x, y).$$

If for some x, y we had $f(x) - f(y) > d(x, y)$, then for all y', f extremal implies

$f(y) + f(y') \geq d(y, y')$. So

$$\begin{aligned} f(x) + f(y') &\geq f(x) + d(y, y') - f(y) \\ &= d(x, y) + d(y, y') + [f(x) - f(y) - d(x, y)] \\ &\geq d(x, y') + [f(x) - f(y) - d(x, y)]. \end{aligned}$$

Since we assumed $f(x) - f(y) - d(x, y) > 0$, we have

$$f(x) > \sup_{y' \in X} d(x, y') - f(y').$$

Contradiction to (2.3). So f must satisfy (2.1) and hence is Katětov.

On the other hand, we also have an important class of Katětov functions defined from (2.1). As with (2.3), this definition uses the $f(x) \leq d(x, y) + f(y)$ form of $f(x) - f(y) \leq d(x, y)$ (definition from Gao and Kechris [7]):

Definition 2.1. For a metric space (X, d) with $X \neq \emptyset$, a function $f \in E(X)$ is said to be *finitely supported* if there is a finite set $Y \subseteq X$ such that

$$f(x) = \min_{y \in Y} d(x, y) + f(y)$$

for all $x \in X$. In this case, Y is called a *support of* f and we say that f is *n-supported* if f has a support of cardinality n.

Denote by $E(X, \omega)$ the set of all all finitely supported Katětov functions and by $E(X, n)$ the set of all n-supported Katětov functions.

The finitely supported functions were originally defined by Katětov [4] for his construction of the Urysohn space. What is more interesting to us is that Melleray [5] proved $E(X)$ is separable if and only if $E(X, \omega)$ is dense in $E(X)$. We will return to this in Section 3.4.

If we view the two side of (1.2) as opposites, then the definitions of the extremal functions (the form given in (2.3)) and the finitely supported functions have symmetric counterparts. We obtain these symmetric definitions by interchanging (2.1) and (2.2) and taking the corresponding extremal values. In this way, the definition of extremal functions in (2.3) yields the symmetric statement

$$\forall x \in X \; f(x) = \inf_{y \in X} d(x, y) + f(y).$$

Unlike the extremal functions, functions satisfying this condition do not have to be Katětov. This is because

$$f(x) = \inf_{y \in X} d(x, y) + f(y) \Rightarrow f(x) - c = \inf_{y \in X} d(x, y) + f(y) - c$$

for any $c \in \mathbb{R}$. For any $x, y \in \mathbb{R}^n$, we can take c large enough so that $f(x) - c + f(y) - c < d(x, y)$, failing (1.2). However, we do have that all Katětov functions satisfy this inf condition, i.e.

$$f \in E(X) \Rightarrow \forall x \in X \; f(x) = \inf_{y \in X} d(x, y) + f(y).$$

This follows immediately from the definitions:

$$f(x) = d(x, x) + f(x) \geq \inf_{y \in X} d(x, y) + f(y)$$

and $f(x) \leq \inf_{y \in X} d(x, y) + f(y)$ since $f(x) - f(y) \leq d(x, y)$ for all y.

The corresponding symmetric statement to the definition of finitely supported functions gives what we call the *finitely witnessed* functions. Though not assumed to be so in the definition, these finitely witnessed functions will be extremal (following easily from the definition), justifying the notation:

Definition 2.2. For a metric space $X \neq \emptyset$, a function $f \in E(X)$ is said to be *finitely witnessed* if there is a finite set $Y \subseteq X$ such that

$$f(x) = \max_{y \in Y} d(x, y) - f(y)$$

for all $x \in X$. In this case, Y is called a *witness set of* f and we say that f is n*-witnessed* if f has a witness set of cardinality n.

Denote by $\varepsilon(X, \omega)$ the set of all finitely witnessed Katětov functions and by $\varepsilon(X, n)$ the set of all n-witnessed Katětov functions.

For both the finitely supported functions and the finitely witnessed functions, assuming the function is Katětov is necessary as part of the definition. Without this assumption, we get functions that may not be Katětov. These generalized definitions are as follows:

Definition 2.3. Given any finite set $F \subseteq X$ and a function $\alpha : F \to \mathbb{R}$, define the *min function on* (F, α) as

$$m_{F,\alpha}(x) = \min_{y \in F} d(x, y) + \alpha(y)$$

and define the *max function on* (F, α) as

$$M_{F,\alpha}(x) = \max_{y \in F} d(x, y) - \alpha(y).$$

We will use $m(X, \omega) := \bigcup_n m(X, n)$ to denote the set of all min functions and $M(X, \omega) := \bigcup_n M(X, n)$ to denote the set of all max functions, where

$$\begin{aligned} m(X, n) &:= \{m_{F,\alpha} : F \subseteq X, |F| = n, \alpha : F \to \mathbb{R}\}, \\ M(X, n) &:= \{M_{F,\alpha} : F \subseteq X, |F| = n, \alpha : F \to \mathbb{R}\}, \end{aligned}$$

for each n.

It follows immediately from the definitions that

$$E(X, \omega) \subseteq m(X, \omega) \text{ and } \varepsilon(X, \omega) \subseteq M(X, \omega),$$

since for all n, $E(X, n) \subseteq m(X, n)$ and $\varepsilon(X, n) \subseteq M(x, n)$. In general, these inclusions are strict, and $m(X, \omega)$, $M(X, \omega)$ are not contained in $E(X)$, $\varepsilon(X)$, respectively. So we ask:

- Under what conditions is a min function Katětov?

- Under what conditions is a min function extremal?
- If a min function is Katětov, is it automatically finitely supported?

The same questions may be asked of the max functions, with "finitely witnessed" in place of "finitely supported". Propositions 2.5 and 2.6 answer all of the above questions. To state the propositions, we will need the following metric space notation (definition from Mineyev [8]):

Definition 2.4. The *double difference* in a metric space (X, d) is the function $\langle \cdot, \cdot | \cdot, \cdot \rangle : X^4 \to \mathbb{R}$ defined by

$$\langle x, y' | y, x' \rangle := \frac{1}{2}(d(x, y) + d(x', y') - d(x, x') - d(y, y')).$$

The following results are proved at the end of this chapter after series of lemmas.

Proposition 2.5. *For any finite $F \subseteq X$ and $\alpha : F \to \mathbb{R}$,*

1. $m_{F,\alpha} \in E(X) \Leftrightarrow \forall y, y' \in F \; \alpha(y) + \alpha(y') \geq d(y, y')$;
2. $M_{F,\alpha} \in E(X) \Leftrightarrow \forall x, x' \in X \; \exists y, y' \in F$
$$2 \langle x, y' | y, x' \rangle \geq \alpha(y) + \alpha(y') - d(y, y');$$
3. $m_{F,\alpha} \in \varepsilon(X) \Leftrightarrow m_{F,\alpha} \in E(X)$ *and* $m_{F,\alpha} = M_{F,\alpha}$;
4. $M_{F,\alpha} \in \varepsilon(X) \Leftrightarrow M_{F,\alpha} \in E(X)$ *and* $\forall y, y' \in F \; \alpha(y) + \alpha(y') \geq d(y, y')$.

Proof. See page 18 below. □

Proposition 2.6. *For any finite $F \subseteq X$ and $\alpha : F \to \mathbb{R}$,*

1. *$m_{F,\alpha} \in E(X) \Rightarrow m_{F,\alpha} \in E(X, |F|)$ and F is a support of $m_{F,\alpha}$;*

2. *$M_{F,\alpha} \in \varepsilon(X) \Rightarrow M_{F,\alpha} \in \varepsilon(X, |F|)$ and F is a witness set of $M_{F,\alpha}$.*

Thus, for any n, $E(X, n) = E(X) \cap m(X, n)$ and $\varepsilon(X, n) = \varepsilon(X) \cap M(X, n)$, so

$$E(X, \omega) = E(X) \cap m(X, \omega)$$

and

$$\varepsilon(X, \omega) = \varepsilon(X) \cap M(X, \omega).$$

Proof. See page 19 below. □

We prove statements 1 and 2 in Proposition 2.5 by considering when min and max functions satisfy each of the inequalities (2.1) and (2.2) separately. From the discussion thus far, we would expect (2.1) to be more easily satisfied. Indeed, this is the case:

Lemma 2.7. *For any finite $F \subseteq X$ and $\alpha : F \to \mathbb{R}$, for all $x, x' \in X$,*

1. *$m_{F,\alpha}(x) - m_{F,\alpha}(x') \leq d(x, x')$ and*

2. *$M_{F,\alpha}(x) - M_{F,\alpha}(x') \leq d(x, x')$.*

Proof. We prove the first statement; the second is proved in a similar manner. Fix any $x, x' \in X$ and let $y, y' \in Y$ be such that $m_{F,\alpha}(x) = d(x, y) + \alpha(y)$ and

$m_{F,\alpha}(x') = d(x', y') + \alpha(y')$. By the min construction in the definition of $m_{F,\alpha}$, we must have

$$d(x, y) + \alpha(y) \leq d(x, y') + \alpha(y').$$

Thus

$$\begin{aligned} m_{F,\alpha}(x) - m_{F,\alpha}(x') &= d(x, y) + \alpha(y) - (d(x', y') + \alpha(y')) \\ &\leq d(x, y') + \alpha(y') - d(x', y') - \alpha(y') \\ &\leq d(x, y') - d(x', y') \\ &\leq d(x, x'). \end{aligned}$$

So the first statement for min functions is proved. The second statement for max functions is proved similarly. □

Now we turn to inequality (2.2). For min functions, this inequality is satisfied if and only if it is satisfied by α on F:

Lemma 2.8. *For any finite $F \subseteq X$ and $\alpha : F \to \mathbb{R}$, $m_{F,\alpha}$ satisfies*

$$m_{F,\alpha}(x) + m_{F,\alpha}(x') \geq d(x, x')$$

for all $x, x' \in X$ if and only if α satisfies

$$\alpha(y) + \alpha(y') \geq d(y, y')$$

for all $y, y' \in F$.

Proof. ($\Rightarrow$) By our assumptions, $m_{F,\alpha}(y) + m_{F,\alpha}(y') \geq d(y, y')$ for all $y, y' \in F$. By definition of min function, for any $y_0 \in F$, we have

$$m_{F,\alpha}(y_0) = \min_{y \in F} d(y_0, y) + \alpha(y) \leq d(y_0, y_0) + \alpha(y_0) = \alpha(y_0).$$

So α satisfies $\alpha(y)+\alpha'(y') \geq d(y, y)$ for all $y, y' \in F$ since $m_{F,\alpha}$ does and $\alpha \geq m_{F,\alpha}$ on F.

($\Leftarrow$) Fix any $x, x' \in X$. By the definition of $m_{F,\alpha}$ we may fix y and y' in F such that $m_{F,\alpha}(x) = d(x, y) + \alpha(y)$ and $m_{F,\alpha}(x') = d(x', y') + \alpha(y')$. Then

$$\begin{aligned} m_{F,\alpha}(x) + m_{F,\alpha}(x') &= d(x, y) + \alpha(y) + d(x', y') + \alpha(y') \\ &\geq d(x, y) + d(x', y') + d(y, y') && \text{(by the hypothesis on } \alpha) \\ &\geq d(x, x') && \text{(by triangle inequality)} \end{aligned}$$

as needed. $\square$

For a max function to satisfy (2.2), the condition is more complicated. However, it can be expressed in terms of a familiar expression on metric spaces, the double difference (recall Definition 2.4):

Lemma 2.9. *Given any finite $F \subseteq X$ and $\alpha : F \to \mathbb{R}$, $M_{F,\alpha}$ satisfies*

$$\forall x, x' \in X \; M_{F,\alpha}(x) + M_{F,\alpha}(x') \geq d(x, x')$$

if and only if

$$\forall x, x' \in X \; \exists y, y' \in F \;\; 2\langle x, y' | y, x' \rangle \geq \alpha(y) + \alpha(y') - d(y, y').$$

Proof. The proof follows directly from the definitions:

$$\begin{array}{rc}
\forall x, x' \in X & M_{F,\alpha}(x) + M_{F,\alpha}(x') \geq d(x, x') \\
\Updownarrow & \text{(by Def. 2.3)} \\
\forall x, x' \in X \; \exists y, y' \in F & d(x, y) - \alpha(y) + d(x', y') - \alpha(y') \geq d(x, x') \\
\Updownarrow & \\
\forall x, x' \in X \; \exists y, y' \in F & d(x, y) + d(x', y') - d(x, x') \geq \alpha(y) + \alpha(y') \\
\Updownarrow & \text{(by Def. 2.4)} \\
\forall x, x' \in X \; \exists y, y' \in F & 2\langle x, y' | y, x' \rangle + d(y, y') \geq \alpha(y) + \alpha(y') \\
\Updownarrow & \\
\forall x, x' \in X \; \exists y, y' \in F & 2\langle x, y' | y, x' \rangle \geq \alpha(y) + \alpha(y') - d(y, y').
\end{array}$$

□

Notice that Lemmas 2.7, 2.8, and 2.9 state exactly when min and max functions are Katětov, proving the first two statements of Proposition 2.5. The following lemma shows when a max function is extremal, and when it is, that the defining set F is a witness set, thus proving statement 4 in Proposition 2.5 and statement 2 in Proposition 2.6.

Lemma 2.10. *Given any finite $F \subseteq X$ and $\alpha : F \to \mathbb{R}$,*

$$M_{F,\alpha} \in \varepsilon(X) \Leftrightarrow M_{F,\alpha} \in E(X) \wedge \forall y,y' \in F \ \alpha(y) + \alpha(y') \geq d(y,y').$$

Moreover, if $M_{F,\alpha} \in \varepsilon(X)$, then F is a witness set of $M_{F,\alpha}$.

Proof. ($\Rightarrow$) We prove this direction by proving the contrapositive. Since $\varepsilon(X) \subseteq E(X)$, then

$$M_{F,\alpha} \notin E(X) \Rightarrow M_{F,\alpha} \notin \varepsilon(X).$$

Suppose we had $y, y' \in F$ such that $\alpha(y) + \alpha(y') < d(y, y')$. We claim $M_{F,\alpha}$ is not minimal at y, so $M_{F,\alpha} \notin \varepsilon(X)$. Take any $x \in X$. By definition of max function, $M_{F,\alpha}(x) \geq d(x, y) - \alpha(y)$, so

$$M_{F,\alpha}(x) + M_{F,\alpha}(y) \geq d(x, y) + M_{F,\alpha}(y) - \alpha(y).$$

Also by definition of the max function,

$$M_{F,\alpha}(y) \geq d(y, y') - \alpha(y') > \alpha(y),$$

so $M_{F,\alpha}(y) - \alpha(y) > 0$. Thus, $M_{F,\alpha}$ is not minimal at y as claimed.

($\Leftarrow$ and the "moreover" part) We show that for all $x \in X$,

$$M_{F,\alpha}(x) = \max_{y \in F} d(x, y) - M_{F,\alpha}(y). \tag{2.4}$$

Then (2.4) and $M_{F,\alpha} \in E(X)$ imply $M_{F,\alpha}$ is extremal and F is a witness set of it. By $M_{F,\alpha} \in E(X)$, we have that for all $x \in X$,

$$M_{F,\alpha}(x) \geq \max_{y \in F} d(x, y) - M_{F,\alpha}(y).$$

On the other hand, our hypothesis that $\alpha(y) + \alpha(y') \geq d(y, y')$ for all $y, y' \in F$ implies $M_{F,\alpha}(y) \leq \alpha(y)$ for all $y \in F$. Hence for all $x \in X$,

$$\begin{aligned} M_{F,\alpha}(x) &= \max_{y \in F} d(x, y) - \alpha(y) \\ &\leq \max_{y \in F} d(x, y) - M_{F,\alpha}(y). \end{aligned}$$

So (2.4) is proved. □

We are ready to finish the proofs of Propositions 2.5 and 2.6. The following piece of notation will help make these proofs, as well as several proofs in Chapter 3, more concise. It comes from a remark of Isbell [1]:

"*If X is compact, then for any f in $\varepsilon(X)$ and x in X there is, by minimality, some y in X such that $f(x) + f(y) = d(x, y)$. In general, we have only $f(x) + f(y) < d(x, y) + \delta$, where δ is any positive number and y depends on δ.*"

Definition 2.11. Given a metric space X and $f \in \varepsilon(X)$, we say that a sequence $(y_i)_i$ in X is a *witness for the minimality of f at x* if

$$\lim_{i \to \infty} f(x) + f(y_i) - d(x, y_i) = 0$$

and denote it by $(y_i)_i \underset{f}{\triangleq} x$.

We say that a point y in X is a *witness for the minimality of f at x* if the constant sequence $(y)_i$ witnesses the minimality of f at x and denote it by $y \underset{f}{\triangleq} x$.

Note that $f(x) + f(y) = d(x, y)$ is an equivalent definition of $y \underset{f}{\doteq} x$.

Proof. (of Proposition 2.5) By the previous lemmas, it remains to prove statement 3:

$$m_{F,\alpha} \in \varepsilon(X) \Leftrightarrow m_{F,\alpha} \in E(X) \text{ and } m_{F,\alpha} = M_{F,\alpha}.$$

($\Leftarrow$) Since $m_{F,\alpha} \in E(X)$, statement 1 of Proposition 2.5 gives for all $y, y' \in F$, $\alpha(y) + \alpha(y') \geq d(y, y')$. Then statement 4 of Proposition 2.5 gives $M_{F,\alpha} \in \varepsilon(X)$. Since $m_{F,\alpha} = M_{F,\alpha}$, so $m_{F,\alpha} \in \varepsilon(X)$ as needed.

($\Rightarrow$) To simplify notation, we let $m = m_{F,\alpha}$ and $M = M_{F,\alpha}$. Since $\varepsilon(X) \subseteq E(X)$, $m \in E(X)$ is clear. For any $y \in F$, by m a min function,

$$m(y) = \min_{y' \in F} d(y, y') + \alpha(y') \leq \alpha(y).$$

Fix any $x \in X$. By m Katětov,

$$\begin{aligned} m(x) &\geq \max_{y \in F} d(x, y) - m(y) \\ &\geq \max_{y \in F} d(x, y) - \alpha(y) \\ &= M(x). \end{aligned}$$

Since m is extremal, there exists some $(y_i)_i$ such that $(y_i)_i \underset{m}{\doteq} x$. Since F is finite, by definition of min function, we may assume that for a fixed $y \in F$, $m(y_i) = d(y_i, y) + \alpha(y)$ for all i. Fix $y' \in F$ such that $m(x) = d(x, y') + \alpha(y')$.

Then for all i,

$$\begin{aligned}
d(x, y_i) &\le d(x, y') + d(y', y) + d(y, y_i) \\
&\le d(x, y') + m(y') + m(y) + d(y, y_i) \qquad\qquad (\text{by } m \in E(X)) \\
&\le d(x, y') + \alpha(y') + \alpha(y) + d(y, y_i) \\
&= m(x) + m(y_i).
\end{aligned}$$

Since $m(x) + m(y_i) - d(x, y_i) \to 0$, we must have $d(x, y) = d(x, y') + d(y', y)$ and $\alpha(y') + \alpha(y) = d(y', y)$. Thus,

$$\begin{aligned}
m(x) &= d(x, y') + m(y') \\
&= d(x, y') + d(y', y) - \alpha(y) \\
&= d(x, y) - \alpha(y) \\
&\le M(x).
\end{aligned}$$

So $m(x) = M(x)$ as needed. □

Proof. (of Proposition 2.6) Statement 1: We need to show that if $m_{F,\alpha} \in E(X)$, then for all $x \in X$, $m_{F,\alpha}(x) = \min_{y \in F} d(x, y) + m_{F,\alpha}(y)$. Consider $m_{G,\alpha'}$ where

$$G = \{y \in F : m_{F,\alpha}(y) = \alpha(y)\}$$

and α' is α restricted to G. Notice that $m_{G,\alpha'} = \alpha'$ on G, so it follows directly from the definition of min function that for all $x \in X$,

$$m_{G,\alpha'}(x) = \min_{y \in G} d(x, y) + m_{G,\alpha'}(y).$$

Then since $F \supseteq G$ and supersets of supports are also supports, it is enough to show that $m_{G,\alpha'} = m_{F,\alpha}$.

Let $x \in X$ be given. If $m_{F,\alpha}(x) = d(x, y) - \alpha(y)$ for some $y \in G$, then $m_{F,\alpha}(x) = m_{G,\alpha'}(x)$ as needed. Suppose not. Then $m_{F,\alpha}(x) = d(x, y) - \alpha(y)$ for some $y \in F \setminus G$. Then by definition of G, $m_{F,\alpha}(y) \neq \alpha(y)$. So there is some $y' \in F$ such that

$$m_{F,\alpha}(y) = d(y, y') + \alpha(y') < \alpha(y).$$

But then

$$d(x, y') + \alpha(y') < d(x, y') + \alpha(y) - d(y, y') \leq d(x, y) + \alpha(y),$$

so $m_{F,\alpha}(x) < d(x, y) + \alpha(y)$. Contradiction. Thus, $m_{F,\alpha}(x) = m_{G,\alpha'}(x)$ as needed.

Statement 2 follows from Lemma 2.10. Then Statements 1 and 2 of Proposition 2.6 give us

$$m(X, n) \cap E(X) \subseteq E(X, n)$$

and

$$M(X, n) \cap \varepsilon(X) \subseteq \varepsilon(X, n)$$

for all n. Since the opposite containments follow directly from the definitions, we have the equalities as needed. $\square$

CHAPTER 3

SEPARABILITY OF THE SPACE OF EXTREMAL FUNCTIONS

Although we do not know exactly when $\varepsilon(X)$ is separable for general metric spaces, we do have the following starting points from Isbell [1]:

1. the map $e : X \to \varepsilon(X)$ defined by $e(x)(y) = d(x, y)$, for all $x, y \in X$, is an isometric embedding; and

2. if X is compact then $\varepsilon(X)$ is compact.

The first statement gives us $X \subseteq \varepsilon(X)$. Thus, we want to focus on metric spaces which are separable, otherwise the injective envelope would be automatically non-separable. However, if the space is not only separable, but compact, then the injective envelope is separable by the second statement. So we want to consider metric spaces that are somewhere between compact and non-separable. We will focus on two particular classes of metric spaces, the proper spaces and the bounded metric spaces. Each of these classes contains spaces that have separable injective envelopes and spaces that have non-separable injective envelopes. Notice that the intersection of these two classes, the bounded proper spaces, are exactly the compact metric spaces.

In the class of proper spaces, we will consider subspaces of finite dimensional real Banach spaces. Lacey and Cohen [6] characterized the injective envelopes for real and complex Banach spaces of any dimension, which also answers the question of separability of the injective envelope for these spaces. However, their methods only apply to Banach spaces, not to subspaces such as $\mathbb{Z}^n$ in $\mathbb{R}^n$. In Section 3.2, we show exactly which norms on $\mathbb{R}^n$ will produce injective envelopes that are separable. Our method also applies to the non-Banach subspace $\mathbb{Z}^n$ of $\mathbb{R}^n$ since we view the injective envelope as the space of extremal functions.

From the discussion at the beginning of Chapter 2, we know that $\varepsilon(X) \subseteq E(X)$. Melleray [5] characterized Polish metric spaces X for which $E(X)$ is separable. One of the conditions Melleray proved to be equivalent to $E(X)$ being separable is that every infinite sequence in X admits what he called an *inline* subsequence. For a bounded Polish metric space, if it is not compact, then it contains an infinite sequence that does not admit an inline subsequence, implying its space of Katětov maps is non-separable. Hence, considering the space $E(X)$ for a non-compact bounded metric space X does not give us any information about separability of $\varepsilon(X)$. We consider the question of separability of the space of extremal functions over bounded metric spaces in Section 3.3. The main result there, Theorem 3.33, states that the existence of a certain sequence of ordered pairs in a bounded metric space is sufficient for the injective envelope to be non-separable, and that for a particular kind of bounded metric space, the existence of such a sequence is also necessary for the injective envelope to be separable.

In [5], Melleray also proved that $E(X)$ separable is equivalent to $E(X, \omega)$ dense in $E(X)$. This leads us to wonder if there is a subspace of $\varepsilon(X)$ which behaves similarly, i.e., a subspace of $\varepsilon(X)$ which is dense in $\varepsilon(X)$ if and only if $\varepsilon(X)$ is separable. We consider several natural subspaces, including the space $\varepsilon(X, \omega)$ defined in Chapter 2. Unfortunately, none of these spaces are as desired, and Section 3.4 contains this discussion and the counterexamples.

We start in Section 3.1 with some general properties of $\varepsilon(X)$ in terms of preservation of separability. These properties will be used in the proofs in Section 3.2 and Section 3.3.

3.1 General Facts Regarding Preservation of Separability of the Space of Extremal Functions

Here we prove some general facts about when separability of $\varepsilon(X)$ is preserved. The first is that for a metric space X, $\varepsilon(X)$ is isomorphic to $\varepsilon(Y)$ (as metric spaces) for any dense subspace Y of X. Next, we show that adjoining a compact space to a metric space does not change separability of the injective envelope. Lastly, we show that if a sequence of metrics $(d_i)_i$ on the same underlying set X converges uniformly (as functions on X^2) to a metric d on X and $\varepsilon(X, d_i)$ are separable for all i, then $\varepsilon(X, d)$ is also separable.

First, we establish an extension property of the extremal functions. This will be used heavily in proofs throughout this section and in the rest of Chapter 3.

Lemma 3.1. *Any extremal function on a subspace of X extends to an extremal function on all of X.*

Proof. Let $Y \subseteq X$ be any subspace and let $f \in \varepsilon(Y)$ be given. We use the following two facts:

- Any Katětov map on a subspace of X can be extended to a Katětov map on X (Melleray [5]).

- There is an extremal function below any function satisfying (1.1) for all $x, y \in X$ (Isbell [1]).

These facts allow us to first extend f to some $f' \in E(X)$ and then take $f'' \in \varepsilon(X)$ below f'. It remains to show that f'' extends f. We have

$$f = f' \upharpoonright Y \geq f'' \upharpoonright Y$$

by choice of f' and f''. On the other hand, by the definition of being extremal, f'' cannot be strictly less than f at any $y \in Y$ and still satisfy (1.1) for all $x, y \in Y$. So $f = f'' \upharpoonright Y$ as needed. □

Proposition 3.2. *For a metric space* (X, d) *and any dense subspace* Y *of* X, $\varepsilon(X)$ *and* $\varepsilon(Y)$ *are isomorphic metric spaces.*

Proof. For any $g \in \varepsilon(X)$, consider its restriction $g \upharpoonright Y$. We claim that this restriction map is an isometry from $\varepsilon(X)$ onto $\varepsilon(Y)$.

First, we show that for all $g \in \varepsilon(X)$, if $f = g \upharpoonright Y$ then $f \in \varepsilon(Y)$. Fix any $x \in Y$, we need to show that there exists a sequence $(y_i)_i$ in Y such that $(y_i)_i \underset{f}{\frown} x$ (recall Definition 2.11). Since $g \in \varepsilon(X)$, we may fix $(x_i)_i$ in X such that $(x_i)_i \underset{g}{\frown} x$. Since Y is dense in X, then for each i we may fix $y_i \in Y$ such that $d(x_i, y_i) < 2^{-i}$.

Note that for all i,

$$\begin{aligned} |(d(x,y_i)-f(y_i))-(d(x,x_i)-g(x_i))| &= |(d(x,y_i)-g(y_i))-(d(x,x_i)-g(x_i))| \\ &= |(d(x,y_i)-d(x,x_i))+(g(x_i)-g(y_i))| \\ &\leq |(d(x,y_i)-d(x,x_i))|+|(g(x_i)-g(y_i))| \\ &\leq |(d(x,y_i)-d(x,x_i))|+d(x_i,y_i). \end{aligned}$$

By our choice of y_i, the last expression above converges to 0 as i goes to infinity. Hence, $(d(x,y_i)-f(y_i))_i$ converges to the same limit as $(d(x,x_i)-g(x_i))_i$, which is $g(x)=f(x)$. So $(y_i)_i \underset{f}{\simeq} x$ as needed, and $g \mapsto g \restriction Y$ is indeed a map from $\varepsilon(X)$ to $\varepsilon(Y)$.

By Lemma 3.1, for all $f \in \varepsilon(Y)$, there must be $g \in \varepsilon(X)$ such that $g \restriction Y = f$. Thus, the restriction map is surjective. It remains to show that the restriction map is injective and isometric. Both of these will follow from the following: for any $g, g' \in \varepsilon(X)$ if $f = g \restriction Y$ and $f' = g' \restriction Y$ then $d_E(g,g') = d_E(f,f')$. Since f and f' are restrictions of g and g', respectively, it follows that $d_E(g,g') \geq d_E(f,f')$. Thus, it is enough to show that for all $x \in X$, $|g(x)-g'(x)| \leq d_E(f,f')$. Fix any $\delta > 0$. Since Y is dense in X, we may fix $y \in Y$ such that $d(x,y) < \frac{\delta}{2}$. Since extremal functions are 1-Lipschitz, we have

$$\begin{aligned} d_E(f,f') &\geq |f(y)-f'(y)| \\ &= |g(y)-g'(y)| \\ &\geq |g(x)-g'(x)|-2d(x,y) \\ &\geq |g(x)-g'(x)|-\delta. \end{aligned}$$

Since δ can be taken arbitrarily close to 0, $d_E(f,f') \geq |g(x), g'(x)|$ as needed. $\square$

Thus, going down to any dense subset preserves separability of the space of extremal functions. We can also go up to supersets: Adjoining a compact space to any metric space will not change the separability of the space of extremal functions over it.

Proposition 3.3. *For a metric space X and compact space K, if $\varepsilon(X)$ is separable then $\varepsilon(X \cup K)$ is also separable, where the metric on $X \cup K$ is any metric that extends the metric on X and the metric on K.*

We will prove Proposition 3.3 assuming X and K are disjoint, since by Lemma 3.1, if $\varepsilon(X)$ is separable, then so is $\varepsilon(X \setminus K)$, so we may work with $X \setminus K$ in place of X. We first prove the proposition in the case that K is just a point, then finitely many points, and finally for general compact spaces. The first two steps are done in the following lemmas.

Lemma 3.4. *For a metric space X and any point $x_0 \notin X$, if $\varepsilon(X)$ is separable then $\varepsilon(X \cup \{x_0\})$ is also separable, where the metric on $X \cup \{x_0\}$ is any metric that extends the metric on X.*

Proof. Let d be any metric on $X \cup \{x_0\}$ that extends the metric on X. Since $\varepsilon(X)$ is separable, we may fix $\{f_i : i < \omega\}$ a countable dense subset of $\varepsilon(X)$. For each $r \in \mathbb{R}$ and $i \in \mathbb{N}$, we define f_i^r as follows:

$$f_i^r(x_0) = r,$$

and for all $x \in X$,

$$f_i^r(x) = \max\{f_i(x), d(x, x_0) - r\}.$$

We note that $\{f_i^r : r \in \mathbb{R}, i \in \mathbb{N}\}$ is separable since it contains $\{f_i^q : q \in \mathbb{Q}, i \in \mathbb{N}\}$ as a countable dense subset. We will show that $\varepsilon(X \cup \{x_0\})$ is contained in the closure of $\{f_i^r : r \in \mathbb{R}, i \in \mathbb{N}\}$, so it is separable. For this, the following claim will suffice.

Claim: For any $f \in \varepsilon(X \cup \{x_0\})$ and any $\epsilon > 0$, there is some i and $r \in \mathbb{R}$ such that $d_E(f, f_i^r) \leq \epsilon$.

Proof of Claim: Consider $f \upharpoonright X$. It may not be extremal on X, but it is Katětov, so we may fix $f' \in \varepsilon(X)$ such that f' is below f on X (recall proof of Lemma 3.1). Take f_i such that $d_E(f', f_i) < \epsilon$ and let $r = f(x_0)$. We show that $d_E(f, f_i^r) \leq \epsilon$. By definition, $f(x_0) = r = f_i^r(x_0)$. Take any $x \in X$, we need to show that $|f(x) - f_i^r(x)| \leq \epsilon$. From the definition of f_i^r, we have two cases.

[Case $f_i^r(x) = f_i(x)$] In this case, if $f'(x) = f(x)$, then we are done since we chose f_i such that $|f_i(x) - f'(x)| \leq \epsilon$. If $f'(x) \neq f(x)$, then we claim $f(x) = d(x, x_0) - f(x)$. This is enough since definition of f_i^r and the case assumption gives

$$f_i^r(x) = f_i(x) \geq d(x, x_0) - f(x_0) = f(x),$$

while our choice of f_i and f' gives $f'(x) + \epsilon \geq f_i(x)$ and $f(x) \geq f'(x)$. So

$$f'(x) + \epsilon \geq f_i^r(x) \geq f'(x),$$

and $f_i^r(x)$ is within ϵ of $f(x)$. Now to show $f(x) = d(x, x_0) - f(x)$. Since f' is below

f, we must have $f'(x) < f(x)$. If $f(x) \neq d(x, x_0) + f(x_0)$, then $f \in \varepsilon(X \cup \{x_0\})$ implies there exists $y \in X$ such that $f(x) + f(y) < d(x, y) + (f(x) - f'(x))$. Then $f'(y) \leq f(y)$ gives

$$\begin{aligned} f(x) + f(y) &< d(x, y) + (f(x) - f'(x)) \\ f(y) + f'(x) &< d(x, y) \\ f'(y) + f'(x) &< d(x, y), \end{aligned}$$

which contradicts f' being extremal on X. So we must have $f(x) = d(x, x_0) - f(x_0)$ as needed.

[Case $f_i^r(x) = d(x, x_0) - r$] This means that

$$f_i^r(x) = d(x, x_0) - f(x_0) \geq f_i(x).$$

By f Katětov and choice of f_i, we have

$$f(x) - \epsilon \leq f_i(x) \leq d(x, x_0) - f(x_0) \leq f(x).$$

So $f_i^r(x)$ is within ϵ of $f(x)$.

In either case, $f_i^r(x)$ is within ϵ of $f(x)$, so $d_E(f_i^r, f) \leq \epsilon$ as needed. □

Lemma 3.5. *For a metric space X and finitely many points $x_0, \ldots, x_n \notin X$, $\varepsilon(X)$ is separable if and only if $\varepsilon(X \cup \{x_0, \ldots, x_n\})$ is separable, where the metric on $X \cup \{x_0, \ldots, x_n\}$ is any metric that extends the metric on X.*

Proof. If $\varepsilon(X)$ is non-separable, then $\varepsilon(X \cup \{x_0, \dots, x_n\})$ is non-separable since an extremal function on X extends to an extremal function on $X \cup \{x_0, \dots, x_n\}$. Suppose $\varepsilon(X)$ is separable. Then applying Lemma 3.4 recursively $n+1$ times gives $\varepsilon(X \cup \{x_0, \dots, x_n\})$ is separable. □

Now we can complete the proof of Proposition 3.3.

Proof. (of Proposition 3.3) Let d be any metric on $X \cup K$ that extends the metric on X and the metric on K. Assume that $\varepsilon(X)$ is separable. We will build a countable dense subset of $\varepsilon(X \cup K)$. All subsets of $X \cup K$ in this proof are taken with the inherited metric from d.

For each n, let $F_n \subseteq K$ be a finite subset of K which is 2^{-n} dense in K; i.e., for each $x \in K$ there is $y \in F_n$ such that $d(x, y) < 2^{-n}$. By Lemma 3.5, we know that each $\varepsilon(X \cup F_n)$ is sparable. We fix $A_n \subseteq \varepsilon(X \cup F_n)$ a countable dense subspace, for each n. By Lemma 3.1, each $g \in A_n$ extends to an extremal function over $X \cup K$. We fix $B_n \subseteq \varepsilon(X \cup K)$ such that B_n is countable and for each $g' \in A_n$, there is some $g \in B_n$ such that g extends g'. We claim that $\bigcup_n B_n$ is dense in $\varepsilon(X \cup K)$.

Let $f \in \varepsilon(X \cup K)$ and $\epsilon > 0$ be given. Pick n such that $2^{-n} < \frac{\epsilon}{6}$. Consider $f \restriction X \cup F_n$. It may not be extremal, but it is Katětov, so we may fix $f' \in \varepsilon(X \cup F_n)$ which is below f (recall proof of Lemma 3.1). By choice of A_n and B_n, we may take $g' \in A_n$ such that $d_E(g', f') < \frac{\epsilon}{6}$ and $g \in B_n$ such that g extend g'. We will show that $d_E(f, g) \leq \epsilon$.

First, we show that $f' \geq f - \frac{\epsilon}{3}$. Assume towards a contradiction that for some $x \in X \cup F_n$, $f'(x) < f(x) - \frac{\epsilon}{3}$. By f extremal, we may fix $y \in X \cup K$ such that

$$f(x) + f(y) < d(x, y) + (f(x) - \frac{\epsilon}{3} - f'(x)).$$

By choice of F_n, we may fix $y' \in X \cup F_n$ such that $d(y, y') < \frac{\epsilon}{6}$. Since f is Katětov, $f(y') \leq d(y, y') + f(y)$. Then our choice of y and y' implies

$$\begin{aligned}
f(x) + f(y') &\leq f(x) + f(y) + d(y, y') \\
&< d(x, y) + (f(x) - \frac{\epsilon}{3} - f'(x)) + \frac{\epsilon}{6} \\
&\leq d(x, y') + d(y, y') + f(x) - f'(x) - \frac{\epsilon}{6} \\
&< d(x, y') + f(x) - f'(x) \\
f(y') &< d(x, y') - f'(x).
\end{aligned}$$

On the other hand, f' is below f, so

$$f'(y') \leq f(y') < d(x, y') - f'(x) \Rightarrow f'(x') + f'(y') < d(x, y').$$

Contradiction to f' extremal, so $f'(x) \geq f(x) - \frac{\epsilon}{3}$ as needed.

Now we show $d_E(f, g) \leq \epsilon$. Fix any $x \in X \cup K$. By choice of F_n, we may fix $y \in X \cup F_n$ such that $d(x, y) < \frac{\epsilon}{6}$. Since $y \in X \cup F_n$ and g extends g', we have $g(y) = g'(y)$. Thus, by the triangle inequality,

$$|g(x) - f(x)| \leq |g(x) - g(y)| + |g'(y) - f'(y)| + |f'(y) - f(y)| + |f(y) - f(x)| \quad (3.1)$$

By f and g Katětov, we have $|g(x) - g(y)| \leq d(x, y) < \frac{\epsilon}{6}$ and $|f(y) - f(x)| \leq$

$d(x,y) < \frac{\epsilon}{6}$. By our choice of g', $d_E(f', g') < \frac{\epsilon}{6}$, so $|g'(y) - f'(y)| < \frac{\epsilon}{6}$. Lastly, we showed above that $f' \geq f - \frac{\epsilon}{3}$, so $|f'(y) - f(y)| \leq \frac{\epsilon}{3}$. Thus, the right hand side of (3.1) is bounded above by

$$\frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} < \epsilon.$$

Hence, $|g(x) - f(x)| \leq \epsilon$ as needed. □

Notice that the extremal functions are defined in terms of the metric on the space. So perturbing the metric on X will change the space $\varepsilon(X)$. For metrics converging uniformly, we have preservation of separability of the space of extremal functions.

Definition 3.6. For two metrics d_1 and d_2 on the same space X, we define the *uniform distance* between d_1 and d_2 by

$$|d_1 - d_2| = \sup_{x,y \in X} |d_1(x,y) - d_2(x,y)|.$$

For a sequence of metrics d_i on the same space X, we say that the d_i's *converge uniformly* to a metric d on X if

$$\lim_{i \to \infty} |d - d_i| = 0$$

and denote it by $d_i \to d$.

Proposition 3.7. *Suppose there is a sequence of metrics $(d_i)_i$ on a space X such that $d_i \to d$ for some metric d on X, and each $\varepsilon(X, d_i)$ is separable, then $\varepsilon(X, d)$ is also separable.*

Proof. Since each $\varepsilon(X, d_i)$ is extremal, we may fix for each i a countable set $\{f_n^i : n \in \omega\}$ which is dense in $\varepsilon(X, d_i)$. We claim that $\varepsilon(X, d)$ is contained in the closure of the countable set $\{f_n^i : i \in \omega, n \in \omega\}$ in the ambient space of all real-valued functions on X. This will follow easily from the following lemma, so $\varepsilon(X, d)$ is separable. $\square$

Lemma 3.8. *If d_1 and d_2 are both metrics on X and $|d_1 - d_2| < \epsilon$, then for every $f \in \varepsilon(X, d_1)$ there is $g \in \varepsilon(X, d_2)$ such that $d_E(f, g) \leq 2\epsilon$.*

Proof. Let $f \in \varepsilon(X, d_1)$ be given. Since $|d_1 - d_2| < \epsilon$, then the function g' defined by $g'(x) = f(x) + \frac{\epsilon}{2}$ for all $x \in X$ satisfies

$$\begin{aligned} g'(x) + g'(y) &= f(x) + f(y) + \epsilon \\ &\geq d_1(x, y) + \epsilon \\ &\geq d_2(x, y) \end{aligned}$$

for all $x, y \in X$. Thus, by the fact from Isbell [1] used in the proof of Lemma 3.1, there is some $g \in \varepsilon(X, d_2)$ such that g is below g'. We will show that this g is within 2ϵ of f. Let $x \in X$ be given. By the definition of g' and choice of g, we have that

$$g(x) \leq g'(x) = f(x) + \frac{\epsilon}{2}. \tag{3.2}$$

On the other hand, since f is extremal, we have that for all $\delta > 0$ there is $y \in X$ such that $f(x) + f(y) \leq d_1(x, y) + \delta$. So

$$
\begin{aligned}
g(x) + g(y) &\geq d_2(x, y) && \text{(by } g \text{ extremal)} \\
g(x) + g'(y) &\geq d_1(x, y) - \epsilon && \text{(by } |d_1 - d_2| < \epsilon \text{ and } g \leq g'\text{)} \\
g(x) + f(y) + \frac{\epsilon}{2} &\geq d_1(x, y) - \epsilon && \text{(by (3.2)))} \\
g(x) &\geq d_1(x, y) - f(y) - \frac{3\epsilon}{2} \\
g(x) &\geq f(x) - \delta - \frac{3\epsilon}{2}.
\end{aligned}
$$

Since δ can be chosen arbitrarily small, we have $g(x) \geq f(x) - \frac{3\epsilon}{2}$. Thus, by (3.2) and what we just proved, $d_E(f, g) \leq 2\epsilon$ as needed. □

3.2 Separability of the Space of Extremal Functions over Subspaces of Finite Dimensional Real Banach Spaces

For Banach spaces, separability of the injective envelope is determined by the shape of the unit sphere in the dual, as shown by Lacey and Cohen [6]. Lindenstrauss proved that for a Banach space, being infinite dimensional or finite dimensional and the unit sphere having infinitely many extreme points is equivalent to the existence of a certain infinite sequence in the space (Theorem 7.7 in [9]). It can be shown that the existence of this infinite sequence leads to non-separability of the space of extremal functions, which was pointed out to me by Henson. By Lemma 3.1, we know that if a Banach space has a separable injective envelope, then so will any subspace of it. What about separability of the injective envelope of subspaces of Banach spaces with non-separable injective envelopes? Lacey and Cohen's methods cannot be applied to subspaces which are not Banach since they worked with the injective envelope as a linear space. However, our methods here can be applied to subspaces of $\mathbb{R}^n$ such as $\mathbb{Z}^n$ because we work with the injective envelope as the space of extremal functions.

As usual, the metric d induced by N is given by $d(x, y) = N(x - y)$ for all $x, y \in \mathbb{R}^n$. If A, B are sets in $\mathbb{R}^n$, then the distance between A and B is given by

$$d(A, B) = \inf \{d(a, b) : a \in A, b \in B\}.$$

If A is just a single point a, we may write $d(a, B)$ for $d(A, B)$. We will use $\mathbf{0}$ to denote the origin of the Banach space.

Definition 3.9. Given a norm N on $\mathbb{R}^n$, we have the unit sphere

$$S_{n-1} := \{x \in \mathbb{R}^n : N(x) = 1\} = \{x \in \mathbb{R}^n : d(\mathbf{0}, x) = 1\}.$$

Definition 3.10. For any sphere S, we say that a point $x \in S$ is *extreme* if x is not contained in the interior of any line segment in S.

Theorem 3.11. *For any n and norm N, $\varepsilon(\mathbb{R}^n, N)$ and $\varepsilon(\mathbb{Z}^n, N)$ are separable if and only if S_{n-1} has finitely many extreme points.*

Furthermore, in the case that S_{n-1} has finitely many extreme points, it is a polytope with and even number, say $2k$, of $(n-1)$-dimensional faces, and there is an isometry between $\varepsilon(\mathbb{R}^n, N)$ and $(\mathbb{R}^k, \|\cdot\|_\infty)$.

The proof of Theorem 3.11 will be given in two parts. To prove separability, we will build an isometry between $\varepsilon(\mathbb{R}^n, N)$ and $(\mathbb{R}^k, \|\cdot\|_\infty)$ where $2k$ is the number of $(n-1)$-dimensional faces of S_{n-1}. This clearly implies separability of $\varepsilon(\mathbb{R}^n, N)$ and hence separability of $\varepsilon(\mathbb{Z}^n, N)$. To prove non-separability, we will first prove a general result which builds uncountably many functions in $\varepsilon(X)$ for any X from a function in $\varepsilon(X)$ satisfying particular conditions on a sequence of ordered pairs in X. Then, we construct such a function in $\varepsilon(\mathbb{Z}^n, N)$ when S_{n-1} has infinitely many extreme points. This gives $\varepsilon(\mathbb{Z}^n, N)$ and $\varepsilon(\mathbb{R}^n, N)$ are non-separable.

For this proof, we need to understand $f \in \varepsilon(\mathbb{R}^n, N)$ geometrically. We also make some definitions and notation that will make the proof more concise.

Definition 3.12. For any $f \in E(\mathbb{R}^n, N)$ and any $x \in \mathbb{R}^n$. The *sphere given by f at x* is

$$S(f, x) := \{z \in \mathbb{R}^n : d(x, z) = f(x)\}.$$

The Katětov condition

$$|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$$

for all $x, y \in \mathbb{R}^n$ means exactly that

$$S(f, x) \cap S(f, y) \neq \emptyset$$

for all $x, y \in \mathbb{R}^n$. If f is extremal, then minimality with respect to

$$d(x, y) \leq f(x) + f(y)$$

for all $x, y \in \mathbb{R}^n$ means exactly that for every $S(f, x)$ there is a sequence $(S(f, y_i))_i$ such that the spheres $S(f, y_i)$ are becoming tangent to $S(f, x)$. We will make this notion precise in Definition 3.17 and Lemma 3.18.

We identify each $f \in E(\mathbb{R}^n, N)$ with the collection of spheres it assigns to each point in $\mathbb{R}^n$. When the unit sphere is a polytope, we will eventually identify each $f \in \varepsilon(\mathbb{R}^n, N)$ with its action on the faces of the unit sphere. This will become the embedding in Theorem 3.11. A first step towards this is to note that any sphere may be obtained from S_{n-1} by a scaling and a translation:

Definition 3.13. For any sphere S with radius r and center x, *the point on* S *corresponding to* z for any $z \in S_{n-1}$ is

$$z(S) := rz + x.$$

When $S = S(f, x)$, we will use $z(f, x)$ in place of $z(S(f, x))$.

The same scaling and translation from Definition 3.13 will make any supporting hyperplane of S_{n-1} into a supporting hyperplane of the corresponding sphere.

Definition 3.14. A hyperplane is said to *support* a convex set S in $\mathbb{R}^n$ if

- S is contained in one of the closed half-spaces of the hyperplane, and

- at least one point of S is on the hyperplane.

Let S be a sphere centered at x with radius r and P be any hyperplane supporting S_{n-1}. *The supporting hyperplane of S corresponding to P* is

$$P(S) := rP + x = \{rz + x : z \in P\}.$$

When $S = S(f, x)$, we will use $P(f, x)$ in place of $P(S(f, x))$.

Notice that $P(S)$ will be parallel to P for any sphere S. In fact, parallel hyperplanes will be used heavily in our proofs in this section. So we establish the following notation:

Definition 3.15. Two hyperplanes are *parallel* if they are translates of each other, and is denoted by $\cdot \parallel \cdot$. A sequence of parallel hyperplanes $(P_i)_i$ *converges* to a parallel hyperplane P if $\lim_{i\to\infty} d(P_i, P) = 0$, and is denoted by $P_i \to P$.

Definition 3.16. For any hyperplane P_0 containing the origin $\mathbf{0}$ in $(\mathbb{R}^n, N)$, we define a map

$$\iota : \{Q : Q \parallel P_0\} \to \mathbb{R}$$

as follows:

1. Let L_0 be the unique line perpendicular to P_0 through $\mathbf{0}$. L_0 will intersect each Q parallel to P_0 at exactly one point.

2. Let $\iota' : L_0 \to \mathbb{R}$ be such that $\iota'(\mathbf{0}) = 0$, and ι' preserves the linear structure of L_0 and $\mathbb{R}$.

3. For each hyperplane Q parallel to P_0, let $\iota(Q) = \iota'(q)$ where $\{q\} = Q \cap L_0$.

For any $P \parallel Q \parallel P_0$, we write $P \leq_\iota Q$ if and only if $\iota(P) \leq \iota(Q)$ in $\mathbb{R}$ with the usual ordering. We write $P <_\iota Q$ for $P \leq_\iota Q$ and $P \neq Q$. If $P \parallel P' \parallel P_0$, we define the closed interval between P and P' to be

$$[P, P'] := \{Q : Q \parallel P_0, P \leq_\iota Q \leq_\iota P'\}.$$

From our definition of ι, it follows that $d(P, Q) = |\iota(P) - \iota(Q)|$ for any $P \parallel Q$, and $P_i \to P$ if and only if $\iota(P_i) \to \iota(P)$ for any sequence $(P_i)_i$ of hyperplanes parallel to P.

Now we can state precisely what the minimality requirement for extremal functions means geometrically:

Definition 3.17. Fix any $f \in \varepsilon(\mathbb{R}^n, N)$ and $x \in \mathbb{R}^n$.

- We say that $z \in S_{n-1}$ is a *tangent point of x (under f)* if there is a sequence $(y_i)_i$ such that $(y_i)_i \underset{f}{\simeq} x$ (see Definition 2.11) and

$$-z(f, y_i) \to z(f, x).$$

- We say that a hyperplane P is a *tangent hyperplane of x (under f)* if P supports S_{n-1} and there is some $z \in P \cap S_{n-1}$ such that z is a tangent point of x.

The phrase "under f" will only be used when we are working with more than one extremal function.

Lemma 3.18. *For every $f \in \varepsilon(\mathbb{R}^n, N)$, every $x \in \mathbb{R}^n$ has a tangent point and a tangent hyperplane. Moreover, if P is tangent hyperplane of x under f, then there is a sequence $(y_i)_i$ such that $-P(f, y_i) \to P(f, x)$.*

Proof. Fix $f \in \varepsilon(\mathbb{R}^n, N)$ and $x \in \mathbb{R}^n$. First, we note that if x has a tangent point, then it has a tangent hyperplane since any point on S_{n-1} is contained in some supporting hyperplane of S_{n-1}. So we only need to show that x has a tangent point.

By the remark from Isbell [1] stated before Definition 2.11, we can always fix $(y_i)_i$ such that $(y_i)_i \underset{f}{\frown} x$. For each i, let $z_i \in S_{n-1}$ be such that $z_i(f, x), -z_i(f, x)$, and $z_i(f, y_i), -z_i(f, y_i)$, are the points where the line determined by x and y_i intersects the spheres $S(f, x)$ and $S(f, y_i)$, respectively. Here, we are using the sphere's symmetry about the origin, which is implied by the norm property $N(z) = N(-z)$. Then $f(x) + f(y_i) - d(x, y_i) \to 0$ implies either

$$d(z_i(f, x), -z_i(f, y_i)) \to 0$$

or

$$d(-z_i(f, x), z_i(f, y_i)) \to 0.$$

Without loss of generality, we assume the former holds. By compactness of the sphere, we may fix a subsequence of $(y_i)_i$ such that the corresponding subsequence of z_i's converge to some $z \in S_{n-1}$. Then $d(z_i(f,x), -z_i(f,y_i)) \to 0$ implies the corresponding subsequence of $-z_i(f,y_i)$'s will converge to $z(f,x)$, so z is a tangent point of x as needed.

Fix P any tangent hyperplane of x under f. By Definition 3.17, there is some $z \in P \cap S_{n-1}$ and $(y_i)_i \underset{f}{\simeq} z$ such that $-z(f,y_i) \to z(f,x)$. Then $P(f,x) \parallel P(f,y_i)$ for all i, $-z_i(f,y_i) \in -P(f,y_i)$ for all i, $z(f,x) \in P(f,y)$, and $-z_i(f,y_i) \to z(f,x)$ imply $-P(f,y_i) \to P(f,x)$ as needed. $\square$

Although in Definition 3.17, the tangent point is from S_{n-1} and the tangent hyperplane supports S_{n-1}, this is enough by the canonical way of obtaining points and supports of any sphere from points and supports of S_{n-1} given in Definition 3.13 and Definition 3.14. In fact, a *complete* set of supporting hyperplanes of S_{n-1} is enough to serve as tangent hyperplanes for all points in $\mathbb{R}^n$:

Definition 3.19. For any norm N on $\mathbb{R}^n$, and any family $\mathcal{H}$ of hyperplanes, we say that $\mathcal{H}$ is a *complete support* for N if every $P \in \mathcal{H}$ is a supporting hyperplane of S_{n-1} and $S_{n-1} \subseteq \bigcup \mathcal{H}$.

Notice that by the proof of Lemma 3.18, if $\mathcal{H}$ is a complete support for N, then every $x \in \mathbb{R}^n$ will have at least one tangent hyperplane in $\mathcal{H}$.

With the above terminology, we are ready to begin the proof of Theorem 3.11 in the case that S_{n-1} has only finitely many extreme points. We first prove some lemmas and then use them to construct the embedding.

Lemma 3.20. *For any $f \in \varepsilon(\mathbb{R}^n, N)$ and any $x \in \mathbb{R}^n$, if P is a tangent hyperplane of x, then*

$$P(f,x) \cap S(f,y) \neq \emptyset$$

for all $y \in \mathbb{R}^n$.

Proof. Let f, x, P be as in the statement of the lemma. By Lemma 3.18, we may fix a sequence $(x_i)_i$ such that $-P(f,x_i) \to P(f,x)$. Notice that $-P \parallel P$, so either $-P \leq_\iota P$ or $P \leq_\iota -P$. Without loss of generality, we may assume $-P \leq_\iota P$, since the $P \leq_\iota -P$ case is symmetric.

Fix any $y \in \mathbb{R}^n$. As f is Katětov, we must have $S(f,x) \cap S(f,y) \neq \emptyset$. Since $S(f,x)$ lies between $P(f,x)$ and $-P(f,x)$, and $S(f,y)$ lies between $P(f,y)$ and $-P(f,y)$, then either

$$P(f,x) \cap S(f,y) \neq \emptyset$$

or

$$P(f,y) \cap S(f,x) \neq \emptyset.$$

If the former holds, then we are done. Assume not, then $P(f,y) \cap S(f,x) \neq \emptyset$ and $P(f,x) \cap S(f,y) = \emptyset$ imply $-P(f,x) \leq_\iota P(f,y) <_\iota P(f,x)$. We also have $-P(f,x_i) \leq_\iota P(f,x)$ for all i, since otherwise we would have

$$P(f,x) <_\iota -P(f,x_i) \leq_\iota P(f,x_i),$$

which implies $S(f,x) \cap S(f,x_i) = \emptyset$, contradiction to f Katětov. By our choice of x_i, $-P(f,x_i) \to P(f,x)$. So we may fix i large enough such that $P(f,y) <_\iota$

$-P(f, x_i) \leq_\iota P(f, x)$. This means

$$S(f, x_i) \cap S(f, x) = \emptyset,$$

contradiction to f Katětov. So we must have had $P(f, x) \cap S(f, y) \neq \emptyset$. $\square$

Lemma 3.20 leads to the following corollary:

Corollary 3.21. *For any $f \in \varepsilon(\mathbb{R}^n, N)$, if $x, y \in \mathbb{R}^n$ has tangent hyperplanes P, Q, respectively, with $P \parallel Q$, then $P(f, x) = Q(f, y)$.*

Proof. Fix $f \in \varepsilon(\mathbb{R}^n, N)$, $x, y \in \mathbb{R}^n$, and P, Q tangent hyperplane of x and y as in the statement of the corollary. Without loss of generality, we may assume $-Q \leq_\iota Q$, since the $Q \leq_\iota -Q$ case is symmetric.

By Lemma 3.18, we may fix a sequence $(y_i)_i$ such that $-Q(f, y_i) \to Q(f, y)$. By Lemma 3.20, $P(f, x) \cap S(f, y) \neq \emptyset$ and $P(f, x) \cap S(f, y_i) \neq \emptyset$ for all i. This means

$$P(f, x) \in [-Q(f, y), Q(f, y)] \cap \left(\bigcap_i [-Q(f, y_i), Q(f, y_i)] \right).$$

Since $-Q(f, y_i) \to Q(f, y)$, the intersection above contains only $Q(f, y)$. So $P(f, x) = Q(f, y)$. $\square$

The following lemma gives us an action of f on supporting hyperplanes of S_{n-1}, for any $f \in \varepsilon(\mathbb{R}^n, N)$.

Lemma 3.22. *For any* $f \in \varepsilon(\mathbb{R}^n, N)$ *and any hyperplane* P *supporting* S_{n-1}*, there is a unique* P' *parallel to* P *satisfying:*

1. *For all* $x \in \mathbb{R}^n$*,* $P' \cap S(f,x) \neq \emptyset$*.*
2. *If* $-P \leq_\iota P$*, then* $\{P'\} = \bigcap_{x\in\mathbb{R}^n} [P', P(f,x)]$*.*
3. *If* $P \leq_\iota -P$*, then* $\{P'\} = \bigcap_{x\in\mathbb{R}^n} [P(f,x), P']$*.*

Proof. First, we note that uniqueness is implied by the conditions P' must satisfy, so we only need to prove the existence of such a P'. We will do so only for the case $-P \leq_\iota P$, since the proof for the case $P \leq_\iota -P$ is symmetric.

Fix any $x, y \in \mathbb{R}^n$, f is Katětov means $S(f,x) \cap S(f,y) \neq \emptyset$. In particular,

$$[-P(f,x), P(f,x)] \cap [-P(f,y), P(f,y)] \neq \emptyset,$$

implying

$$[\iota(-P(f,x)), \iota(P(f,x))] \cap [\iota(-P(f,y)), \iota(P(f,y))] \neq \emptyset,$$

ι from Definition 3.16. By compactness, the intersection of all such intervals in $\mathbb{R}$ is a nonempty closed interval, say $[p, p']$, for some real numbers $p \leq p'$. Let P' be the unique hyperplane parallel to P such that $\iota(P') = p'$. Then clearly P' is as needed. $\square$

Definition 3.23. For any $f \in \varepsilon(\mathbb{R}^n, N)$ and hyperplane P supporting S_{n-1}, we say that f *acts on* P by defining $f(P)$ to be the unique hyperplane P' parallel to P as given by Lemma 3.22.

Note that if x has P as a tangent hyperplane under f, then by Lemma 3.18,

$$\bigcap_{y\in\mathbb{R}^n}[-P(f,y),P(f,y)] = \{P(f,x)\}$$

assuming $-P \leq_\iota P$. If $P \leq_\iota -P$, then the symmetric argument gives the same conclusion. By definition, $f(P)$ and $f(-P)$ intersect $S(f,y)$ for all y. Thus,

$$f(P) \in \bigcap_{y\in\mathbb{R}^n}[-P(f,y),P(f,y)] \Rightarrow f(P) = P(f,x)$$

and

$$f(-P) \in \bigcap_{y\in\mathbb{R}^n}[-P(f,y),P(f,y)] \Rightarrow f(-P) = P(f,x).$$

In particular, $f(P) = f(-P)$.

The next lemma will be used to show that our embedding is isometric.

Lemma 3.24. *For any $f, g \in \varepsilon(\mathbb{R}^n, N)$ and any complete support $\mathcal{H}$,*

$$d_E(f,g) = \sup\{d(f(P),g(P)) : P \in \mathcal{H}\}.$$

Proof. Fix any $f, g \in \varepsilon(\mathbb{R}^n, N)$ and any $x \in \mathbb{R}^n$. Without loss of generality, assume $f(x) \leq g(x)$. By the note after Definition 3.19, there is some $P \in \mathcal{H}$ such that P is a tangent hyperplane of x under g. By the note after Definition 3.23, $g(P) = P(g,x)$. Since $f(P) \cap S(f,x) \neq \emptyset$ by Lemma 3.22, and $g(P)$ supports $S(g,x)$, we must have have $d(f(P),g(P)) \geq g(x) - f(x)$. Thus, for all $x \in \mathbb{R}^n$, there is $P \in \mathcal{H}$ such that $d(f(P), g(P)) \geq g(x) - f(x)$, so

$$d_E(f, g) \leq \sup \{d(f(P), g(P)) : P \in \mathcal{H}\}. \tag{3.3}$$

Fix any $f, g \in \varepsilon(\mathbb{R}^n, N)$ and any $P \in \mathcal{H}$. If $f(P) = g(P)$, then $d_E(f, g) \geq d(f(P), g(P))$. Suppose $f(P) \neq g(P)$. Without loss of generality, we may assume $f(P) <_\iota g(P)$, as the case for $g(P) <_\iota f(P)$ is symmetric. We show $d_E(f, g) \geq d(f(P), g(P))$ by showing $d_E(f, g) \geq d(f(P), g(P)) - \epsilon$ for all $\epsilon > 0$. Fix $\epsilon > 0$. We have the following two cases:

[Case $-P \leq_\iota P$] By Definition 3.23 and Lemma 3.22, we may find $x \in \mathbb{R}^n$ such that $f(P) \leq_\iota P(f, x)) <_\iota g(P)$ with $d(f(P), P(f, x)) < \epsilon$. On the other hand, Definition 3.23 and Lemma 3.22 also give $S(g, x) \cap g(P) \neq \emptyset$, so

$$g(x) \geq f(x) + d(f(P), g(P)) - \epsilon.$$

[Case $P \leq_\iota -P$] By Definition 3.23 and Lemma 3.22, we may find $x \in \mathbb{R}^n$ such that $f(P) <_\iota P(g, x) \leq_\iota g(P)$ with $d(g(P), P(g, x)) < \epsilon$. On the other hand, Definition 3.23 and Lemma 3.22 also give $S(f, x) \cap f(P) \neq \emptyset$, so

$$f(x) \geq g(x) + d(f(P), g(P)) - \epsilon.$$

In either case, $d_E(f,g) \geq d(f(P),g(P)) - \epsilon$ as needed. Since $P \in \mathcal{H}$ and $\epsilon > 0$ were chosen arbitrarily, we have

$$d_E(f,g) \geq \sup\{d(f(P),g(P)) : P \in \mathcal{H}\}. \tag{3.4}$$

Thus, the lemma is proved by (3.3) and (3.4). □

Now we are ready to put everything together to construct the isometry for Theorem 3.11.

Proof. ($\Leftarrow$ direction and "furthermore" part of Theorem 3.11) Assume S_{n-1} only has finitely many extreme points. Since S_{n-1} is convex, it is the convex hull of its set of extreme points, hence a polytope. Observe that $N(z) = N(-z)$ implies S_{n-1} has symmetry about the origin. This means if P is a support of S_{n-1}, then $-P$ is a support of S_{n-1}. By symmetry, S_{n-1} must have an even number of $(n-1)$-dimensional faces and opposite faces are parallel. Let

$$\mathcal{H} := \{P_i, -P_i : 1 \leq i \leq k\}$$

be the hyperplanes containing these $(n-1)$-dimensional faces. Note that $\mathcal{H}$ is a complete support.

Fix any $f \in \varepsilon(\mathbb{R}^n, N)$. We claim that for all i, either P_i or $-P_i$ will be a tangent hyperplane of some $x \in \mathbb{R}^n$. Fix any $1 \leq i \leq k$. By Lemma 3.22, we may fix P' parallel to P_i and $-P_i$ such that for all $x \in \mathbb{R}^n$, $S(f,x) \cap P' \neq \emptyset$. For every $j \neq i$, $f(P_j)$ and $f(-P_j)$ are not parallel to P', so must intersect P'. So for each

$j \neq i$, we may fix points $x_j \in P' \cap f(P_j)$ and $x'_j \in P' \cap f(-P_j)$. For every $x \in \mathbb{R}^n$, let S_x be the sphere centered at x with radius $d(x, P')$. This definition guarantees either $P_i(S_x) = P'$ or $-P_i(S_x) = P'$. Fix x_0 such that $d(x_0, P')$ is large enough so x_j and x'_j are in the interior of the face of S_{x_0} contained in P' for all $j \neq i$. By choice of P', $S(f, x_0) \cap P' \neq \emptyset$ implies

$$f(x_0) \geq d(x_0, P') > d(x_0, f(P_j))$$

for all $j \neq i$. Thus, $P(f, x_0) \neq \pm P_j$ for all $j \neq i$, which means $\pm P_j$ for all $j \neq i$ is not a tangent hyperplane of x_0. Since $\mathcal{H}$ is a complete support, x_0 must have P_i or $-P_i$ as a tangent hyperplane.

By the note after Definition 3.23, we have $f(P_i) = f(-P_i)$ for all i. Then by Lemma 3.24,

$$f \mapsto (f(P_1), \ldots, f(P_k))$$

is an isometric embedding of $\varepsilon(\mathbb{R}^n, N)$ into $(\mathbb{R}^k, \|\cdot\|_\infty)$. It remains to show that this map is onto. Fix any k hyperplanes $Q_1, \ldots, Q_k$ in $\mathbb{R}^n$ such that Q_i is parallel to P_i for $i = 1, \ldots, k$. We need $f \in \varepsilon(\mathbb{R}^k, N)$ such that $f(P_i) = Q_i$ for all $1 \leq i \leq k$. By choosing the points recursively, we may fix distinct points x_i and y_i for $1 \leq i \leq k$ satisfying:

1. The line segment determined by x_i and y_i is perpendicular to Q_i and intersects Q_i.

2. For all $j \neq i$, $d(x_i, Q_i) \geq d(x_i, Q_j)$ and $d(y_i, Q_i) \geq d(y_i, Q_j)$.

3. For all $j < i$, $d(x_i, Q_i) + d(x_j, Q_j) \geq d(x_i, x_j)$ and $d(x_i, Q_i) + d(y_j, Q_j) \geq d(x_i, y_j)$.

4. For all $j < i$, $d(y_i, Q_i) + d(x_j, Q_j) \geq d(y_i, x_j)$ and $d(y_i, Q_i) + d(y_j, Q_j) \geq d(y_i, y_j)$.

Clearly, condition 1 can be satisfied. Condition 2 can be satisfied since for all $i \neq j$, $Q_i \cap Q_j \neq \emptyset$. Condition 3 can be satisfied recursively as follows: by condition 2, for any $j < i$, the sphere centered at x_j with radius $d(x_j, Q_j)$, say S_j, must intersect Q_i. Thus, we may choose x_i far enough from Q_i such that the sphere centered at x_i with radius $d(x_i, Q_i)$ contains a point where S_j intersects Q_i for all $j < i$. Clearly, this x_i will satisfy condition 3. Condition 4 is satisfied similarly. Let f' be the partial function on $\{x_i, y_i : 1 \leq i \leq k\}$ defined by $f'(x_i) = d(x_i, Q_i)$ and $f'(y_i) = d(y_i, Q_i)$ for each $1 \leq i \leq k$. Conditions 1, 3, and 4, guarantee that f' is extremal. By Lemma 3.1, there is $f \in \varepsilon(\mathbb{R}^n, N)$ extending f'. Condition 1 and the fact that Q_i is parallel to P_i guarantees that $S(f, x_i)$ and $S(f, y_i)$ are tangent along Q_i for each $1 \leq i \leq k$. Hence $f(P_i) = Q_i$ for all $1 \leq i \leq k$ as needed. $\square$

Now for the "only if" direction of Theorem 3.11. First, we will show how to use one particular function in $\varepsilon(X)$ to build uncountably many functions in $\varepsilon(X)$ whose distances from each other have a lower bound.

Definition 3.25. Let (X, d) be any metric space. A function $f \in \varepsilon(X)$ is said to be *unconstrained* if there exists a sequence $(x_i, y_i)_i$ in $X \times X$ and $\rho > \delta \geq 0$ such that for all $i \neq j$

$$\begin{aligned} f(x_i) + f(y_i) - d(x_i, y_i) &\leq \delta, \\ f(x_i) + f(y_j) - d(x_i, y_j) &\geq \rho, \text{ and} \\ f(x_i) + f(x_j) - d(x_i, x_j) &\geq 2\rho. \end{aligned}$$

We call ρ and δ the *slack* of f on $(x_i, y_i)_i$.

Lemma 3.26. *Let (X, d) be any metric space. If there exists an unconstrained $f \in \varepsilon(X)$, then $\varepsilon(X)$ is non-separable.*

Proof. Fix f unconstrained with slack ρ and δ on $(x_i, y_i)_i$. For every $\alpha \in 2^\omega$, we will define an $f_\alpha \in \varepsilon(X)$. First, we define a partial function on $\{x_i, y_i : i \in \omega\}$ as follows:

$$f'_\alpha(x_i) = \begin{cases} d(x_i, y_i) - f(y_i) & \text{if } \alpha(i) = 0 \\ f(x_i) - \rho & \text{if } \alpha(i) = 1 \end{cases}$$

and $f'_\alpha(y_i) = d(x_i, y_i) - f'_\alpha(x_i)$. Notice that for all i, $f'_\alpha(y_i) \geq f(y_i)$ and $f'_\alpha(x_i) \geq f(x_i) - \rho$. So f'_α is extremal, since for all $i \neq j$,

$$\begin{aligned} f'_\alpha(x_i) + f'_\alpha(x_j) &\geq (f(x_i) - \rho) + (f(x_j) - \rho) \\ &= f(x_i) + f(x_j) - 2\rho \\ &\geq d(x_i, x_j), \end{aligned}$$

$$\begin{aligned} f'_\alpha(x_i) + f'_\alpha(y_j) &\geq (f(x_i) - \rho) + f(y_j) \\ &= f(x_i) + f(y_j) - \rho \\ &\geq d(x_i, y_j), \end{aligned}$$

$$\begin{aligned} f'_\alpha(y_i) + f'_\alpha(y_j) &\geq f(y_i) + f(y_j) \\ &\geq d(y_i, y_j), \end{aligned}$$

and

$$f'_\alpha(x_i) + f'_\alpha(y_i) = d(x_i, y_i).$$

By Lemma 3.1, for each α, we may fix an extremal function f_α extending f'_α to all of X. Notice that f_α must agree with f'_α on the x_i and y_i's since f'_α was already extremal.

Now, if $\alpha \neq \beta \in 2^\omega$, then without loss of generality, we may fix i such that $\alpha(i) = 0$ and $\beta(i) = 1$. Then $f_\alpha(x_i) = d(x_i, y_i) - f(y_i)$ and $f_\beta(x_i) = f(x_i) - \rho$. Hence

$$\begin{aligned} d_E(f_\alpha, f_\beta) &\geq f_\alpha(x_i) - f_\beta(x_i) \\ &= (d(x_i, y_i) - f(y_i)) - (f(x_i) - \rho) \\ &= d(x_i, y_i) - (f(x_i) + f(y_i)) + \rho \\ &\geq \rho - \delta > 0 \end{aligned}$$

Thus, $\{f_\alpha : \alpha \in 2^\omega\} \subseteq \varepsilon(X)$ witness that $\varepsilon(X)$ is non-separable. □

The following technical lemma will allow us to make the recursive step in building (x_i, y_i) in $\mathbb{R}^n$ to show non-separability of $\varepsilon(\mathbb{R}^n, N)$. In what follows, we will use $\widehat{x} = \frac{x}{N(x)}$ to denote the normalized x.

Lemma 3.27. *Let $z \neq z'$ be any two points on S_{n-1} which are not antipodal. For any $0 < \lambda < 1$, let L_λ be the line determined by z' and $(\widehat{\lambda z + (1-\lambda)z'})$ (since z and z' are not antipodal, $(\widehat{\lambda z + (1-\lambda)z'}) \neq z'$ for all $0 < \lambda < 1$). For any $R > 0$, let $y_R = z + Rz'$. Then for any $0 < \lambda < 1$ there is some R_λ such that for all*

$R \geq R_\lambda$,

$$1 + R - N(y_R) \geq d(z, L_\lambda).$$

Proof. We work on the 2-dimensional subspace of $(\mathbb{R}^n, N)$ spanned by z and z', i.e., the subspace given by all points of the form $rz + r'z'$ for some $r, r' \in \mathbb{R}$. Throughout this proof, when we say a sphere S, we mean the intersection of S with this subspace. All the other points and lines used in this proof are contained in this subspace.

Fix any $0 < \lambda < 1$. Let $x = \lambda z + (1-\lambda)z'$ and let

$$L = L_\lambda = \{z' + r(z' - \widehat{x}) : r \in \mathbb{R}\}.$$

Let L' be the unique line parallel to L and containing z, i.e.,

$$L' = L + (z - z').$$

For each $R > 0$, let S_R be the sphere centered at y_R with radius R and let $z_R = \widehat{y_R}$. Note that the line determined by $\mathbf{0}$ and y_R intersects S_{n-1} at z_R, and by symmetry, intersects S_R at $-z_R(S_R)$. By definition of y_R, $z \in S_R$, so $S_R \cap S_{n-1} \neq \emptyset$, and

$$d(\mathbf{0}, z_R) = d(\mathbf{0}, -z_R(S_R)) + d(-z_R(S_R), z_R). \tag{3.5}$$

By our definition of $-z_R(S_R)$, either $-z_R(S_R)$ is between $\mathbf{0}$ and y_R or $\mathbf{0}$ is between $-z_R(S_R)$ and y_R. In the former case,

$$d(\mathbf{0}, y_R) = d(\mathbf{0}, -z_R(S_R)) + d(y_R, -z_R(S_R)), \tag{3.6}$$

so

$$\begin{aligned}
1 + R - N(y_R) &= d(\mathbf{0}, z_R) + d(y_R, -z_R(S_R)) - d(\mathbf{0}, y_R) \\
&= d(\mathbf{0}, -z_R(S_R)) + d(-z_R(S_R), z_R) + d(y_R, -z_R(S_R)) - d(\mathbf{0}, y_R) && \text{(by (3.5))} \\
&= d(\mathbf{0}, y_R) + d(z_R, -z_R(S_R)) - d(\mathbf{0}, y_R) && \text{(by (3.6))} \\
&= d(z_R, -z_R(S_R)).
\end{aligned}$$

In the latter case, $d(-z_R(S_R), y_R) = d(-z_R(S_R), \mathbf{0}) + d(\mathbf{0}, y_R)$ implies

$$d(\mathbf{0}, y_R) = d(-z_R(S_R), y_R) - d(-z_R(S_R), \mathbf{0}), \tag{3.7}$$

so

$$\begin{aligned}
1 + R - N(y_R) &= d(\mathbf{0}, z_R) + d(-z_R(S_R), y_R) - d(\mathbf{0}, y_R) \\
&= d(\mathbf{0}, -z_R(S_R)) + d(-z_R(S_R), z_R) + d(-z_R(S_R), y_R) - d(\mathbf{0}, y_R) && \text{(by (3.5))} \\
&= d(z_R, -z_R(S_R)) + 2d(\mathbf{0}, -z_R(S_R)) && \text{(by (3.7))} \\
&\geq d(z_R, -z_R(S_R)).
\end{aligned}$$

Hence, in either case, it is enough to show that there is R_λ such that for all $R \geq R_\lambda$, the points z_R and $-z_R(S_R)$ are not both contained in the same open half-plane bounded by L and not both contained in the same open half-plane bounded by L', in particular,

$$d(z_R, -z_R(S_R)) \geq d(L, L') = d(z, L).$$

For the remainder of this proof, when I say a line l *separates* a and b, I mean that a and b are not contained in the same open half-plane bounded by l, where a and

b can be points or lines parallel to l. Notice that l separates a and b includes the situation when one or both of a and b are on l. Thus, we want to show that there is R_λ such that for all $R \geq R_\lambda$, z_R and $-z_R(S_R)$ are separated by L and L'.

First, note that if $R \geq 2$, then

$$N(y_R) = N(z + Rz') \geq N(Rz') - N(z) = R - 1 \geq 1.$$

Let $R_0 = \frac{1-\lambda}{\lambda}$. We claim that $R_\lambda = \max(2, R_0)$ is as needed. Take any $R \geq R_\lambda$. By choice of R_0,

$$y_{R_0} = z + \frac{1-\lambda}{\lambda} z' = \frac{1}{\lambda}(\lambda z + (1-\lambda)z') = \frac{1}{\lambda} x,$$

so $\widehat{x} = \widehat{y_{R_0}} = z_{R_0}$. Then $R \geq R_\lambda \geq R_0$ implies $\frac{1}{R+1} \leq \lambda$, and

$$y_R = z + Rz' = (R+1)(\frac{1}{R+1} z + (1 - \frac{1}{R+1})z')$$

implies $z_R = \widehat{y_R}$ is on the minor arc of S_{n-1} bounded by z' and $\widehat{x}$. By construction, z is on the major arc of the unit sphere bounded by z' and $\widehat{x}$. Notice that L intersects S_{n-1} at exactly z' and $\widehat{x}$, so convexity of the sphere implies z_R and z are separated by L. Since $z \in L'$ and $L' \parallel L$, then z_R and L' are separated by L. Thus, it is enough to show that $-z_R(S_R)$ and L are separated by L', since this would imply $-z_R(S_R)$ and z_R are separated by both L and L'.

First, we show L' intersects S_R at exactly $-z'(S_R)$ and $-\widehat{x}(S_R)$. By definition, these points are on S_R, so it remains to show that they are on L':

$$\begin{aligned} -z'(S_R) &= R(-z') + y_R \\ &= -Rz' + z + Rz' \\ &= z \\ &\in L', \end{aligned}$$

$$\begin{aligned} -\widehat{x}(S_R) &= R(-\widehat{x}) + y_R \\ &= -R\widehat{x} + z + Rz' \\ &= z' + R(z' - \widehat{x}) + (z - z') \\ &\in L + (z - z') \\ &= L'. \end{aligned}$$

By symmetry, $-z_R(S_R)$ is on the minor arc of S_R bounded by $-z'(S_R)$ and $-\widehat{x}(S_R)$. By construction, L and L' both intersect S_{n-1}, so L separates L' and z_R. This along with $z_R \in S_{n-1}$ implies L also separates L' and rz_R for any $r \geq 1$. Since $R \geq L_\lambda \geq 2$ implies $N(y_R) \geq 1$, so L separates L' and $N(y_R)z_R = N(y_R)\widehat{y_R} = y_R$. This means L intersects the sphere S_R on the major arc bounded by $-z'(S_R)$ and $-\widehat{x}(S_R)$. Then convexity of the sphere implies $-z_R(S_R)$ and L are separated by L' as needed. □

Now we are ready to finish the proof of Theorem 3.11.

Proof. ($\Rightarrow$ direction of Theorem 3.11) We will build recursively a sequence $(x_i, y_i)_i$

in $\mathbb{Z}^n \times \mathbb{Z}^n$ and a function $f \in \varepsilon(\mathbb{Z}^n, N)$ such that

1. $\forall i\ f(x_i) + f(y_i) = d(x_i, y_i)$

2. $\forall i \neq j\ f(x_i) + f(x_j) \geq d(x_i, x_j) + 1$

3. $\forall i \neq j\ f(x_i) + f(y_j) \geq d(x_i, y_j) + 1$.

Notice that f will be unconstrained with slack $\rho = \frac{1}{2}$ and $\delta = 0$ on $(x_i, y_i)_i$. So by Lemma 3.26, $\varepsilon(\mathbb{R}^n, N)$ is non-separable as needed.

Since S_{n-1} has infinitely many extreme points and is compact, we may fix a point z_0 which is a cluster point for the set of extreme points of S_{n-1}. Take $(z'_i)_{i>0}$ a sequence of extreme points converging to z_0 and for each $i > 0$, let

$$z''_i = (z'_i + \widehat{2^{-i}(z'_i} - z_0)).$$

Since z'_i is extreme and the sphere is convex, $(z'_i + 2^{-i}(z'_i - z_0)) \notin S_{n-1}$. This implies z''_i is not collinear with z_0 and $(z'_i + 2^{-i}(z'_i - z_0))$, so z'_i is not on the line segment from z_0 to z''_i. On the other hand, by definition of z''_i, there is some point on the line segment from z_0 to z''_i whose normalization equals z'_i. So the line segment from z_0 to z''_i is not contained in S_{n-1}. Of course, the z''_i are not necessarily in $\mathbb{Z}^n$. We note that

$$\mathbf{Z} := \{\widehat{z} : z \in \mathbb{Z}^n\}$$

is dense in S_{n-1}. Thus, for each $i > 0$ we may fix $z_i \in \mathbf{Z}$ such that $d(z_i, z''_i) < 2^{-i}$, and the line segment from z_0 to z_i is not contained in S_{n-1}. Then the sequence $(z_i)_{i>0}$ is in $\mathbf{Z} \subseteq S_{n-1}$ and converging to z_0.

We define $L_\lambda(i,j)$ to be the line determined by z_j and $(\widehat{\lambda z_i + (1-\lambda) z_j})$ for each $i \neq j$ and $0 < \lambda < 1$. For each $i \neq j$, note that $d(z_i, L_\lambda(i,j)) \leq 2$, so we may define $D(i,j)$ to be

$$D(i,j) = \sup_{0<\lambda<1} d(z_i, L_\lambda(i,j)).$$

In fact, it can be shown that $D(i,j) = \lim_{\lambda \to 0} d(z_i, L_\lambda(i,j))$, but this is not needed for the proof. By choice of z_i, the line segment from z_0 to z_i is not contained in S_{n-1}, which implies $D(i,0) > 0$ for all $i > 0$. For any fixed i, $z_j \to z_0$ implies

$$\lim_{j\to\infty} D(i,j) = D(i,0) > 0.$$

So by going to a subsequence, we may assume $(z_i)_i$ satisfies

1. for any fixed i, $\inf_{j>i} D(i,j) > 0$,
2. for all $i \neq j$, z_i and z_j are not antipodal, and
3. for all $i \neq j$, $d(z_i, z_j) \leq 1$.

We will show that there exists a sequence of increasing positive real numbers (r_i) such that (x_i, y_i) and f defined as follows are as needed:

- $x_i = r_i z_i$,
- $y_i = -r_i z_i$, and
- $f(x_i) = f(y_i) = \frac{d(x_i,y_i)}{2} = r_i$.

These definitions translate conditions $1-3$ on f directly into conditions on the r_i's. Clearly, f will satisfy condition 1 just from its definition. So, as we define the r_i's recursively, we only need to guarantee conditions 2 and 3.

We construct the r_i's recursively. At each stage i, we choose r_i so that

- $r_i \geq 1$,
- $r_i D(i,j) > 1 \ \forall j > i$,
- $r_j + r_i - d(-r_j z_j, r_i z_i) \geq 1 \ \forall j < i$, and
- $r_i z_i \in \mathbb{Z}^n$.

The first condition can be satisfied trivially. Since we assumed $\inf_{j>i} D(i,j) > 0$, then the second condition can be satisfied for all large enough r_i. The last condition can be fulfill along with the other conditions since for every $z \in \mathbf{Z}$, the set

$$\{r \in \mathbb{R} : rz \in \mathbb{Z}^n\}$$

is unbounded. So it remains to satisfy the third condition.

We satisfy the third condition by applying Lemma 3.27. For all $j < i$, by condition 2 on $(z_i)_i$, z_i and z_j are not antipodal. By the second condition of the recursive construction, $r_j D(j,i) > 1$ for all $j < i$. So we may choose $\epsilon_0 > 0$ small enough such that $r_j D(j,i) - r_j \epsilon_0 \geq 1$ for all $j < i$. By definition of $D(j,i)$, there is $L_\lambda(j,i)$ such that $d(z_j, L_\lambda(j,i)) \geq D(j,i) - \epsilon_0$. Applying Lemma 3.27 to $z = z_j$, $z' = z_i$, and $L_\lambda = L_\lambda(j,i)$, we obtain R_j such that for all $R \geq R_j$

$$1 + R - N(z_j + Rz_i) \geq d(z_j, L_\lambda(j,i)) \geq D(j,i) - \epsilon_0. \tag{3.8}$$

Multiplying (3.8) by r_j gives for all $R \geq R_j$,

$$r_j + r_j R - N(r_j z_j + r_j R z_i) \geq r_j D(j,i) - r_j \epsilon_0 \geq 1. \tag{3.9}$$

Now take any $R \geq (\max_{j<i} r_j)(\max_{j<i} R_j)$. For each $j < i$, we have $\frac{R}{r_j} \geq R_j$, so (3.9) gives that

$$\begin{aligned}
r_j + r_j \frac{R}{r_j} - N(r_j z_j + r_j \frac{R}{r_j} z_i) &\geq 1 \\
r_j + R - N(r_j z_j + R z_i) &\geq 1 \\
r_j + R - d(\mathbf{0}, r_j z_j + R z_i) &\geq 1 \\
r_j + R - d(-r_j z_j, R z_i) &\geq 1.
\end{aligned}$$

Hence, if $r_i \geq (\max_{j<i} r_j)(\max_{j<i} R_j)$, then r_i satisfies the third recursive condition.

With the r_i's build as above, we show that the $(x_i, y_i)_i$ and f we defined from them are as needed. Condition 1 is satisfied by construction. Consider condition 2: for all $i \neq j$, $r_i + r_j \geq d(x_i, x_j) + 1$. By the first condition of the recursive construction, $r_i \geq 1$ for all i. By condition 3 on $(z_i)_i$, $d(z_i, z_j) \leq 1$. So

$$\begin{aligned}
d(x_i, x_j) &= d(r_i z_i, r_j z_j) \\
&\leq d(r_j z_j, r_i z_j) + d(r_i z_j, r_i z_i) && \text{(by triangle inequality)} \\
&= (r_j - r_i) N(z_j) + r_i d(z_j, z_i) && \text{(by linearity of norm)} \\
&\leq (r_j - r_i) + r_i && \text{(by } d(z_i, z_j) \leq 1\text{)} \\
&\leq r_j + r_i - 1 && \text{(by } r_i \geq 1\text{)}
\end{aligned}$$

as needed. To satisfy condition 3, we need to have for all $i \neq j$,

$$r_i + r_j \geq d(x_i, y_j) + 1. \tag{3.10}$$

Notice that by symmetry of our construction, we have $d(x_i, y_j) = d(y_i, x_j)$ for all

i, j. So it is enough if we satisfy (3.10) for all $j < i$. Fix any $j < i$, this is exactly the third condition of the recursive construction of r_i:

$$r_i + r_j - d(x_i, y_j) = r_j + r_i - d(-r_j z_j, r_i z_i) \geq 1.$$

Lastly, $x_i = r_i z_i \in \mathbb{Z}^n$ implies $y_i = -x_i \in \mathbb{Z}^n$. The theorem is proved as we have produced $(r_i)_i$, and hence $(x_i, y_i)_i$ and f, as stated. $\square$

Melleray [5] proved that $E(X)$ is separable if and only if every sequence in X admits what he called an *inline* subsequence. By a result of Lindenstrauss (Theorem 7.7 in [9]), we know that every sequence in $(\mathbb{R}^n, N)$ will admit such an inline subsequence whenever the unit sphere given by N is a polytope. On the other hand, $\varepsilon(\mathbb{R}^n, N) \subseteq E(\mathbb{R}^n, N)$ gives us $E(\mathbb{R}^n, N)$ is non-separable whenever S_{n-1} is not a polytope by Theorem 3.11. So we arrive at the following Theorem:

Theorem 3.28. *For any n and norm N, $E(\mathbb{R}^n, N)$ is separable if and only if S_{n-1} has finitely many extreme points.*

Corollary 3.29. *For any n and norm N, the following are equivalent:*

1. *$\varepsilon(\mathbb{Z}^n, N)$ is separable.*

2. *$\varepsilon(\mathbb{R}^n, N)$ is separable.*

3. *$E(\mathbb{Z}^n, N)$ is separable.*

4. $E(\mathbb{R}^n, N)$ is separable.

Proof. Equivalence of 1 and 2 follows from Theorem 3.11, and equivalence of 2 and 4 follows from Theorem 3.11 and Theorem 3.28. So 1, 2, and 4 are all equivalent. Since $\varepsilon(\mathbb{Z}^n, N) \subseteq E(\mathbb{Z}^n, N)$, then 3 implies 1 is obvious. On the other hand, Katětov functions can always be extended (recall proof of Lemma 3.1), so we also have 4 implies 3. $\square$

From Theorem 3.11, we know that $\varepsilon(\mathbb{R}^n, \|\cdot\|_1)$ is separable, while $\varepsilon(\mathbb{R}^n, \|\cdot\|_p)$ is non-separable for any $1 < p < \infty$ and $n \geq 2$. However, applying the $\|\cdot\|_1$ norm to an infinite dimensional space makes the space of extremal functions non-separable. In fact, Lacy and Cohen [6] showed that all infinite dimensional Banach spaces have non-separable injective envelopes. Recall that our proof of non-separability in Theorem 3.11 was based on the existence of an unconstrained function. This leads us to ask:

Question 1. *Does every metric space, or every Banach space, with non-separable injective envelope have an unconstrained extremal function?*

The following example shows this is true for the infinite dimensional Banach space ℓ^1. In Section 3.3, we will see that this is also true for a certain class of bounded metric spaces. However, the question is still open in general.

Example 3.30. *The Banach space ℓ^1 has an unconstrained extremal function over it.*

Proof. We will construct a sequence $(x_n, y_n)_n$ in ℓ^1 and an unconstrained function $f \in \varepsilon(\ell^1)$ with slack $\rho = \frac{1}{2}$ and $\delta = 0$ on (x_n, y_n). Then Lemma 3.26 gives $\varepsilon(\ell^1)$ is non-separable as needed.

We fix e_n for all $n \geq 2$ to be

$$e_n = (\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^{n-1}}, \frac{-1}{2^n}, \frac{1}{2^{n+1}}, \ldots).$$

We will let $x_n = 2^{n-2}e_n$, $y_n = -2^{n-2}e_n$, and $f(x_n) = f(y_n) = 2^{n-2}$. Note that each e_n has norm 1, so

$$f(x_n) + f(y_n) = d(x_n, y_n)$$

for all $n \geq 2$. Thus, it remains to show that

1. $2^{m-2} + 2^{n-2} \geq d(x_n, x_m) + 1$; and
2. $2^{m-2} + 2^{n-2} \geq d(x_n, y_m) + \frac{1}{2}$.

for all $n \neq m$.

Take any $2 \leq m < n$. The first inequality is satisfied:

$$\begin{aligned} d(x_m, x_n) &= (2^{n-2} - 2^{m-2})(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{m-1}} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^{n+1}} + \cdots) \\ &\quad + (2^{n-2} + 2^{m-2})(\frac{1}{2^m} + \frac{1}{2^n}) \\ &= 2^{n-2} + 2^{m-2} - 2^{m-1}(1 - (\frac{1}{2^m} + \frac{1}{2^n})) \\ &\leq 2^{n-2} + 2^{m-2} - 1. \end{aligned}$$

The second inequality is also satisfied:

$$\begin{aligned} d(x_m, y_n) &= (2^{m-2} + 2^{n-2})(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{m-1}} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^{n+1}} + \cdots) \\ &\quad +(2^{n-2} - 2^{m-2})(\frac{1}{2^m} + \frac{1}{2^n}) \\ &= (2^{m-2} + 2^{n-2}) - 2^{m-1}(\frac{1}{2^m} + \frac{1}{2^n}) \\ &\leq (2^{m-2} + 2^{n-2}) - \frac{1}{2}. \end{aligned}$$

The case for $d(x_n, y_m)$ is symmetric to the above. □

3.3 Separability of the Space of Extremal Functions over Bounded Metric Spaces

Now we turn our attention to metric spaces with bounded metric. Recall in Lemma 3.26, we proved that a metric space has a non-separable space of extremal functions if there exist an unconstrained extremal function on the space. For bounded metric spaces, we will show that the existence of an unconstrained extremal function is equivalent to the existence of a sequence of ordered pairs on which the distances satisfy certain inequalities. We also prove that that every sequence of ordered pairs in a bounded metric space admits a subsequence with distances converging to constants. In fact, this is true of sequences of n-tuples for any n. Lastly, we show that for bounded metric spaces whose points can be arranged into a sequence of n-tuples with distances converging to constants, the converse of Lemma 3.26 also holds. Before we can state the main theorem of this section, we need to establish some terminology.

Definition 3.31. A sequence of ordered pairs $(x_i, y_i)_i$ in a bounded metric space

(X,d) is said to be *slack* if

$$\begin{aligned}
\liminf_i d(x_i,y_i) &> \frac{1}{2}(\limsup_{i\neq j} d(x_i,x_j) + \limsup_{i\neq j} d(y_i,y_j)), \text{ and} \\
\liminf_i d(x_i,y_i) &> \limsup_{i\neq j} d(x_i,y_j).
\end{aligned}$$

Note that since the metric d is bounded, for every sequence of pairs in X, the lim inf and lim sup's in Definition 3.31 must exist. Thus, every sequence of pairs in X is either slack or fails one of the inequalities in Definition 3.31.

Definition 3.32. Let (X,d) be a metric space. Let $(x_i^1, x_i^2, \ldots, x_i^n)_i$ be any sequence of n-tuples from X. For any $\epsilon > 0$, we say that $(x_i^1, x_i^2, \ldots, x_i^n)_i$ is ϵ-*constant* if there are constants H_k for all $1 \leq k \leq n$ and $D_{k,l}$, $V_{k,l}, V_{l,k}$ for all $1 \leq k < l \leq n$ such that for some N, for all $N \leq i < j$,

$$\begin{aligned}
|d(x_i^k, x_j^k) - H_k| &< \epsilon \\
|d(x_i^k, x_i^l) - D_{k,l}| &< \epsilon \\
|d(x_i^k, x_j^l) - V_{k,l}| &< \epsilon \\
|d(x_i^l, x_j^k) - V_{l,k}| &< \epsilon.
\end{aligned}$$

If $(x_i^1, x_i^2, \ldots, x_i^n)_i$ is ϵ-constant for all $\epsilon > 0$, then we say it is *approximately constant.*

Notice that if a sequence of ordered pairs $(x_i, y_i)_i$ is approximately constant with limiting distances D, V_X, V_Y, H_X, and H_Y, then $(x_i, y_i)_i$ is slack if and only if $D > \frac{1}{2}(H_X + H_Y)$, $D > V_X$, and $D > V_Y$.

Theorem 3.33. *Let (X, d) be a bounded metric space. If $(x_i^1, x_i^2, \ldots, x_i^n)_i$ is an approximately constant sequence in X, then $\varepsilon(\left\{x_i^k : 1 \le k \le n, i < \omega\right\}, d)$ is non-separable if and only if for some $k \neq l$, $(x_i^k, x_i^l)_i$ is slack. Moreover, every sequence of n-tuples in X admits an approximately constant subsequence.*

To prove the "moreover" statement of Theorem 3.33, we will use a diagonal argument. First, we establish a lemma which will be used in the inductive step of our recursive construction.

Lemma 3.34. *Given a bounded metric space (X, d). For every sequence of n-tuples $(x_i^1, x_i^2, \ldots, x_i^n)_i$ in X and $\epsilon > 0$ there is a strictly increasing $\alpha : \mathbb{N} \to \mathbb{N}$ such that $(x_{\alpha(i)}^1, x_{\alpha(i)}^2, \ldots, x_{\alpha(i)}^n)_i$ is ϵ-constant.*

Proof. Let M be a bound for d. Fix $0 < r_1 < r_2 < \ldots < r_m < M$ such that the intervals $I_p = (r_p - \epsilon, r_p + \epsilon)$, for $p = 1, \ldots, m$, cover $[0, M]$. We define $\phi(r) := \min\{r_p : r \in I_p\}$ for all $r \in [0, M]$. Notice that $d(\phi(r), r) < \epsilon$ for all $r \in [0, M]$. So it is enough to find a subsequence such that the relevant corresponding distances all have the same ϕ-value, since these ϕ-values will be constants witnessing that the subsequence is ϵ-constant. We find such a subsequence by applying the Infinite Ramsey Theorem.

Consider the $m^{n\times(n-1)}$-coloring on $\mathbb{N}$ given by

$$i \mapsto (\phi(d(x_i^k, x_i^l)))_{1\leq k<l\leq n},$$

for all $i \in \mathbb{N}$. By the Infinite Ramsey Theorem, there is some infinite $A \subseteq \mathbb{N}$ where the coloring is monochromatic, say of the color $(D_{k,l})_{1\leq k<l\leq n}$. Let $\alpha' : \mathbb{N} \to A$ be the unique strictly increasing onto map.

Next, consider the m^n-coloring on 2-element subsets of $\alpha'[\mathbb{N}]$ given by

$$\{i, j\} \mapsto (\phi(d(x_i^k, x_j^k)))_{1\leq k\leq n},$$

for all $i < j \in \alpha'[\mathbb{N}]$. By the Infinite Ramsey Theorem, there must be an infinite subset A' of $\alpha'[\mathbb{N}]$ where the coloring is monochromatic, say of the color $(H_1, \ldots, H_n)$. Let $\alpha'' : \mathbb{N} \to A'$ be the unique strictly increasing onto map.

Lastly, consider the $m^{n\times(n-1)}$-coloring on 2-element subsets of $\alpha''[\mathbb{N}]$ given by

$$\{i, j\} \mapsto (\phi(d(x_i^k, x_j^l)), \phi(d(x_i^l, x_j^k)))_{1\leq k<l\leq n}$$

for all $i < j \in \alpha''[\mathbb{N}]$. By the Infinite Ramsey Theorem, there must be an infinite subset A'' of $\alpha''[\mathbb{N}]$ where the coloring is monochromatic, say of the color $(V_{k,l}, V_{l,k})_{1\leq k<l\leq n}$. Then the unique strictly increasing onto map $\alpha : \mathbb{N} \to A''$ will give us a subsequence which is ϵ-constant with respect to the constants H_k, $D_{k,l}$, $V_{k,l}$, and $V_{l,k}$ as defined above. $\square$

Now, we use Lemma 3.34 recursively to build nesting subsequences such that the diagonal sequence will be approximately constant.

Lemma 3.35. *Every sequence of n-tuples in a bounded metric space (X, d) for any n admits an approximately constant subsequence.*

Proof. Fix any sequence of n-tuples $(x_i^1, x_i^2, \ldots, x_i^n)_i$ in X. Let α_0 be the identity function from $\mathbb{N}$ to $\mathbb{N}$. We construct strictly increasing maps $\alpha_m : \mathbb{N} \to \alpha_{m-1}[\mathbb{N}]$ for all $m > 0$ recursively as follows. At each stage $m > 0$, apply Lemma 3.34 to $(x^1_{\alpha_{m-1}(i)}, x^2_{\alpha_{m-1}(i)}, \ldots, x^n_{\alpha_{m-1}(i)})_i$ to obtain a strictly increasing $\alpha_m : \mathbb{N} \to \alpha_{m-1}[\mathbb{N}]$ and $D^m_{k,l}, V^m_{l,k}, H^m_k$, such that for all $1 \leq k \leq n$ and for all $i < j$,

$$|d(x^k_{\alpha_m(i)}, x^k_{\alpha_m(j)}) - H^m_k| < 2^{-m};$$

and for all $1 \leq k < l \leq n$ and for all $i < j$,

$$\begin{aligned} |d(x^k_{\alpha_m(i)}, x^l_{\alpha_m(i)}) - D^m_{k,l}| &< 2^{-m} \\ |d(x^k_{\alpha_m(i)}, x^l_{\alpha_m(j)}) - V^m_{k,l}| &< 2^{-m} \\ |d(x^l_{\alpha_m(i)}, x^k_{\alpha_m(j)}) - V^m_{l,k}| &< 2^{-m}. \end{aligned}$$

Notice that the sequence of α_m's gives nesting subsequences and $2^{-m} \to 0$. Hence, the sequence $(H^m_k)_{m>0}$ for each $1 \leq k \leq n$ and the sequences $(D^m_{k,l})_{m>0}$, $(V^m_{k,l})_{m>0}$, $(V^m_{l,k})_{m>0}$ for each $1 \leq k < l \leq n$ are cauchy, and therefore must converge. Let $H_k, D_{k,l}, V_{k,l}, V_{l,k}$ be the corresponding limits.

Let us now take α to be the diagonal of these subsequences, i.e., $\alpha(m) = \alpha_m(m)$. We claim that $(x^1_{\alpha(i)}, x^2_{\alpha(i)}, \ldots, x^n_{\alpha(i)})_i$ is approximately constant. Let $\epsilon > 0$ be given. Fix N large enough such that for all $m \geq N$, $|C^m - C| < \frac{\epsilon}{2}$, for C any of the constants, and $2^{-m} < \frac{\epsilon}{2}$. Fix any $1 \leq k < l \leq n$. For all $m \geq N$, we have

$$\begin{aligned}
|d(x^k_{\alpha(m)}, x^l_{\alpha(m)}) - D_{k,l}| &\leq |d(x^k_{\alpha_m(m)}, x^l_{\alpha_m(m)}) - D^m_{k,l}| + |D^m_{k,l} - D| \\
&< 2^{-m} + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

So $(x^1_{\alpha(i)}, x^2_{\alpha(i)}, \ldots, x^n_{\alpha(i)})_i$ is ϵ-constant with respect to $D_{k,l}$. Take any $1 \leq k \leq n$ and $N \leq m < m'$. By construction of $\alpha_{m'}$, there is some $m'' \geq m'$ such that $\alpha_{m'}(m') = \alpha_m(m'')$. Thus,

$$\begin{aligned}
|d(x^k_{\alpha(m)}, x^k_{\alpha(m')}) - H_k| &\leq |d(x^k_{\alpha_m(m)}, x^k_{\alpha_{m'}(m')}) - H^m_k| + |H^m_k - H_k| \\
&< |d(x^k_{\alpha_m(m)}, x^k_{\alpha_m(m'')}) - H^m_k| + \frac{\epsilon}{2} \\
&< 2^{-m} + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

So $(x^1_{\alpha(i)}, x^2_{\alpha(i)}, \ldots, x^n_{\alpha(i)})_i$ is ϵ-constant with respect to H_k. Also, for $N \leq m < m' \leq m''$ as above and any $1 \leq k < l \leq n$,

$$\begin{aligned}
|d(x^k_{\alpha(m)}, x^l_{\alpha(m')}) - V_{k,l}| &\leq |d(x^k_{\alpha_m(m)}, x^l_{\alpha_{m'}(m')}) - V^m_{k,l}| + |V^m_{k,l} - V_{k,l}| \\
&< |d(x^k_{\alpha_m(m)}, x^l_{\alpha_m(m'')}) - V^m_{k,l}| + \frac{\epsilon}{2} \\
&< 2^{-m} + \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

So $(x^1_{\alpha(i)}, x^2_{\alpha(i)}, \ldots, x^n_{\alpha(i)})_i$ is ϵ-constant with respect to $V_{k,l}$. A calculation symmetric to the above also shows $(x^1_{\alpha(i)}, x^2_{\alpha(i)}, \ldots, x^n_{\alpha(i)})_i$ is ϵ-constant with respect to $V_{l,k}$. Since $\epsilon > 0$ was chosen arbitrarily, $(x^1_{\alpha(i)}, x^2_{\alpha(i)}, \ldots, x^n_{\alpha(i)})_i$ is approximately constant as needed. □

So the "moreover" statement of Theorem 3.33 is proved. For the main statement, we first show that a slack sequence of ordered pairs in X always gives rise to an unconstrained extremal function on X, and vice versa, in bounded metric spaces. By Lemma 3.26, we have the "if" direction. Then we prove the "only if" direction by embedding the space of extremal functions into a separable space.

Lemma 3.36. *Let (X, d) be a bounded metric space. Then X contains a slack sequence of ordered pairs if and only if $\varepsilon(X)$ contains an unconstrained function.*

Proof. ($\Rightarrow$) Fix $(x_i, y_i)_i$ a slack sequence of ordered pairs in X. Let $D = \liminf_i d(x_i, y_i)$, $H_X = \limsup_{i \neq j} d(x_i, x_j)$, $H_Y = \limsup_{i \neq j} d(y_i, y_j)$, $V_X = \limsup_{j>i} d(x_i, y_j)$, and $V_Y = \limsup_{j>i} d(x_j, y_i)$. Since $(x_i, y_i)_i$ is slack, we may fix $\epsilon > 0$ such that

$$D \geq \frac{1}{2}(H_X + H_Y) + \epsilon, \tag{3.11}$$

$$D \geq V_X + \epsilon, \tag{3.12}$$

$$D \geq V_Y + \epsilon. \tag{3.13}$$

Fix N such that the liminf and limsup's are within $\frac{\epsilon}{4}$ of their respective limits beyond N, i.e.,

$$\inf_{i \geq N} d(x_i, y_i) \geq D - \frac{\epsilon}{4}, \tag{3.14}$$

$$\sup_{j>i>N} d(x_i, x_j) \leq H_X + \frac{\epsilon}{4}, \tag{3.15}$$

$$\sup_{j>i>N} d(y_i, y_j) \leq H_Y + \frac{\epsilon}{4}, \tag{3.16}$$

$$\sup_{j>i>N} d(x_i, y_j) \leq V_X + \frac{\epsilon}{4}, \tag{3.17}$$

$$\sup_{j>i>N} d(y_i, x_j) \leq V_Y + \frac{\epsilon}{4}. \tag{3.18}$$

We define f by

$$f(x_i) := \frac{H_x}{2} + \frac{\epsilon}{4}$$

$$f(y_i) := d(x_i, y_i) - f(x_i)$$

for all i. We now show that by our choice of N, f is extremal and satisfies the inequalities from Definition 3.25 for f to be unconstrained on $(x_i, y_i)_{i>N}$ with slack $\delta = 0$ and $\rho = \frac{\epsilon}{8}$. Fix any $j > i > N$. We have

$$f(x_i) + f(y_i) = d(x_i, y_i) \Rightarrow f(x_i) + f(y_i) - d(x_i, y_i) \leq 0$$

by definition of f. Then $f(x_i) + f(x_j) - d(x_i, x_j) \geq 2(\frac{\epsilon}{8})$ follows from

$$\begin{aligned} f(x_i) + f(x_j) &= H_x + \frac{\epsilon}{2} \\ &\geq d(x_i, x_j) + \frac{\epsilon}{4}, && \text{(by (3.15))} \end{aligned}$$

and $f(x_i) + f(y_j) - d(x_i, y_j) \geq \frac{\epsilon}{8}$ follows from

$$\begin{aligned} f(x_i) + f(y_j) &= (\frac{H_X}{2} + \frac{\epsilon}{4}) + d(x_j, y_j) - \frac{H_X}{2} + \frac{\epsilon}{4}) \\ &\geq D - \frac{\epsilon}{4} && \text{(by (3.14))} \\ &\geq V_X + \frac{3\epsilon}{4} && \text{(by (3.12))} \\ &\geq d(x_i, y_j) + \frac{\epsilon}{2} && \text{(by (3.17))} \end{aligned}$$

Symmetrically, (3.14), (3.13), and (3.18) will give us $f(x_j) + f(y_i) \geq d(x_j, y_i) + \frac{\epsilon}{2}$, which implies $f(x_j) + f(y_i) - d(x_j, y_i) \geq \frac{\epsilon}{8}$. Lastly, to show that f is Katětov, we note that

$$
\begin{aligned}
f(y_i) + f(y_j) &= d(x_i, y_i) - (\frac{H_X}{2} + \frac{\epsilon}{4}) + d(x_j, y_j) - (\frac{H_X}{2} + \frac{\epsilon}{4}) \\
&\geq 2D - \frac{\epsilon}{2} - (H_X + \frac{\epsilon}{2}) && \text{(by (3.14))} \\
&\geq (H_X + H_Y + 2\epsilon) - H_X - \epsilon && \text{(by (3.11))} \\
&= H_Y + \epsilon \\
&\geq d(y_i, y_j). && \text{(by (3.16))}
\end{aligned}
$$

So f extremal and unconstrained as needed.

($\Leftarrow$) Fix $f \in \varepsilon(X)$ an unconstrained function with slack $0 \leq \delta < \rho$ on $(x_i, y_i)_i$. Definition 3.25 gives that for all $i < j$

$$
\begin{aligned}
f(x_i) + f(y_i) - d(x_i, y_i) &\leq \delta, && (3.19) \\
f(x_i) + f(x_j) - d(x_i, x_j) &\geq 2\rho, && (3.20) \\
f(x_i) + f(y_j) - d(x_i, y_j) &\geq \rho, && (3.21) \\
f(x_j) + f(y_i) - d(x_j, y_i) &\geq \rho. && (3.22)
\end{aligned}
$$

By Lemma 3.35, we may assume that (x_i, y_i) is an approximately constant sequence. Fix constants D, V_X, V_Y, H_X, H_Y, and N large enough such that for all

$N < i < j$,

$$|d(x_i, y_i) - D| \leq \frac{\rho - \delta}{4}, \tag{3.23}$$

$$|d(x_i, x_j) - H_X| \leq \frac{\rho - \delta}{4}, \tag{3.24}$$

$$|d(y_i, y_j) - H_Y| \leq \frac{\rho - \delta}{4}, \tag{3.25}$$

$$|d(x_i, y_j) - V_X| \leq \frac{\rho - \delta}{4}, \tag{3.26}$$

$$|d(x_j, y_i) - V_Y| \leq \frac{\rho - \delta}{4}. \tag{3.27}$$

We claim that $(x_i, y_i)_{i>N}$ is slack.

First, for any $j > i > N$, we have by (3.23)

$$\begin{aligned}
2D &\geq d(x_i, y_i) + d(x_j, y_j) - \frac{\rho - \delta}{2} \\
&\geq f(x_i) + f(y_i) + f(x_j) + f(y_j) - 2\delta - \frac{\rho - \delta}{2} && \text{(by (3.19))} \\
&\geq d(x_i, x_j) + d(y_i, y_j) + 2\rho - 2\delta - \frac{\rho - \delta}{2} && \text{(by (3.20) and } f \text{ Katětov)} \\
&\geq H_X + H_Y - \frac{\rho - \delta}{2} + \frac{3(\rho - \delta)}{2} && \text{(by (3.24) and (3.25))} \\
&= H_X + H_Y + (\rho - \delta),
\end{aligned}$$

so $D > \frac{1}{2}(H_X + H_Y)$. Thus, it remains to show that $D > V_X$ and $D > V_Y$. Assume towards a contradiction that $D \leq V_X$. Take any $i > N$, we have by (3.23)

$$\begin{aligned} f(x_i) + f(y_{i+1}) &\geq d(x_i, y_{i+1}) + \rho \\ &\geq V_X - \frac{\rho - \delta}{4} + \rho && \text{(by (3.26))} \\ &\geq D - \frac{\rho - \delta}{4} + \rho \\ &\geq d(x_i, y_i) - \frac{\rho - \delta}{2} + \rho && \text{(by (3.23))} \\ &\geq f(x_i) + f(y_i) + \frac{\rho - \delta}{2} && \text{(by (3.19))} \\ \Rightarrow f(y_{i+1}) &\geq f(y_i) + \frac{\rho - \delta}{2}. \end{aligned}$$

So $(f(y_i))_{i>N}$ is increasing without bound. On the other hand, f is extremal means it is bounded above by the same bound on d. Contradiction. We arrive at the same contradiction for $(f(x_i))_{i>N}$ when $D \leq V_Y$. So $D > V_X$ and $D > V_Y$ as needed, and hence $(x_i, y_i)_{i>N}$ is slack. □

For the "only if" direction of Theorem 3.33, we first prove the result for sequences of n-tuples with constant distances, and then use Proposition 3.7 to obtain it for approximately constant sequences.

Lemma 3.37. *Let $(x_i^1, x_i^2, \ldots, x_i^n)$ be any sequence of n-tuples in a bounded metric space (X, d) such that for all $1 \leq k \leq n$, for all $i < j$, for some H_k,*

$$d(x_i^k, x_j^k) = H_k;$$

and for all $1 \leq k < l \leq n$, for all $i < j$, for some $D_{k,l}, V_{k,l}, V_{l,k}$,

$$\begin{aligned} d(x_i^k, x_i^l) &= D_{k,l} \\ d(x_i^k, x_j^l) &= V_{k,l} \\ d(x_i^l, x_j^k) &= V_{l,k}. \end{aligned}$$

Then $\varepsilon(\{x_i^k : 1 \leq k \leq n, i \in \mathbb{N}\}, d)$ is non-separable if and only if $(x_i^k, x_i^l)_i$ is slack for some $1 \leq k < l \leq n$.

Proof. The "if" direction follows from Lemma 3.36 and Lemma 3.26. We prove the "only if" direction by proving the contrapositive. Assume that $(x_i^k, x_i^l)_i$ is not slack for all $1 \leq k < l \leq n$. Let L denote the subspace $\{x_i^k : 1 \leq k \leq n, i \in \mathbb{N}\}$ with the inherited metric from (X, d). We will show that given $f \in \varepsilon(L)$, for all $1 \leq k \leq n$, the sequence $(f(x_i^k))_i$ is eventually monotone. Since the metric on L is bounded and f is extremal, f must also be bounded. For all $1 \leq k \leq n$, the space of bounded eventually monotone functions on $\{x_i^k : i \leq \omega\}$ is separable, so $\varepsilon(L)$ can be embedded into the cross product of n separable spaces, hence is separable.

Fix any $f \in \varepsilon(L)$. Assume towards a contradiction that for some $1 \leq k \leq n$, f is not eventually monotone on $(x_i^k)_i$. Thus, we may fix $\rho : \mathbb{N} \to \mathbb{N}$ strictly increasing such that for all $i \geq 1$,

$$f(x^k_{\rho(2i-1)}) < f(x^k_{\rho(2i)}) \text{ and } f(x^k_{\rho(2i)}) > f(x^k_{\rho(2i+1)}).$$

Since f is extremal, then for each $i \geq 1$, there must be sequence $(y_j)_j$ such that $(y_j)_j \underset{f}{\frown} x^k_{\rho(2i)}$ (recall Definition 2.11). By going to a subsequence, we may assume that each $(y_j)_j$ is of the form $(x^l_{\sigma(j)})_j$ for some $1 \leq l \leq n$ and $\sigma : \mathbb{N} \to \mathbb{N}$ either constant or strictly increasing. Since there are infinitely many $i \geq 1$ and only finitely many $1 \leq l \leq n$, we may fix an infinite $I \subseteq \mathbb{N}$ such that for all $i \in I$, there is some $\sigma : \mathbb{N} \to \mathbb{N}$ either constant or strictly increasing such that $(x^l_{\sigma(j)})_j \underset{f}{\simeq} x^k_{\rho(2i)}$ for the same $1 \leq l \leq n$.

First, we show that $l \neq k$. Fix any $i \geq 1$. By choice of ρ and f Katětov,

$$f(x^k_{\rho(2i)}) + f(x^k_{\rho(2i-1)}) > f(x^k_{\rho(2i+1)}) + f(x^k_{\rho(2i-1)}) \geq H_k;$$

and for all $j \neq \rho(2i-1)$,

$$\begin{aligned} f(x^k_{\rho(2i)}) + f(x^k_j) &= [f(x^k_{\rho(2i)}) - f(x^k_{\rho(2i-1)})] + [f(x^k_{\rho(2i-1)}) + f(x^k_j)] \\ &\geq [f(x^k_{\rho(2i-1)}) - f(x^k_j)] + H_k \\ &> H_k. \end{aligned}$$

Hence, minimality of f at $x^k_{\rho(2i)}$ is not witnessed by any x^k_j.

So we have $l \neq k$. Fix any $i \in I$ and $\sigma : \mathbb{N} \to \mathbb{N}$ either constant or strictly increasing, such that $(x^l_{\sigma(j)})_j \underset{f}{\simeq} x^k_{\rho(2i)}$. We claim that σ must be constant and $\sigma(j) = \rho(2i)$. We show that all other cases give us contradictions:

[Case σ **strictly increasing**] In this case, $(x^l_{\sigma(j)})_j \underset{f}{\simeq} x^k_{\rho(2i)}$ means

$$\lim_{j\to\infty} f(x^k_{\rho(2i)}) + f(x^l_{\sigma(j)}) = V_{k,l}.$$

Also, σ strictly increasing means we may fix j large enough such that $\sigma(j) >$

$\rho(2i-1)$ and

$$\begin{aligned} f(x^k_{\rho(2i)}) + f(x^l_{\sigma(j)}) &< V_{k,l} + [f(x^k_{\rho(2i)}) - f(x^k_{\rho(2i-1)})] \\ f(x^k_{\rho(2i-1)}) + f(x^l_{\sigma(j)}) &< V_{k,l} \\ f(x^k_{\rho(2i-1)}) + f(x^l_{\sigma(j)}) &< d(x^k_{\rho(2i-1)}, x^l_{\sigma(j)}). \end{aligned}$$

Contradiction to f Katětov.

[Case σ **constant and** $\sigma(j) < \rho(2i)$] In this case, $(x^l_{\sigma(j)})_j \underset{f}{\frown\!\!\!=} x^k_{\rho(2i)}$ means for all j

$$f(x^k_{\rho(2i)}) + f(x^l_{\sigma(j)}) = V_{l,k}.$$

But then

$$\begin{aligned} f(x^k_{\rho(2i+1)}) + f(x^l_{\sigma(j)}) &< f(x^k_{\rho(2i)}) + f(x^l_{\sigma(j)}) \\ &= V_{l,k} \\ &= d(x^k_{\rho(2i+1)}, x^l_{\sigma(j)}), \end{aligned}$$

contradiction to f Katětov.

[Case σ **constant and** $\sigma(j) > \rho(2i)$] This case is symmetric to the above. Using $V_{k,l}$ and $x^k_{\rho(2i-1)}$ in the inequalities, we arrive at the contradiction

$$f(x^k_{\rho(2i-1)}) + f(x^l_{\sigma(j)}) < d(x^k_{\rho(2i-1)}, x^l_{\sigma(j)}).$$

Thus, σ is constantly $\rho(2i)$, and

$$f(x^k_{\rho(2i)}) + f(x^l_{\rho(2i)}) = D_{k,l}.$$

Here, we are assuming $k < l$ since technically we didn't define $D_{k,l}$ when $k > l$. However, in the case that $k > l$, we simply use $D_{l,k}$ instead of $D_{k,l}$ and the proof will go through just the same.

Fix any $i \in I$. By our choice of ρ, we have

$$\begin{aligned} f(x^k_{\rho(2i-1)}) + f(x^l_{\rho(2i)}) &< f(x^k_{\rho(2i)}) + f(x^l_{\rho(2i)}) \\ V_{k,l} &< D_{k,l}, \end{aligned}$$

and

$$\begin{aligned} f(x^k_{\rho(2i+1)}) + f(x^l_{\rho(2i)}) &< f(x^k_{\rho(2i)}) + f(x^l_{\rho(2i)}) \\ V_{l,k} &< D_{k,l}. \end{aligned}$$

Since I is infinite, there is $j \in I$ such that $j \neq i$. As in the argument for $l \neq k$ above, $f(x^k_{\rho(2i)}) + f(x^k_{\rho(2j)}) > H_k$. Then

$$\begin{aligned} f(x^k_{\rho(2i)}) + f(x^k_{\rho(2j)}) + f(x^l_{\rho(2i)}) + f(x^l_{\rho(2j)}) &> H_k + H_l \\ 2D_{k,l} &> H_k + H_l. \end{aligned}$$

This contradicts our assumption that $(x^k_i, x^l_i)_i$ is not slack. So f is eventually monotone on $(x^k_i)_i$, and $\varepsilon(L)$ is separable. $\square$

Using Lemma 3.37 and Proposition 3.7, we can to finish the proof of Theorem 3.33.

Proof. (of Theorem 3.33) It remains to prove the "only if" part of the main statement. Fix any approximately constant n-tuple $(x_i^1, x_i^2, \ldots, x_i^n)_i$ in X such that for all $1 \leq k < l \leq n$, $(x_i^k, x_i^l)_i$ is not slack. Let H_k, $D_{k,l}, V_{k,l}, V_{l,k}$ be its limiting distances, so for all $1 \leq k < l \leq n$, either $D_{k,l} \leq V_{k,l}$, $D_{k,l} \leq V_{l,k}$, or $D_{k,l} \leq \frac{1}{2}(H_k + H_l)$. Let L be the set $\left\{x_i^k : 1 \leq k \leq n, i \in \mathbb{N}\right\}$, we do not affix a metric to it. Let d' be the metric that makes $(x_i^1, x_i^2, \ldots, x_i^n)_i$ into a sequence with constant distances $H_k, D_{k,l}, V_{k,l}$, and $V_{l,k}$. By Lemma 3.37, $\varepsilon(L, d')$ is separable.

For each $m \in \mathbb{N}$, let $B_m = \left\{x_i^k : 1 \leq k \leq n, i \leq m\right\}$. We define a metric d_m for each m such that d_m agrees with d, the original metric, on B_m and d_m agrees with d', the constant metric, on $L \setminus B_n$, i.e.,

$$d_m \restriction B_m = d \restriction B_m \text{ and } d_m \restriction (L \setminus B_m) = d' \restriction (L \setminus B_m).$$

Since $\varepsilon(L, d')$ is separable, then $\varepsilon(L \setminus B_m, d' \restriction L \setminus B_m)$ is also separable for each m by Lemma 3.1. Then Proposition 3.3 gives $\varepsilon(L, d_m)$ is separable for each m since (L, d_m) is the space $(L \setminus B_m, d' \restriction L \setminus B_m)$ adjoined by a finite set B_m. By choice of d' and construction of d_m,

$$|d_m - d| = |d' \restriction (X \setminus B_m) - d \restriction (X \setminus B_m)| \to 0.$$

Thus, by Proposition 3.7, we have $\varepsilon(L, d)$ is separable. $\square$

3.4 Subspaces of the Space of Extremal Functions

From Melleray [5], we know that $E(X)$ is separable if and only if $E(X) = \overline{E(X, \omega)}$. We wonder if there is a subspace of $\varepsilon(X)$ which is dense in $\varepsilon(X)$ exactly when $\varepsilon(X)$ is separable. Since $E(X, \omega)$ is such a subspace of $E(X)$ and $\varepsilon(X)$ is contained in $E(X)$, we might conjecture that $E(X, \omega) \cap \varepsilon(X)$ works. The following example, also due to Melleray, shows that $\varepsilon(X)$ can be separable without being the closure of $\varepsilon(X) \cap E(X)$.

Example 3.38. ***[Melleray]*** *The discrete countable space X with distance 1 everywhere has $\varepsilon(X) = \varepsilon(X, 2)$, so is separable. But $\varepsilon(X)$ is not the closure of $\varepsilon(X) \cap E(X, \omega)$.*

Proof. For each $x \in X$ and each $0 \le r \le \frac{1}{2}$, define the function $f_{x,r}$ by

$$f_{x,r}(y) = \begin{cases} r & \text{if } y = x \\ 1 - r & \text{if } y \neq x. \end{cases}$$

Clearly, each $f_{x,r}$ is in $\varepsilon(X, 2)$. We show that

$$\varepsilon(X) = \left\{ f_{x,r} : x \in X, 0 \le r \le \frac{1}{2} \right\}, \tag{3.28}$$

so $\varepsilon(X) = \varepsilon(X, 2)$.

Fix any $f \in \varepsilon(X)$. There exists some $x_0 \in X$ such that $f(x_0) \le \frac{1}{2}$. Otherwise, the constant $\frac{1}{2}$ function, $f_{x,\frac{1}{2}}$ for any $x \in X$, is in $\varepsilon(X)$ and strictly below f, contradicting $f \in \varepsilon(X)$. Fix x_0 with $f(x_0) \le \frac{1}{2}$. We claim $f = f_{x_0,r}$ where

$r = f(x_0)$. Since f is Katětov, we have for all $y \neq x_0$,

$$\begin{aligned} f(y) + f(x_0) &\geq d(y, x_0) \\ f(y) &\geq 1 - f(x_0) \end{aligned}$$

which implies $f \geq f_{x_0,r}$. Since f and $f_{x_0,r}$ are both extremal, then $f = f_{x_0,r}$. So (3.28) is proved.

To show that $\varepsilon(X)$ is not the closure of $\varepsilon(X) \cap E(X, \omega)$, we show

$$\varepsilon(X) \cap E(X, \omega) \subseteq \{f_{x,0} : x \in X\},$$

so the constant $\frac{1}{2}$ function is in $\varepsilon(X)$ but not in the closure of $\{f_{x,0} : x \in X\}$. By (3.28), it is enough to show that for all $f_{x,r} \in E(X, \omega)$, $r = 0$. Since $f_{x,r}$ has $\{x, y\}$ as a witness set, for any $y \neq x$, by Proposition 2.5, if $f_{x,r} \in E(X, \omega)$, then $\{x, y\}$ is also a support of $f_{x,r}$ for any $y \neq x$. Fix $y \neq x$ and $z \neq x, y$. By definition of $f_{x,r}$, $f_{x,r}(y) = 1 - r$ and $f_{x,r}(z) = 1 - r$. On the other hand, $\{x, y\}$ being a support means

$$\begin{aligned} f_{x,r}(z) &= \min(1 + f_{x,r}(x), 1 + f_{x,r}(y)) \\ &= \min(1 + r, 2 + r) \\ &= 1 + r. \end{aligned}$$

Thus,

$$1 + r = f_{x,r}(z) = 1 - r$$

implies $r = 0$ as needed. $\square$

In Example 3.38, we see that the space of finitely supported extremal functions is not dense in $\varepsilon(X)$, but we do have that every extremal function is finitely witnessed. Since we developed the finitely witness extremal functions as analogs to the finitely supported Katětov functions in Chapter 2, we might conjecture that $\varepsilon(X)$ is separable if an only if it is the closure of $\varepsilon(X, \omega)$, as was the case for $E(X)$ and $E(X, \omega)$. This turns out to be false. In fact, the following example shows something even stronger: $\varepsilon(X)$ can be separable and not contained in the closure of a larger space, $M(X, \omega)$.

Example 3.39. *Consider the metric space $X = \{x_i, y_i : i < \omega\}$ with metric d such that*

$$\begin{aligned}
\forall i < \omega \quad & d(x_i, y_i) = 1 \\
\forall i \neq j \quad & d(x_i, x_j) = 1 \\
\forall i \neq j \quad & d(y_i, y_j) = \tfrac{1}{2} \\
\forall i < j \quad & d(x_i, y_j) = \tfrac{1}{2} \\
\forall i < j \quad & d(y_i, x_j) = 1
\end{aligned}$$

For this space, $\varepsilon(X)$ is separable but $\varepsilon(X) \not\subseteq \overline{M(X, \omega)}$.

Proof. Notice that $(x_i, y_i)_i$ is an approximately constant sequence which is not slack. Thus, by Theorem 3.33, $\varepsilon(X)$ is separable. To show that $\varepsilon(X) \not\subseteq \overline{M(X, \omega)}$ we consider the function $f \in \varepsilon(X)$ where $f(x_i) = \frac{7}{10}$ and $f(y_i) = \frac{3}{10}$ for all i. We will show that for all $g \in M(X, \omega)$, $d_E(f, g) \geq \frac{1}{20}$.

Take any $g \in M(X, \omega)$. Let F and α be such that $g = M_{F,\alpha}$. If there is some

$z \in F$ where $|f(z) - g(z)| \geq \frac{1}{20}$, then we are done since this implies $d_E(f,g) \geq \frac{1}{20}$. So we assume that on F, g is within $\frac{1}{20}$ of f. Since F is finite, we may fix N such that

$$F \subseteq \{x_i, y_i : i \leq N\}.$$

Fix any $m > N$. If $|g(y_m) - f(y_m)| \geq \frac{1}{20}$, then $d_E(f,g) \geq \frac{1}{20}$ and we are done. Suppose $|g(y_m) - f(y_m)| < \frac{1}{20}$. By definition, $f(y_m) = \frac{3}{10}$, so

$$g(y_m) > f(y_m) - \frac{1}{20} = \frac{3}{10} - \frac{1}{20} = \frac{1}{4}. \tag{3.29}$$

Since $g = M_{F,\alpha}$, there is some $z \in F$ such that $g(y_m) = d(z, y_m) - \alpha(z)$. By choice of N, $z = x_i$ or $z = y_i$ for some $i \leq N < m$. In either case, $d(z, y_m) = \frac{1}{2}$ and $d(z, x_m) = 1$. Then

$$\begin{aligned} g(y_m) &= d(z, y_m) - \alpha(z) \\ &= \frac{1}{2} - \alpha(z) \end{aligned}$$

and (3.29) implies $\alpha(z) < \frac{1}{4}$, which implies

$$\begin{aligned} g(x_m) &\geq d(z, x_m) - \alpha(z) \\ &= 1 - \alpha(z) \\ &> \frac{3}{4}. \end{aligned}$$

Then

$$g(x_m) - f(x_m) > \frac{3}{4} - \frac{7}{10} = \frac{1}{20}$$

gives $d_E(g,f) > \frac{1}{20}$ as needed. □

From Example 3.39, we see that $\varepsilon(X,\omega)$ is too small to capture when $\varepsilon(X)$ is separable. The requirement of having finitely many points witnessing the minimality of $f \in \varepsilon(X)$ is too restrictive. Recall that in general, minimality at a point is witnessed by a sequence of points. Thus, being finitely witnessed is doubly restrictive: (1) the function has a finite set of witnesses and (2) each point has a single point witnessing minimality at it. Let us expand the set of finitely witnessed functions by keeping the requirement that each point as a single point witnessing minimality but not putting any restrictions on the set of these witness points.

Definition 3.40. For a metric space X and $f \in \varepsilon(X)$, we say that f has the *witness property (WP)* if at every $x \in X$ there is $y \in X$ such that $y \underset{f}{\frown} x$, i.e., if f satisfies

$$\forall x \in X \; \exists y \in X \; [f(x) + f(y) = d(x,y)].$$

We let $\varepsilon^{WP}(X)$ denote the family of extremal functions with the witness property.

Notice that every finitely witnessed extremal function has WP by definition, so $\varepsilon(X,\omega) \subseteq \varepsilon^{WP}(X)$. From Isbell's remark cited before Definition 2.11, we know that $\varepsilon(X) = \varepsilon^{WP}(X)$ whenever X is compact. We also know from Isbell [1] that $\varepsilon(X)$ is separable whenever X is compact. Unfortunately, for non-compact spaces, the space of extremal functions having WP can be non-separable. In other words, by allowing infinitely many witnesses, we have ended up with a space which is too big.

Example 3.41. *Consider the metric space* (X, d) *consisting of points* $\{x_0, x_1, x_2, \ldots\}$ *and* $\{y_1, y_2, \ldots\}$ *with distance* d *defined as*

$$\begin{aligned} \forall i \neq j \quad d(x_i, x_j) &= 1 \\ \forall i \neq j \quad d(x_i, y_j) &= 1 \\ \forall 0 < i \neq j \quad d(y_i, y_j) &= 2 \\ \forall 0 < i \quad d(x_i, y_i) &= 2. \end{aligned}$$

For this space

1. $\varepsilon(X)$ *is non-separable,*
2. $\varepsilon(X) \supsetneq \varepsilon^{WP}(X)$*, and*
3. $\varepsilon(X) = \overline{\varepsilon^{WP}(X)} \supsetneq \overline{\varepsilon(X, \omega)}$.

Proof. For 1, note that $(x_i, y_i)_{i>0}$ is a slack sequence. By Theorem 3.33, $\varepsilon(X)$ is non-separable.

For 2, consider the function f defined by

$$\begin{aligned} f(x_0) &= \frac{1}{2} \\ f(x_i) &= \frac{1}{2} + 2^{-i} \text{ (for } i > 0) \\ f(y_i) &= 2 - f(x_i) \text{ (for } i > 0). \end{aligned}$$

First, observe that this definition of f gives $\frac{1}{2} \leq f(x_i)$ for all $i \geq 0$ and $1 \leq f(y_i)$ for all $i > 0$. Using these bounds on f, we can easily check that $f \in E(X)$. To see that $f \in \varepsilon(X)$ we note that for $i > 0$, $x_i \underset{f}{\simeq} y_i$ by definition of f and $(x_i)_{i>0} \underset{f}{\simeq} x_0$ follows from

$$f(x_0) + f(x_i) = d(x_0, x_i) + 2^{-i}$$

for all $i > 0$. The definition of f also gives

$$f(x_0) + f(x_i) = \frac{1}{2} + \frac{1}{2} + 2^{-i} > 1 = d(x_0, x_i)$$
$$f(x_0) + f(y_i) \geq \frac{1}{2} + 1 = \frac{3}{2} > 1 = d(x_0, y_i)$$

for all $i > 0$. Hence, for all $i > 0$, x_i and y_i do not witness the minimality of f at x_0, so $f \notin \varepsilon^{WP}(X)$.

It remains to show 3: $\varepsilon(X) = \overline{\varepsilon^{WP}(X)} \supsetneq \overline{\varepsilon(X, \omega)}$. Since $\varepsilon(X, \omega)$ is separable, its closure cannot be $\varepsilon(X)$, which we just showed is non-separable. Thus, it is enough to show $\varepsilon(X) = \overline{\varepsilon^{WP}(X)}$.

First, we prove a claim: For any $g \in \varepsilon(X)$, g must have the WP at all points other than x_0.

Proof of claim: For the sake of contradiction, suppose g fails to have the WP at x_i, for some $i > 0$. Then we must be able to find either a non-constant sequence $(x_j)_j$ such that $(x_j)_j \underset{g}{\frown} x_i$, i.e.,

$$g(x_j) + g(x_i) \to 1$$

or a non-constant sequence $(y_j)_j$ such that $(y_j)_j \underset{g}{\frown} x_i$, i.e.,

$$g(y_j) + g(x_i) \to 1.$$

In either case, $g(x_i) \leq \frac{1}{2}$: Otherwise, in the former case, we could find $j \neq k$ such that $g(x_j), g(x_k) < \frac{1}{2}$, which fails the Katětov inequality $g(x_j) + g(x_k) \geq d(x_j, x_k) = 1$; or in the latter case, we could find $j \neq k$ such that $g(y_j), g(y_k) < \frac{1}{2}$, which fails $g(y_j)+g(y_k) \geq d(y_j, y_k) = 2$. Then $g(x_i) \leq \frac{1}{2}$ and our assumption that g does not have the WP at x_i imply

$$g(x_i) + g(y_i) > d(x_i, y_i) = 2 \Rightarrow g(y_i) > \frac{3}{2}.$$

So for any $j \neq i$,

$$\begin{aligned} g(x_i) + g(y_j) \geq d(x_i, y_j) = 1 \quad &\Rightarrow \quad g(y_i) + g(y_j) \geq 2 + (\frac{3}{2} - g(y_i)), \\ g(x_i) + g(x_j) \geq d(x_i, x_j) = 1 \quad &\Rightarrow \quad g(y_i) + g(x_j) \geq 2 + (\frac{3}{2} - g(y_i)). \end{aligned}$$

Thus, g is not minimal at y_i, contradiction to g extremal. Similarly, if g fails to have the WP at some y_i, then g will not be minimal at x_i. The claim is proved.

Now take any $g \in \varepsilon(X) \setminus \varepsilon^{WP}(X)$. We claim that g must satisfy

- $g(x_i) + g(y_i) = 2$ for all $i > 0$, and
- $g(x_0) + g(x_{\alpha(i)}) \to 1$ for some $(x_{\alpha(i)})_i$ subsequence of $(x_i)_i$.

Since $g \notin \varepsilon^{WP}(X)$ and by the previous claim has the WP at all points except x_0, g must fail the WP at x_0. Since $f(y_i)+f(y_j) \geq d(y_i, y_j) = 2$ for all $i \neq j$, then for all except one i, $g(y_i) \geq 1$. But $d(x_0, y_i) = 1$ for all i, so any sequence witnessing the minimality of g at x_0 must contain infinitely many x_i's. So there is α such that $(x_{\alpha(i)})_i \underset{f}{\frown} x_0$. Since $d(x_i, x_j) = 1$ for all $i \neq j$, $g(x_0)+g(x_{\alpha(i)}) \to 1$ as needed. On the other hand, $g(x_i) + g(x_j) \geq d(x_i, x_j) = 1$ for all $i \neq j$ and g fails the WP at x_0 imply

$$g(x_0) \leq \frac{1}{2} \text{ and } g(x_i) > \frac{1}{2}$$

for all $i > 0$. By our claim, g has the WP at x_i for all $i > 0$. If y_i does not witness the minimality of g at x_i, and since no x_j for any $j \neq i$ witnesses the minimality of g at x_i, there must be some y_j for some $j \neq i$ such that

$$g(x_i) + g(y_j) = 1.$$

Note that $\{g(x_i) : i > 0\}$ can not have a minimal element, since such a minimal element would witness the minimality of g at x_0. So there is $k > i, j$ such that $g(x_k) < g(x_i)$. Then

$$g(x_k) + g(y_j) < 1 = d(x_k, y_j),$$

contradiction to g extremal. Thus, the single witness to the minimality of f at x_i must be y_i for all $i > 0$. So g is as described.

Now we show $g \in \overline{\varepsilon^{WP}(X)}$ by constructing $g_n \in \varepsilon^{WP}(X)$ such that $g_n \to g$. By the previous claim, we may fix $\alpha : \mathbb{N} \to \mathbb{N}$ strictly increasing such that for all n,

$$g(x_0) + g(x_{\alpha(n)}) < 1 + 2^{-n}.$$

For each n, let

$$g_n(z) = \begin{cases} 1 - g(x_{\alpha(n)}) & z = x_0 \\ g(z) & \text{else} \end{cases}$$

Since g had the WP at all points except x_0 and our definition makes g_n have the WP at x_0 and equal to g everywhere else, $g_n \in \varepsilon^{WP}(X)$ for all n. Then by our definition of g_n, $d_E(g, g_n) < 2^{-n}$ for all n. So g_n converges to g and g is in the closure of $\varepsilon^{WP}(X)$. $\square$

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