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SOME RESULTS ON  $G_\delta$  IDEALS OF COMPACT SETS

BY

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DISSERTATION

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For a compact metric space  $E$ , Solecki has defined a broad natural class of  $G_\delta$  ideals of compact sets on  $E$ , called  $G_\delta$  ideals with property  $(*)$ , and has shown that any ideal  $I$  in this class can be represented through the ideal of nowhere dense subsets of a closed subset  $\mathcal{F}$  of the hyperspace of compact subsets of  $E$ . In this thesis we show that the closed set  $\mathcal{F}$  in this representation can be taken to be closed upwards, i.e., it contains the compact supersets of its members. We examine the behaviour of  $G_\delta$  subsets of  $E$  with respect to the representing sets of  $I$ ; we formulate a conjecture and prove it for several classes of ideals.

*For Mama, my ground — and Kranti, my sky*

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# CHAPTER 1

## Introduction

Let  $E$  be a Polish space and let  $\mathcal{K}(E)$  denote the hyperspace of its compact subsets, equipped with the Vietoris topology. A basis for this topology consists of sets of the form

$$\{F \in \mathcal{K}(E) : F \subseteq U_0, F \cap U_i \neq \emptyset \forall i = 1, \dots, k\},$$

where  $k \in \mathbb{N}$  and  $U_0, U_1, \dots, U_k$  are basic open subsets of  $E$ . We may also assume that the sets  $U_1, \dots, U_k$  are contained in  $U_0$ . Topologized in this way,  $\mathcal{K}(E)$  is itself a Polish space and may be metrized by the Hausdorff metric, denoted  $d_H$ . For any set  $A \subseteq E$  and  $\delta > 0$ , we use the notation  $A + \delta$  for the set  $\bigcup_{x \in A} B(x, \delta)$ ; here  $B(x, \delta)$  is the open ball about  $x$  of radius  $\delta$ . With this notation  $d_H$  may be defined through the following condition: for any nonempty sets  $F$  and  $K$  in  $\mathcal{K}(E)$ ,

$$d_H(F, K) < \delta \iff F \subseteq K + \delta \text{ and } K \subseteq F + \delta,$$

and if either  $F$  or  $K$  is empty then we set  $d_H(F, K) = 1$ . From this point on, let  $E$  be a compact Polish space. For such  $E$ ,  $\mathcal{K}(E)$  is compact as well. A set  $I \subseteq \mathcal{K}(E)$  is an *ideal of compact sets* if it is closed under the operations of taking compact subsets and finite unions. Ideals arise commonly in analysis out of various notions of smallness. An ideal  $I$  is a  $\sigma$ -*ideal of compact sets* if it is also closed under countable unions whenever the union itself is compact. (When the context

is clear we will often simply use the terms ‘ideal’ and ‘ $\sigma$ -ideal’.) The condition of being an ideal or  $\sigma$ -ideal of compact sets is strongly related to the complexity of  $I$ . By results of Kechris and Louveau, and independently, Dougherty (see [6]), we know that if  $I$  is a  $G_\delta$  ideal (by which we mean an ideal that is  $G_\delta$  as a subset of  $\mathcal{K}(E)$ ), it must be a  $\sigma$ -ideal, and by results of Kechris–Louveau–Woodin proved in the seminal paper [8], we know that if a  $\sigma$ -ideal  $I$  is either co-analytic or analytic, it must be either complete co-analytic or simply  $G_\delta$ . In particular, all analytic  $\sigma$ -ideals are  $G_\delta$ .

In this thesis we consider  $G_\delta$   $\sigma$ -ideals of compact sets that also satisfy the following natural condition, formulated by Solecki in [13]: a collection of compact sets  $I \subseteq \mathcal{K}(E)$  has *property*  $(*)$  if, for any sequence of sets  $(K_n)_{n \in \mathbb{N}} \subseteq I$ , there exists a  $G_\delta$  set  $G$  such that  $\bigcup_n K_n \subseteq G$  and  $\mathcal{K}(G) \subseteq I$ .

It is easily seen that if  $I \subseteq \mathcal{K}(E)$  has property  $(*)$ , it must be a  $\sigma$ -ideal. Further, it follows from the results mentioned above and from Theorem 2 below that if  $I$  has  $(*)$  and is analytic or co-analytic, it must be  $G_\delta$ . The converse question — whether every  $G_\delta$  ideal has property  $(*)$  — is a version of a longstanding problem of Kechris. The original problem asked whether for  $I$  a  $G_\delta$   $\sigma$ -ideal of compact subsets of  $2^\omega$ , containing all singletons, there exists a dense  $G_\delta$  set  $G \subseteq 2^\omega$  whose every compact subset is in  $I$ . This question was settled in the negative by Matrai’s construction of a counterexample in [11]. Such a counterexample obviously fails to have property  $(*)$ . Property  $(*)$  does however hold in all natural examples of  $G_\delta$  ideals, including the ideals of compact meager sets, measure-zero sets, sets of topological dimension  $\leq n$  for fixed  $n \in \mathbb{N}$ , and  $\mathcal{Z}$ -sets for  $E = [0, 1]^\omega$ . (See [13] for these and other examples. For a broad survey of the descriptive set theory of families of small sets, see [10].)

Solecki has shown in [13] that  $G_\delta$  ideals with property  $(*)$  are represented via the meager ideal of a closed subset of  $\mathcal{K}(E)$ . (This representation is analogous to

a result of Choquet (see [1]) that establishes a correspondence between alternating capacities of order  $\infty$  on  $E$  and probability Borel measures on  $\mathcal{K}(E)$ .) The following definition is essential to the representation:

**Definition 1.** For  $A \subseteq E$ , let  $A^* = \{K \in \mathcal{K}(E) : K \cap A \neq \emptyset\}$ .

**Theorem 2** (Solecki). *Suppose  $I$  is co-analytic and nonempty. Then the following are equivalent:*

1.  $I$  has property  $(*)$ ;
2. there exists a closed set  $\mathcal{F} \subseteq \mathcal{K}(E)$  such that, for any  $K \in \mathcal{K}(E)$ ,

$$K \in I \iff K^* \cap \mathcal{F} \text{ is meager in } \mathcal{F}.$$

For a co-analytic and nonempty ideal  $I$  with property  $(*)$ , if a closed set  $\mathcal{F} \subseteq \mathcal{K}(E)$  satisfies the second condition of this theorem we will say that  $\mathcal{F}$  represents  $I$ . As an example, let  $\mu$  be an atomless probability measure on  $E$  and let  $I$  be the  $\sigma$ -ideal of compact  $\mu$ -null sets. Fix a basis of the topology on  $E$  closed under finite unions and let  $s \in (0, 1)$  be chosen so that it is not the measure of any basic set. Then the set  $\mathcal{F} = \{K \in \mathcal{K}(E) : \mu(K) \geq s\}$  represents the ideal.

The set  $\mathcal{F}$  in Theorem 2 is not unique. We might hope to determine properties for  $\mathcal{F}$  that make it a more canonical representative, perhaps upto some notion of equivalence. The following property of is of particular interest: we say that a set  $\mathcal{F} \subseteq \mathcal{K}(E)$  is *upward closed* if for any sets  $A, B$  in  $\mathcal{K}(E)$ , if  $A \in \mathcal{F}$  and  $B \supseteq A$ , then  $B \in \mathcal{F}$ . If  $\mathcal{F}$  is upward closed, the map  $K \mapsto K^* \cap \mathcal{F}$ , a fundamental function in this context, is continuous. In the example described above, the set  $\mathcal{F}$  representing the null ideal for the Borel measure is upward closed. The main result of Chapter 2 shows that, provided that the ideal  $I$  contains only sets with empty interiors, the representing set  $\mathcal{F}$  in Theorem 2 may always be chosen to be upward closed.

(The condition that  $I$  contain only meager sets is in fact necessary.) The rest of that chapter deals with a possible alternative to using the sets  $K^*$  to characterize membership in the ideal, and describes natural representatives for certain classes of ideals.

The proof of Theorem 2 shows that for nonempty co-analytic ideals, the two equivalent conditions of the theorem are also equivalent to the following third condition:

3. *There exist closed and upward closed  $\mathcal{F}_n \subseteq \mathcal{K}(E)$ ,  $n \in \mathbb{N}$ , such that  $\forall n$ ,  $\mathcal{F}_{n+1} \subseteq \mathcal{F}_n$  and, for any  $K \in \mathcal{K}(E)$ ,*

$$K \in I \implies \forall n \ K^* \cap \mathcal{F}_n \text{ is meager in } \mathcal{F}_n;$$

$$K \notin I \implies \exists n \ \mathcal{F}_n \subseteq K^*.$$

The naturalness of this condition is indicated in [13], and in fact familiar examples of ideals typically yield sequences  $\mathcal{F}_n$  satisfying this condition from their very definitions. From now on, if  $I$  is a nonempty co-analytic ideal with property (\*), and  $\mathcal{F}_n$ ,  $n \in \mathbb{N}$  are closed subsets of  $\mathcal{K}(E)$  satisfying the condition that for any  $K \in \mathcal{K}(E)$ ,

$$K \in I \iff \forall n \ K^* \cap \mathcal{F}_n \text{ is meager in } \mathcal{F}_n,$$

then we will say that  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  is a *representing sequence* for  $I$ .

Given an ideal of compact sets and the corresponding set  $\mathcal{F} \subseteq \mathcal{K}(E)$  (or sequence  $(\mathcal{F}_n)$ ) that represents it as in Theorem 2, we may examine meagerness of  $A^*$  in the representing sets for sets  $A \subseteq E$  that are not necessarily closed. In Chapter 3 we consider  $G_\delta$  sets  $G$  and investigate when  $G^*$  is meager in  $\mathcal{F}$  (or in each  $\mathcal{F}_n$ ). Even though the notion of smallness used to define the original ideal

of compact sets might apply to  $G_\delta$  sets as well (for example, the set  $G$  may have measure zero, or be meager, or have dimension less than or equal to some fixed  $n \in \mathbb{N}$ ), the smallness of  $G$  as gauged by that original notion typically does not correspond to the meagerness of  $G^*$  in  $\mathcal{F}$  (or, again, in each  $\mathcal{F}_n$ ). We conjecture that each  $G_\delta$  ideal  $I$  of compact sets with property  $(*)$  has a representing sequence of upward closed sets  $\mathcal{F}_n$  such that, for  $G_\delta$  sets  $G$ ,  $G^*$  is meager in each  $\mathcal{F}_n$  exactly when  $G$  can be covered by a sequence of sets in  $I$ . While this conjecture remains unproved in general, we examine it in various contexts where we prove that it holds for the representing sequence arising naturally from the definition of specific ideals. Finally we prove the conjecture for the null ideals of certain submeasures on  $\mathcal{K}(E)$  by constructing a representing sequence which differs from the natural one.

# CHAPTER 2

## Characterizing closed sets in an ideal

### 2.1 Upward closed representatives for $G_\delta$ ideals

Let  $E$  be a compact Polish space and let  $I \subseteq \mathcal{K}(E)$  be an ideal of compact sets with property (\*). Solecki has shown that  $I$  can be represented via the meager ideal of a closed subset of  $\mathcal{K}(E)$  (Theorem 2). In Corollary 4, which follows from Theorem 3 below, we show that as long as the ideal  $I$  contains only meager sets, we may always find a closed set  $\mathcal{F} \subseteq \mathcal{K}(E)$  representing it that is upward closed.<sup>1</sup> We use  $\text{Int}(A)$  to denote the interior of  $A$  in  $E$ .

**Theorem 3.** *For a nonempty closed set  $\mathcal{F} \subseteq \mathcal{K}(E)$ , the following are equivalent:*

- (1)  $\forall K \in \mathcal{K}(E)$ ,  $K$  has nonempty interior  $\Rightarrow K^*$  nonmeager in  $\mathcal{F}$ .
- (2)  $\exists \mathcal{F}' \subseteq \mathcal{K}(E)$ , nonempty, closed and upward closed, such that

$$\forall K \in \mathcal{K}(E) (K^* \text{ nonmeager in } \mathcal{F}' \iff K^* \text{ nonmeager in } \mathcal{F}).$$

*Proof.* It is clear that (2)  $\Rightarrow$  (1) simply because, if  $\mathcal{F}' \subseteq \mathcal{K}(E)$  is nonempty and upward closed, and  $U \subseteq E$  is nonempty and open, then  $\mathcal{F}' \cap U^*$  is nonempty and open in  $\mathcal{F}'$ . To prove the other direction, let  $I = \{K \in \mathcal{K}(E) : K^* \text{ is meager in } \mathcal{F}\}$ . The set  $I$  is a  $\sigma$ -ideal with property (\*). Let  $\{\mathcal{V}_n\}$  be a basis of nonempty

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<sup>1</sup>This result has been published as *A note on  $G_\delta$  ideals of compact sets*, Comment. Math. Univ. Carolin. 50, 4 (2009) 569–573.

sets for the relative topology on  $\mathcal{F}$ , and let  $\mathcal{K}_n = \overline{\mathcal{V}_n}$ . We now have:

$$K \in I \Rightarrow \forall n K^* \text{ meager in } \mathcal{K}_n; \quad (2.1)$$

$$K \notin I \Rightarrow \exists n \mathcal{K}_n \subseteq K^*. \quad (2.2)$$

The proof now divides into two major parts: we first deal with the case when  $I$  contains at least one infinite set, and then give a separate treatment for the case when all sets in  $I$  are finite.

Suppose that  $I$  contains an infinite set. In this case, within this infinite set we fix distinct points  $x$  and  $x_i$ ,  $i \in \mathbb{N}$ , such that  $x_i \rightarrow x$ . Since  $I$  is downward closed,  $\{x\} \in I$  and each  $\{x_i\} \in I$ . Let  $U'_i$  be open such that  $x_i \in U'_i$ ,  $\overline{U'_i} \rightarrow \{x\}$  and the sets  $\overline{U'_i}$  are pairwise disjoint. We will pick a subsequence  $U'_{n_i}$  and define sets  $(U_i, F_i, W_i)$ ,  $i \in \mathbb{N}$ , satisfying each of these conditions:

- $U_i, W_i$  are open,
- $U_i \subseteq U'_{n_i}$ , so the sets  $\overline{U_i}$  are pairwise disjoint,
- $F_i \in \mathcal{K}_i$ ,
- $F_i \subseteq W_i$ ,
- If  $j \leq i$  then  $\overline{W_j} \cap \overline{U_i} = \emptyset$ .

To begin, let  $n_0 = 0$  and note that since  $\{x, x_0\} \in I$ ,  $\mathcal{K}_0 \not\subseteq \{x, x_0\}^*$ . Let  $F_0$  be a set in  $\mathcal{K}_0 \not\subseteq \{x, x_0\}^*$ . Let  $W_0$  be an open superset of  $F_0$  such that  $x, x_0 \notin \overline{W_0}$ , and let  $U_0 \subseteq U'_0$  be an open set containing  $x_0$  such that  $\overline{U_0} \cap \overline{W_0} = \emptyset$ . Pick  $n_1 > 0$  such that for every  $m \geq n_1$ ,  $\overline{W_0} \cap \overline{U'_m} = \emptyset$ . This concludes the 0'th stage of the induction.

For  $i > 0$ , we define  $(U_i, F_i, W_i)$  and pick the number  $n_{i+1}$  as follows. Suppose that  $n_i$  has been defined at the previous stage of the induction and consider  $\mathcal{K}_i$

and  $U'_{n_i}$ . Again, we may pick  $F_i \in \mathcal{K}_i \setminus \{x, x_{n_i}\}^*$ . Let  $W_i \supseteq F_i$  be open such that  $x, x_{n_i} \notin \overline{W_i}$ . Let  $U_i \subseteq U'_{n_i}$  be an open set containing  $x_{n_i}$  such that  $\overline{U_i} \cap \overline{W_i} = \emptyset$ . Pick  $n_{i+1} > n_i$  such that for any  $m \geq n_{i+1}$ ,  $\overline{W_i} \cap \overline{U'_m} = \emptyset$ .

Now note that

$$\begin{aligned} K \in I &\Rightarrow \forall n \ K^* \text{ meager in } \mathcal{K}_n \cap \mathcal{K}(W_n); \\ K \notin I &\Rightarrow \exists n \ \mathcal{K}_n \cap \mathcal{K}(W_n) \subseteq K^*. \end{aligned}$$

In other words, conditions (2.1) and (2.2) hold with the sets  $\mathcal{K}_n$  replaced by the sets  $\mathcal{K}_n \cap \mathcal{K}(W_n)$ . Therefore we may simply assume that  $\mathcal{K}_n \subseteq \mathcal{K}(W_n)$ .

We now define  $\mathcal{L} \subseteq \mathcal{K}(E)$ . For  $n, j \in \mathbb{N}$ , first define closed sets

$$A_{n,j} = \begin{cases} \overline{U_j} & \text{if } j < n, \\ E \setminus \bigcup_{i < n} (U_i + 1/j) & \text{if } j \geq n. \end{cases}$$

Also, for every  $n \in \mathbb{N}$ , let  $U_{n,j}$ ,  $j \in \mathbb{N}$ , be nonempty disjoint open subsets of  $U_n$ . (This is possible because, since  $\{x_n\}$  is in  $I$  it is not open, and thus  $x_n$  must be a limit point of  $E$ .)

Define sets  $\mathcal{L}_{n,j}$  as follows: for  $L \in \mathcal{K}(E)$ ,

$$L \in \mathcal{L}_{n,j} \iff \exists F \in \mathcal{K}_n \text{ such that } F \cap A_{n,j} \subseteq L \text{ and } L \text{ intersects } U_{n,j}.$$

Let  $\mathcal{L} = \bigcup_{n,j} \mathcal{L}_{n,j}$ . Since each  $\mathcal{L}_{n,j}$  is upward closed, so is  $\mathcal{L}$ . We will establish that

$$K \in I \iff K^* \text{ is nowhere dense in } \mathcal{L}. \quad (2.3)$$

First suppose that  $K \in I$ . We want to show that  $\mathcal{L} \setminus K^*$  is dense in  $\mathcal{L}$ . Let  $L_1 \in \mathcal{L}_{n,j}$ , i.e.,  $L_1$  intersects  $U_{n,j}$  and there exists a set  $F \in \mathcal{K}_n$  such that  $F \cap A_{n,j} \subseteq L_1$ . Let  $L \supseteq L_1$  be close to  $L_1$ , satisfying  $L_1 \subseteq \text{Int}(L)$  and  $\overline{\text{Int}(L)} = L$ .

Note that  $L$  is nonmeager in  $U_{n,j}$ .

Consider the set  $\mathcal{D} = \mathcal{K}_n \cap \{F : F \cap A_{n,j} \subseteq \text{Int}(L)\}$ .  $\mathcal{D}$  is a nonempty open subset of  $\mathcal{K}_n$ . (Openness follows from this easily checked fact about  $\mathcal{K}(E)$ : if  $A \subseteq E$  is closed and  $U \subseteq E$  is open, then  $\{F \in \mathcal{K}(E) : F \cap A \subseteq U\}$  is open.) Since  $K \in I$ , we have that  $K^*$  is meager in  $\mathcal{K}_n$ . So  $\mathcal{D} \not\subseteq K^*$  and we may pick some  $F_1$  in  $\mathcal{D} \setminus K^*$ . Now we can remove from  $L$  an open superset  $U$  of  $K$  where  $U$  is chosen small enough so that  $U \cap F_1 = \emptyset$  and  $L \setminus U$  is still nonmeager in  $U_{n,j}$ . The set  $L \setminus U$  is in  $\mathcal{L}_{n,j} \setminus K^*$  and is close to  $L$ , which establishes the implication from left to right in (2.3).

Conversely, suppose  $K \notin I$ . We want to find an open set  $\mathcal{U} \subseteq \mathcal{K}(E)$  such that  $\emptyset \neq \mathcal{U} \cap \mathcal{L} \subseteq K^*$ .

Let  $C = \bigcup_n \overline{U_n} \cup \{x\}$ , a closed set. Write  $K \setminus C = \bigcup_j K_j$ , where  $K_j = K \setminus (C + 1/j)$ , which is closed. Now,

$$K = (K \cap \{x\}) \cup \bigcup_n (K \cap \overline{U_n}) \cup \bigcup_j K_j.$$

Since  $I$  is a  $\sigma$ -ideal and  $\{x\} \in I$ , we have two possible cases: either some  $K \cap \overline{U_n} \notin I$  or some  $K_j \notin I$ .

Case 1: There exists  $n$  such that  $K \cap \overline{U_n} \notin I$ .

In this case we fix such an  $n$ , and fix  $m$  such that  $\mathcal{K}_m \subseteq (K \cap \overline{U_n})^*$ . If  $m \leq n$  then  $\overline{U_n} \cap \overline{W_m} = \emptyset$ . So  $m > n$ . This means that  $\overline{U_n}$  is one of the sets  $A_{m,j}$ . Let  $V \supseteq \overline{U_n}$  be open such that  $V \cap \overline{U_i} = \emptyset$  for all  $i \neq n$  and  $V \cap \overline{W_n} = \emptyset$ . Let  $W = V \cup U_{m,j}$ .

Claim:  $\emptyset \neq \mathcal{L} \cap \mathcal{K}(W) \subseteq K^*$ .

It is clear that  $\mathcal{L}_{m,j} \cap \mathcal{K}(W) \neq \emptyset$ . Let  $L \in \mathcal{K}(W) \cap \mathcal{L}$ . For any  $i \notin \{n, m\}$ ,  $L \cap U_i = \emptyset$ . Also,  $L \cap W_n = \emptyset$  and  $L \cap U_{m,j'} = \emptyset$  for all  $j' \neq j$ . So the only possibility is that  $L \in \mathcal{L}_{m,j}$ , i.e., there exists a set  $F \in \mathcal{K}_m$  such that  $F \cap A_{m,j} = F \cap \overline{U_n} \subseteq L$ .

Since  $F \cap \overline{U_n} \cap K \neq \emptyset$ , we have  $L \cap K \neq \emptyset$ .

Case 2: There exists  $j$  such that  $K_j \notin I$ .

In this case we fix such a  $j$ , and fix  $m$  such that  $\mathcal{K}_m \subseteq K_j^*$ . Fix  $\delta > 0$  such that  $K_j \cap \bigcup_{i < m} \overline{(U_i + \delta)} = \emptyset$  and let  $k \in \mathbb{N}$  such that  $k \geq m$  and  $1/k < \delta$ . Let  $W = (W_m \setminus \bigcup_{i < m} \overline{U_i}) \cup U_{m,k}$ .

Claim:  $\emptyset \neq \mathcal{L} \cap \mathcal{K}(W) \subseteq K^*$ .

It is clear that  $\mathcal{K}(W) \cap \mathcal{L}_{m,k} \neq \emptyset$ . (To get something in this set, we can simply take any  $F \in \mathcal{K}_m$  and join some piece of  $U_{m,k}$  to  $F \cap A_{m,k}$ .) So  $\mathcal{K}(W) \cap \mathcal{L} \neq \emptyset$ .

Now let  $L \in \mathcal{K}(W) \cap \mathcal{L}$ . As before, the only possibility is that  $L \in \mathcal{L}_{m,k}$ , i.e., there exists a set  $F \in \mathcal{K}_m$  such that  $F \cap A_{m,k} = F \setminus \bigcup_{i < m} (U_i + 1/k) \subseteq L$ . Since  $F \in \mathcal{K}_m$ , we have  $F \cap K_j \neq \emptyset$ . Let  $x \in F \cap K_j$ . Since  $1/k < \delta$ , we have  $x \in L$ . Therefore  $L \in K_j^* \subseteq K^*$ .

So in both cases,  $K^*$  contains a nonempty relatively open subset of  $\mathcal{L}$ . Finally, set  $\mathcal{F}' = \overline{\mathcal{L}}$ . This concludes the proof of the theorem for the case when  $I$  has an infinite set.

To deal with the case where  $I$  has only finite sets, we note that in this situation  $I$  is of the form  $\mathcal{K}(A)$ , where  $A$  is a countable  $G_\delta$  set. (In fact,  $A$  is just  $\bigcup I$ , which is  $G_\delta$  since  $I$  is  $G_\delta$ .) In this case, we let  $C_n$ ,  $n \in \mathbb{N}$ , be closed subsets of  $E$  such that  $E \setminus A = \bigcup_i C_i$ , and set  $\mathcal{K}_n = \{C_n\}$ . The sets  $\mathcal{K}_n$  satisfy the conditions (2.1) and (2.2). Now let  $x \in A$ . (If no such  $x$  exists then  $I = \{\emptyset\}$ ; for this ideal we may simply set  $\mathcal{F}' = \{E\}$ .) Since  $\{x\}$  is in  $I$ , it is not open and we may find a sequence of distinct points  $x_i$  in the dense set  $E \setminus A$ , converging to  $x$ . For any  $n$ ,  $C_n$  does not contain  $x$ . So by replacing  $(x_i)$  with a suitable subsequence, we may assume that  $C_n$  is disjoint from  $\{x\} \cup \{x_i : i \geq n\}$ . We may now let  $U'_i$  be open neighbourhoods of  $x_i$  with disjoint closures, and exactly as in the case where  $I$  had an infinite set, proceed to define sets  $(U_i, F_i, W_i)$  satisfying all the listed properties. The construction of these sets succeeds because it remains true that

if  $n_i \geq i$ , then  $\mathcal{K}_i \setminus \{x, x_{n_i}\}^* \neq \emptyset$ .

At this point we deal with two subcases. Suppose first that the sequence  $(x_n)$  contains infinitely many non-isolated points. In this case we assume that in fact each  $x_n$  is non-isolated; this allows us to construct the sets  $U_{n,j}$  and carry out the rest of the proof exactly as before.

Now consider the alternative: all but finitely many  $x_n$  are isolated. In this case we assume that every  $x_n$  is isolated. For  $n \in \mathbb{N}$ , define

$$\mathcal{L}_n = \{F \in \mathcal{K}(E) : C_n \setminus \{x_0, \dots, x_{n-1}\} \subseteq F \text{ and } x_n \in F\},$$

and set  $\mathcal{L} = \bigcup_n \mathcal{L}_n$ , which is obviously upward closed. Now for any  $K \in \mathcal{K}(E)$ ,  $K^*$  is nowhere dense in  $\mathcal{L}$  if and only if  $K \in I$ . To see this, let  $K \in I$ .  $K$  consists of finitely many points of  $A$ , which are all non-isolated. So if  $F \in \mathcal{L}_n$  we may remove a small open superset of  $K$  from  $F$  without removing  $x_n$  or any point of  $C_n$ , resulting in a set in  $\mathcal{L}_n \setminus K^*$  that is close to  $F$ . (Recall that  $x_n \notin A$ .)

Conversely, if  $K \notin I$ , pick  $y \in K \setminus A$ . If  $y = x_n$  for some  $n$ , then  $\{y\}$  is open, and  $\{y\}^* \cap \mathcal{L}$  is a nonempty open subset of  $\mathcal{L}$ , which is all we need. If on the other hand  $y \in E \setminus \{x_n : n \in \mathbb{N}\}$ , fix  $m$  such that  $y \in C_m$ . Consider the open set  $V = W_m \setminus \{x_i : 0 \leq i < m\} \cup \{x_m\}$ ; it is immediate that  $\emptyset \neq \mathcal{K}(V) \cap \mathcal{L} \subseteq \{y\}^*$ . The set  $\mathcal{L}$  is thus as required, and we may set  $\mathcal{F}' = \overline{\mathcal{L}}$ .  $\square$

**Corollary 4.** *Let  $I \subseteq \mathcal{K}(E)$  be a co-analytic ideal with property (\*) containing only meager sets. Then there exists a closed set  $\mathcal{F} \subseteq \mathcal{K}(E)$  such that  $\mathcal{F}$  is upward closed and for any  $K \in \mathcal{K}(E)$ ,*

$$K \in I \iff K^* \cap \mathcal{F} \text{ is meager in } \mathcal{F}.$$

*Proof.* An immediate consequence of Theorem 2 and Theorem 3.  $\square$

We next define a natural operation  $A'$  along the lines of the operation  $A^*$ . We ask for which ideals we might find a closed and upward closed set  $\mathcal{F} \subseteq \mathcal{K}(E)$  such that for closed sets  $A$ , the meagerness of  $A'$  in  $\mathcal{F}$  characterizes membership in  $I$ . It turns out that this condition is so strong, only one ideal satisfies it.

**Theorem 5.** *For a set  $A \subseteq E$ , define*

$$A' = \{K \in \mathcal{K}(E) : A \text{ is nonmeager in } K\}.$$

*Let  $\mathcal{F} \subseteq \mathcal{K}(E)$  be nonempty, closed and upward closed. Let  $I \subseteq \mathcal{K}(E)$  be a  $\sigma$ -ideal of closed sets, and suppose that*

$$\forall K \in \mathcal{K}(E) \quad K \in I \iff K' \cap \mathcal{F} \text{ is meager in } \mathcal{F} \quad (2.4)$$

*Then  $I$  is the ideal of closed meager subsets of  $E$ .*

*Proof.* Fix a basis  $\{W_n\}_{n \in \mathbb{N}}$  of  $E$  consisting of nonempty open sets. For each  $n$ , let  $\{W_{n,k}\}_{k \in \mathbb{N}}$  be a collection of open sets such that  $W_n = \bigcup_k \overline{W_{n,k}}$ . Let  $K \subseteq E$  be closed. Note that for any  $F \in \mathcal{K}(E)$ , we have that  $F \in K'$  if and only if  $K$  has nonempty interior in  $F$ . Therefore we may write

$$K' = \bigcup_n (\{F \in \mathcal{K}(E) : F \cap W_n \neq \emptyset\} \cap \{F \in \mathcal{K}(E) : F \cap W_n \subseteq K\})$$

and if we set

$$\mathcal{A}_{n,k} = \{F \in \mathcal{K}(E) : F \cap \overline{W_{n,k}} \neq \emptyset\} \cap \{F \in \mathcal{K}(E) : F \cap (W_n \setminus K) = \emptyset\}$$

then we have

$$K' = \bigcup_n \bigcup_k \mathcal{A}_{n,k}.$$

Each  $\mathcal{A}_{n,k}$  is a closed set. So for any closed  $\mathcal{F} \subseteq \mathcal{K}(E)$ , the set  $K'$  is meager in  $\mathcal{F}$  if and only if each  $\mathcal{A}_{n,k}$  has empty interior in  $\mathcal{F}$ .

Now fix  $I$  and  $\mathcal{F}$  as in the statement of the theorem and suppose  $K$  is meager. To show that  $K \in I$ , we need to show that each  $\mathcal{A}_{n,k}$  has empty interior in  $\mathcal{F}$ . Let  $\mathcal{U}$  be an open set in  $\mathcal{K}(E)$  of the form  $\mathcal{U} = \{K \in \mathcal{K}(E) : K \subseteq V_0, K \cap V_i \neq \emptyset, i = 1, \dots, N\}$  for some open sets  $V_0, \dots, V_N$  and suppose that  $\mathcal{F} \cap \mathcal{U} \neq \emptyset$ . Let  $F \in \mathcal{F} \cap \mathcal{U}$ . If  $\overline{W_{n,k}} \cap V_0 = \emptyset$ , then in fact  $\mathcal{U} \cap \mathcal{A}_{n,k} = \emptyset$ . If  $\overline{W_{n,k}} \cap V_0 \neq \emptyset$ , then  $W_{n,k} \cap V_0 \neq \emptyset$ . Since  $K$  is meager, there exists an  $x \in (W_{n,k} \cap U_0) \setminus K$ . Let  $F_1 = F \cup \{x\}$ . Then  $F_1 \in \mathcal{U} \cap \mathcal{F} \setminus \mathcal{A}_{n,k}$ .

Conversely, let  $K$  be a nonmeager closed set and pick  $n$  such that  $W_n \subseteq K$ .  $W'_n$  is just  $\{F : F \cap W_n \neq \emptyset\}$ , which is open. Also,  $W'_n \cap \mathcal{F}$  is nonempty because  $\mathcal{F}$  is nonempty and upward closed. So  $W'_n$  is nonmeager in  $\mathcal{F}$ . Since  $K' \supseteq W'_n$ ,  $K'$  is nonmeager in  $\mathcal{F}$ . By (2.4),  $K \notin I$ .  $\square$

**Corollary 6.** *Let  $\mathcal{F} \subseteq \mathcal{K}(E)$  be closed and upward closed. Let  $I$  be a  $\sigma$ -ideal of Borel sets (i.e.,  $I$  consists of Borel sets and is closed under countable union and the taking of Borel subsets), and suppose that (2.4) holds for all closed sets. Then  $I$  is the ideal of meager Borel subsets of  $E$ .*

*Proof.*  $I \cap \mathcal{K}(E)$  is a  $\sigma$ -ideal of compact sets, so by Theorem 5,  $I$  contains all closed meager sets. Since  $I$  is closed under the taking of countable unions and Borel subsets, it must also contain all meager Borel sets. Conversely, suppose  $A \subseteq E$  is nonmeager and Borel. Since  $A$  has the Baire property, for some nonempty open  $U$  and meager Borel set  $M$ , we have  $U \subseteq A \cup M$ . Since  $U$  contains closed nonmeager sets,  $U$  is not in  $I$ . Therefore  $A$  cannot be in  $I$ .  $\square$

## 2.2 Natural representing sequences

We say that a set  $\mathcal{A} \subseteq \mathcal{K}(E)$  is *downward closed* if the following condition holds: for all  $F, K \in \mathcal{K}(E)$ , if  $F \subseteq K \in \mathcal{A}$ , then  $F \in \mathcal{A}$ . Solecki has shown the following equivalence in [13]:

**Theorem 7** (Solecki). *For a nonempty set  $I \subseteq \mathcal{K}(E)$ , the following are equivalent:*

- (i)  *$I$  is a  $G_\delta$  set with property  $(*)$ .*
- (ii) *There exists a sequence of open, downward closed sets  $\mathcal{U}_n \subseteq \mathcal{K}(E)$ ,  $n \in \mathbb{N}$ , such that  $I = \bigcap_n \mathcal{U}_n$  and the sets  $\mathcal{U}_n$  satisfy the condition that*

$$\forall K \in \mathcal{U}_n \exists m \forall L \in \mathcal{U}_m K \cup L \in \mathcal{U}_n. \quad (2.5)$$

For many  $G_\delta$  ideals with property  $(*)$ , there exists a natural countable sequence, say  $(\mathcal{C}_n)_{n \in \mathbb{N}}$ , of collections of open subsets of  $E$  such that any compact set  $K \subseteq E$  is in the ideal if and only if  $K$  can be covered by some member of each  $\mathcal{C}_n$ . The following are examples of ideals defined in this way.

1. For the ideal of closed null sets for a Borel measure  $\mu$ , we have

$$\mathcal{C}_n = \left\{ U : U \text{ is open and } \mu(U) < \frac{1}{n} \right\}.$$

2. For the ideal of closed zero dimensional subsets we have

$$\mathcal{C}_n = \left\{ \bigcup_{i=1}^k U_i : k \in \mathbb{N}, U_i \text{ open, } |U_i| < \frac{1}{n}, U_i \cap U_j = \emptyset \forall i, j, i \neq j \right\},$$

where  $|A|$  denotes the diameter of any set  $A$ .

3. Let  $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$  be a countable family of closed subsets of  $E$  and let  $MGR(\mathcal{A})$  denote the  $\sigma$ -ideal of compact sets that are meager in each

member of  $\mathcal{A}$ . For each  $n$ , let  $\{W_{n,j} : j \in \mathbb{N}\}$  be a basis for  $A_n$  consisting of nonempty relatively open sets. The following sets then serve our purpose: for  $n, j \in \mathbb{N}$ ,

$$\mathcal{C}_{n,j} = \{U \text{ open} : U \subseteq E \setminus \{x\} \text{ for some } x \in W_{n,j}\}$$

and we may of course enumerate the sets  $\mathcal{C}_{n,j}$  as  $\mathcal{C}_n$ .

4. For the Hausdorff measure  $\mu^h$  based on the function  $h \in \mathcal{H}$  (see Section 3.3.5 for details), we have

$$\mathcal{C}_n = \left\{ \bigcup_{i=1}^k U_i : \sum_{i=1}^k h(|U_i|) < \frac{1}{n}, k \in \mathbb{N}, U_i \text{ open} \right\},$$

where  $|A|$  denotes the diameter of a subset  $A$  of  $E$ .

Note that in each case, the sets  $\mathcal{C}_n$  are downward closed collections of open sets, i.e., if  $U, V$  are open such that  $V \subseteq U \in \mathcal{C}_n$ , then  $V \in \mathcal{C}_n$ . Given such a sequence  $(\mathcal{C}_n)$ , consisting of downward closed families of open sets, we may naturally define the sequence  $(\mathcal{U}_n)$  of open subsets of  $\mathcal{K}(E)$  by setting

$$\mathcal{U}_n = \{K \in \mathcal{K}(E) : K \subseteq U \text{ for some } U \in \mathcal{C}_n\}.$$

It is clear that each  $\mathcal{U}_n$  is open and downward closed, and that  $I = \bigcap_n \mathcal{U}_n$ . In each example just mentioned, the sequence  $(\mathcal{U}_n)$  (corresponding to the respective collection  $(\mathcal{C}_n)$  described above) also satisfies condition (2.5) as in Theorem 7, viz.

$$\forall F \in \mathcal{U}_n \exists m \forall K \in \mathcal{U}_m F \cup K \in \mathcal{U}_n.$$

Given an ideal  $I = \bigcap_n \mathcal{U}_n$ , where the sets  $\mathcal{U}_n$  are as above (open, downward closed and satisfying (2.5)), we give in Proposition 9 an explicit sequence of closed

and upward closed sets  $\mathcal{F}_n \subseteq \mathcal{K}(E)$  representing  $I$ . We first present a useful lemma.

**Lemma 8.** *Let  $\mathcal{A} \subseteq \mathcal{K}(E)$  be upward closed and let  $K \in \mathcal{K}(E)$  be meager. Then  $K^*$  has nonempty interior in  $\mathcal{A}$  if and only if, for any nonempty open set  $U \subseteq E$  such that  $\mathcal{K}(U) \cap \mathcal{A} \neq \emptyset$ ,  $\mathcal{K}(U) \cap \mathcal{A} \not\subseteq K^*$ .*

*Proof.* Fix  $\mathcal{A}$  and a meager set  $K$  as in the statement of the lemma and assume the second part of the implication above. Let  $\mathcal{U} \subseteq \mathcal{K}(E)$  be an open set of the form  $\{F \in \mathcal{K}(E) : F \subseteq U_0, F \cap U_i \neq \emptyset \forall 1 \leq i \leq n\}$ , where  $U_0, \dots, U_n$  are open subsets of  $E$ . Suppose that  $\mathcal{U} \cap \mathcal{A} \neq \emptyset$ . Since  $\mathcal{K}(U_0) \cap \mathcal{A} \not\subseteq K^*$ , we have some  $F \in \mathcal{K}(U) \cap \mathcal{A}$  that is disjoint from  $K$ . Since  $K$  is meager, we may pick some  $x_i \in U_i \setminus K$  for each  $i = 1, \dots, n$ . The set  $F \cup \{x_i : 1 \leq i \leq n\}$  is now in  $\mathcal{U} \cap \mathcal{A} \not\subseteq K^*$ . The other direction of the implication is trivial.  $\square$

**Proposition 9.** *Let  $I \subseteq \mathcal{K}(E)$  contain only meager sets and let  $I = \bigcap_n \mathcal{U}_n$ , where the sets  $\mathcal{U}_n$ ,  $n \in \mathbb{N}$ , are open, downward closed subsets of  $\mathcal{K}(E)$  satisfying (2.5). Define, for  $n \in \mathbb{N}$ , the following upward closed subsets of  $\mathcal{K}(E)$ :*

$$\mathcal{A}_n = \{F \in \mathcal{K}(E) : \exists K \in \mathcal{U}_n \text{ such that } E \setminus F \subseteq K\}$$

and let  $\mathcal{F}_n = \overline{\mathcal{A}_n}$ . Then, for any  $K \in \mathcal{K}(E)$ ,

$$K \in I \Rightarrow \forall n \ K^* \text{ is meager in } \mathcal{F}_n;$$

$$K \notin I \Rightarrow \exists n \ \mathcal{F}_n \subseteq K^*.$$

*Proof.* It is immediate that, if  $K \notin I$ , then there exists  $n \in \mathbb{N}$  such that  $\mathcal{A}_n \subseteq K^*$ , and therefore  $\mathcal{F}_n \subseteq K^*$ . (Just pick  $n$  such that  $K \notin \mathcal{U}_n$ . If  $F$  was such that  $K$  was contained in  $E \setminus F$ , then  $E \setminus F$  could not be covered by any member of  $\mathcal{U}_n$ .) Conversely, we also wish to show that if  $K \in I$ , then  $K^*$  has empty interior in

each  $\mathcal{F}_n$ , or equivalently, in each  $\mathcal{A}_n$ . Note that since  $\mathcal{U}_n$  is closed downwards,  $F \in \mathcal{A}_n$  if and only if  $\overline{E \setminus F} \in \mathcal{U}_n$ . If  $\mathcal{U} \subseteq \mathcal{K}(E)$  is nonempty, open and intersects  $\mathcal{A}_n$ , we wish to show that  $\mathcal{U} \cap \mathcal{A}_n \not\subseteq K^*$ . By the preceding lemma, it suffices to restrict our attention to open sets of the form  $\mathcal{K}(U)$ , where  $U \subseteq E$  is open. Let therefore  $U$  be open such that  $\mathcal{K}(U) \cap \mathcal{A}_n \neq \emptyset$ . Let  $F \in \mathcal{K}(U) \cap \mathcal{A}_n$ . Since  $\overline{E \setminus F} \in \mathcal{U}_n$ , we may pick  $m$  such that for any  $L \in \mathcal{U}_m$ , we have  $\overline{E \setminus F} \cup L \in \mathcal{U}_n$ . Since  $K \in \mathcal{U}_m$  and  $\mathcal{U}_m$  is open, we may find an open set  $V \supseteq K$  such that  $\overline{V} \in \mathcal{U}_m$ . Then  $\overline{E \setminus (F \setminus V)} = \overline{(E \setminus F) \cup V} = \overline{E \setminus F} \cup \overline{V} \in \mathcal{U}_n$ , which shows that  $F \setminus V \subseteq \mathcal{K}(U) \cap \mathcal{A}_n \setminus K^*$ .  $\square$

# CHAPTER 3

## Characterizing $G_\delta$ sets in an ideal

We now turn our attention to  $G_\delta$  subsets of the compact metric space  $E$ . In the previous chapter we considered ideals of closed sets  $I$  with property  $(*)$ , along with a corresponding closed and upward closed set  $\mathcal{F} \subseteq \mathcal{K}(E)$  that determined membership of closed sets in the ideal via the map  $K \mapsto K^*$ . Given such a set  $\mathcal{F}$ , we may now look at the meagerness of  $A^*$  in  $\mathcal{F}$  for sets  $A$  that are not closed. Define the set

$$J = \{A \subseteq E : A^* \text{ meager in } \mathcal{F}\}, \quad (3.1)$$

which is clearly closed under the taking of subsets. Since for any sequence of sets  $A_n$  we have  $(\bigcup_n A_n)^* = \bigcup_n A_n^*$ , it is also closed under countable union. A closed set is in  $J$  precisely when it is in  $I$ ; in this chapter we examine conditions that determine membership of  $G_\delta$  sets in  $J$ . We also consider alternatives to the operator  $A \mapsto A^*$ .

### 3.1 General results

As mentioned in the introduction, in many cases the ideal  $I$  and its representing set  $\mathcal{F}$  arise from a notion of smallness (for example, meagerness, having measure zero, being of dimension 0, et cetera) that also applies to  $G_\delta$  sets. The small  $G_\delta$  sets determined by this notion, however, need not correspond to the  $G_\delta$  sets in the collection  $J$  defined in equation (3.1). Below we show that if  $G$  is a nonmeager  $G_\delta$  set and  $\mathcal{F}$  is any closed and upward closed subset of  $\mathcal{K}(E)$ , then  $G^*$  is nonmeager

in  $\mathcal{F}$ . Thus, if the notion of smallness used to define  $I$  yields any  $G_\delta$  sets that are small with respect to that notion and yet are nonmeager, then those  $G_\delta$  sets fail to be in  $J$ . (This is the case, for example, for a comeager  $G_\delta$  set of measure 0, for a Borel probability measure defined on  $E$ .)

We use the following definitions.

**Definition 10.** For any set  $A \subseteq E$ , define

$$A'' = \{K \in \mathcal{K}(E) : K \neq \emptyset \text{ and } A \cap K \text{ comeager in } K\}.$$

For  $W \subseteq E$ , define

$$A''_W = \{K \in \mathcal{K}(E) : K \cap W \neq \emptyset \text{ and } A \cap K \cap W \text{ comeager in } K \cap W\}.$$

The sets  $A''$  and  $A''_W$  are clearly both contained in  $A^*$ . Note also that for any sequence  $(A_n)_{n \in \mathbb{N}}$  of subsets of  $E$ , we have  $(\bigcap_n A_n)'' = \bigcap_n A_n''$  and  $(\bigcap_n A_n)''_W = \bigcap_n (A_n)''_W$ .

**Definition 11.** For any set  $A \subseteq E$ , define

$$A^+ = \{K \in \mathcal{K}(E) : \forall F_n \in I, n \in \mathbb{N}, K \cap A \not\subseteq \bigcup_n F_n\}.$$

Note that if  $A$  is closed and  $I$  has property  $(*)$ , then  $A^+ = \{K \in \mathcal{K}(E) : K \cap A \notin I\}$ .

**Theorem 12.** Let  $\mathcal{F}$  be a closed and upward closed subset of  $\mathcal{K}(E)$ . Let  $G$  be a  $G_\delta$  set. If  $G$  is comeager, then  $G''$  is comeager in  $\mathcal{F}$ . If  $G$  is comeager in an open nonempty set  $W \subseteq E$ , then  $G''_W$  is comeager in  $W^* \cap \mathcal{F}$ .

*Proof.* Let  $G$  be a dense  $G_\delta$  subset of  $E$ , say  $G = \bigcap_n U_n$  with each  $U_n$  open dense. We will show that  $\bigcap_n U_n'' \cap \mathcal{F}$  is comeager in  $\mathcal{F}$ . Let  $\{W_m\}_{m \in \mathbb{N}}$  be a basis of  $E$ .

Fix  $n$  and consider  $U_n''$ . Since any  $K \in \mathcal{K}(E)$  is in  $U_n''$  if and only if  $K \cap U_n$  is dense in  $K$ , we may write

$$U_n'' = \bigcap_m \{K \in \mathcal{K}(E) : (K \cap W_m \neq \emptyset \implies K \cap W_m \cap U_n \neq \emptyset)\}.$$

Fix  $m$  as well now and consider the  $m^{\text{th}}$  set in the countable intersection above; let us call this set  $\mathcal{B}$ . This is a  $G_\delta$  subset of  $\mathcal{K}(E)$  and we show that it is dense in  $\mathcal{F}$ . It will then follow that each  $U_n''$ , and therefore  $G''$ , is comeager in  $\mathcal{F}$ .

Let  $\mathcal{U}$  be an open subset of  $\mathcal{K}(E)$  of the form

$$\{K \in \mathcal{K}(E) : K \subseteq V_0, K \cap V_i \neq \emptyset \forall 1 \leq i \leq N\}$$

for open sets  $V_0, \dots, V_N$ , and suppose  $\mathcal{U} \cap \mathcal{F} \neq \emptyset$ . Let  $K \in \mathcal{U} \cap \mathcal{F}$ . If  $K \cap W_m = \emptyset$ , then  $K \in \mathcal{B} \cap \mathcal{U} \cap \mathcal{F}$ . If  $K \cap W_m \neq \emptyset$ , then  $V_0 \cap W_m \neq \emptyset$ . Since  $U_n$  is dense in  $E$ , we may pick some  $x_0 \in V_0 \cap W_m \cap U_n$ . Let  $L = K \cup \{x_0\}$ . Since  $\mathcal{F}$  is upward closed,  $L \in \mathcal{B} \cap \mathcal{U} \cap \mathcal{F}$ . This shows that  $\mathcal{B}$  is dense in  $\mathcal{F}$ , proving the first part of the theorem.

Now suppose  $G$  is dense in some nonempty open set  $W$ . The argument above with  $G''_W$  in place of  $G''$  and  $W^* \cap \mathcal{F}$  in place of  $\mathcal{F}$ , now shows that  $G''_W$  is comeager in  $W^* \cap \mathcal{F}$ . For the sake of completeness we repeat the argument here. We have  $G''_W = \bigcap_n (U_n)''_W$ , and showing that each  $(U_n)''_W$  is comeager in  $W^* \cap \mathcal{F}$  amounts to showing that for each  $m \in \mathbb{N}$ , the  $G_\delta$  set

$$\{K \in \mathcal{K}(E) : K \cap W \neq \emptyset, K \cap W \cap W_m \neq \emptyset \implies K \cap W \cap W_m \cap U_n \neq \emptyset\}$$

is dense in  $W^* \cap \mathcal{F}$ . Fix  $n$  and  $m$  and call the displayed set  $\mathcal{C}$  for convenience. Let  $\mathcal{U}$  be an open subset of  $\mathcal{K}(E)$  of the form  $\{K \in \mathcal{K}(E) : K \subseteq V_0, K \cap V_i \neq \emptyset \forall 1 \leq i \leq N\}$  for open sets  $V_0, \dots, V_N$ , and suppose  $\mathcal{U} \cap W^* \cap \mathcal{F} \neq \emptyset$ . Let

$K \in \mathcal{U} \cap W^* \cap \mathcal{F}$ . This  $K$  obviously intersects  $W$ . If  $K \cap W \cap W_m = \emptyset$ , then  $K \in \mathcal{C} \cap \mathcal{U} \cap W^* \cap \mathcal{F}$ . If  $K \cap W \cap W_m \neq \emptyset$ , then  $V_0 \cap W \cap W_m \neq \emptyset$ . Since  $U_n$  is dense in  $W$ , we may pick some  $x_0 \in V_0 \cap W \cap W_m \cap U_n$ . Let  $L = K \cup \{x_0\}$ . Since  $\mathcal{F}$  is upward closed,  $L \in \mathcal{C} \cap \mathcal{U} \cap W^* \cap \mathcal{F}$ , and we are done.  $\square$

**Corollary 13.** *Let  $\mathcal{F}$  be a closed and upward closed subset of  $\mathcal{K}(E)$ . Let  $G$  be a  $G_\delta$  set. If  $G$  is comeager, then  $G^*$  is comeager in  $\mathcal{F}$ . If  $G$  is nonmeager, then  $G^*$  is nonmeager in  $\mathcal{F}$ .*

*Proof.* This follows from the previous theorem and a couple of observations: the first is that  $G''$  and  $G''_W$  are contained in  $G^*$  (for any  $W$ ); the second is that if  $W \subseteq E$  is an open nonempty set in which  $G$  is dense, then  $W^* \cap \mathcal{F}$  is an open nonempty subset of  $\mathcal{F}$ . (It is nonempty because  $\mathcal{F}$  is upward closed.)  $\square$

Theorem 15 shows that under some additional assumptions, an analogous result is true of  $G^+$  as well.

**Lemma 14.** *Let  $E$  be a compact Polish space. Fix a clopen set  $W \subseteq E$  and an open set  $\mathcal{U} \subseteq \mathcal{K}(E)$ . Then the set  $\{F \in \mathcal{K}(E) : F \setminus W \in \mathcal{U}\}$  is open.*

*Proof.* We may write  $\mathcal{U} = \bigcup_n \mathcal{U}_n$ , where each  $\mathcal{U}_n$  is the finite intersection of sets in the usual subbasis of  $\mathcal{K}(E)$ . (This subbasis consists of sets of the form  $\mathcal{K}(U)$  and  $U^*$  for open  $U \subseteq E$ .) It therefore suffices to show the lemma holds for subbasic open sets  $\mathcal{U}$ , i.e., if  $U \subseteq E$  is an open set, then the sets  $\{F \in \mathcal{K}(E) : F \setminus W \subseteq U\}$  and  $\{F \in \mathcal{K}(E) : F \setminus W \in U^*\}$  are open. To see this, we simply note that

$$\{F \in \mathcal{K}(E) : F \setminus W \subseteq U\} = \{F \in \mathcal{K}(E) : F \subseteq U \cup W\}$$

and

$$\{F \in \mathcal{K}(E) : F \setminus W \in U^*\} = \{F \in \mathcal{K}(E) : F \in (U \setminus W)^*\}.$$

$\square$

**Theorem 15.** *Let  $E$  be zero dimensional. Let  $\mathcal{F} \subseteq \mathcal{K}(E)$  be closed and upward closed, and let  $I = \{K \in \mathcal{K}(E) : K^* \text{ meager in } \mathcal{F}\}$ . Suppose that  $\mathcal{F}$  also satisfies the following two conditions:*

- (A) *If  $K \in \mathcal{F}$  and  $A \subseteq E$  is a clopen set such that  $K \cap A \in I$ , then  $K \setminus A \in \mathcal{F}$ .*
- (B) *Every basic open subset of  $\mathcal{F}$  contains a  $\subseteq$ -minimal element with no isolated points.*

*Then, if  $G$  is comeager, then  $G^+$  is comeager in  $\mathcal{F}$ . If  $G$  is nonmeager, then  $G^+$  is nonmeager in  $\mathcal{F}$ .*

*Proof.* Let  $\{W_n\}_{n \in \mathbb{N}}$  be a basis of  $E$  consisting of clopen sets and let  $G$  be a dense  $G_\delta$  subset of  $E$ . We have shown in the proof of Theorem 12 that  $G''$  is comeager in  $\mathcal{F}$ . So to show that  $G^+$  is comeager in  $\mathcal{F}$ , it suffices to show that  $G'' \setminus G^+$  is meager in  $\mathcal{F}$ . For any  $K \in G'' \setminus G^+$ ,  $K \cap G$  is dense in  $K$  and can be covered by  $\bigcup_n F_n$  for some closed  $F_n \in I$ . Now  $\bigcup_n F_n$  is dense in  $K$ , and therefore  $\bigcup_n F_n^\circ$  is dense in  $K$ , where the interior indicated is the interior in  $K$ . This means that  $K$  has a dense relatively open subset covered by countably many sets in  $I$ . Since  $I$  has property (\*),  $K$  must have a nonempty relatively clopen subset in  $I$ . The set  $\mathcal{F} \cap G'' \setminus G^+$  is therefore contained in the complement in  $\mathcal{F}$  of the set

$$P = \{K \in \mathcal{F} : \forall n, \text{ if } W_n \cap K \neq \emptyset, \text{ then } W_n \cap K \notin I\}$$

and thus it suffices to show that the set  $P$  is comeager in  $\mathcal{F}$ . Condition (A) implies that  $P$  contains the set

$$Q = \{K \in \mathcal{F} : \forall n \text{ if } W_n \cap K \neq \emptyset, \text{ then } K \setminus W_n \notin \mathcal{F}\}$$

and we shall show that  $Q$  is comeager in  $\mathcal{F}$ . By Lemma 14,  $Q$  is  $G_\delta$ , and so it suffices to show that  $Q$  is dense in  $\mathcal{F}$ , i.e.,  $Q$  intersects every nonempty open subset

of  $\mathcal{F}$ . Let  $\mathcal{U} \subseteq \mathcal{K}(E)$  be an open such that  $\mathcal{U} \cap \mathcal{F} \neq \emptyset$ ; since  $E$  is zero dimensional we may assume that  $\mathcal{U} = \{K \in \mathcal{K}(E) : K \subseteq U_1 \cup \dots \cup U_k, K \cap U_i \neq \emptyset \forall 1 \leq i \leq k\}$  for some disjoint nonempty open sets  $U_1, \dots, U_k$ .

By condition (B), we can find in  $\mathcal{U} \cap \mathcal{F}$  a  $\subseteq$ -minimal element, say  $K$ , with no isolated points. We claim that  $K \in Q$ . Fix  $n$  such that  $W_n \cap K \neq \emptyset$ . Let  $x \in W_n \cap K$  and let  $U_i$  be the open set such that  $x \in U_i$ . Since  $x$  is not an isolated point of  $K$ , we may find an open set  $V$  such that  $x \in V \subseteq W_n \cap U_i$  and  $K \setminus V$  still intersects  $U_i$ . Since  $K \setminus V \in \mathcal{U}$ , it must be that  $K \setminus V \notin \mathcal{F}$  (because  $K$  was minimal in  $\mathcal{U} \cap \mathcal{F}$ .) Since  $\mathcal{F}$  is upward closed, we also have  $K \setminus W_n \notin \mathcal{F}$ . So the claim holds and  $Q$  is dense in  $\mathcal{F}$ .

Now suppose  $G$  is dense in some nonempty clopen set  $W$ . Along the same lines as above, consider the set  $G''_W$  defined in Theorem 12. For any  $K \in G''_W \setminus G^+$ , the set  $K \cap G$  is nonempty and dense in  $K \cap W$  and can be covered by  $\bigcup_n F_n$  for a sequence of closed sets  $F_n \in I$ . Since  $\bigcup_n F_n$  is dense in  $K \cap W$ , so is  $\bigcup_n F_n^o$ , where the interior indicated is the interior in  $K$ . In other words,  $K \cap W$  has a dense relatively open subset that is covered by a countable union of sets in  $I$ . So

$$G''_W \setminus G^+ \subseteq \{K : K \text{ has a nonempty relatively clopen subset in } I\}.$$

We have already shown that this set is meager in  $\mathcal{F}$  and  $G''_W$  is comeager in  $W^* \cap \mathcal{F}$ . So  $G^+$  is comeager in  $W^* \cap \mathcal{F}$ .  $\square$

Note that the set  $\mathcal{F}$  described in Chapter 2 that characterizes null sets for an atomless Borel probability measure does satisfy the conditions of Theorem 15. In fact, if  $E$  is zero dimensional, then each closed set in the representing sequence described in Proposition 9 satisfies condition (A) of the theorem. Recall from that proposition that for an ideal  $I$  we obtained a representing sequence  $(\mathcal{F}_n)$  by first defining sets  $\mathcal{A}_n$  and then setting  $\mathcal{F}_n = \overline{\mathcal{A}_n}$  for  $n \in \mathbb{N}$ . It is straightforward (and

illustrated in the proof of Proposition 9) that Condition (A) holds for each  $\mathcal{A}_n$ . The fact that it also holds for  $\mathcal{F}_n$  then follows from the lemma below.

**Lemma 16.** *Let  $E$  be zero dimensional and let  $F, F_n \in \mathcal{K}(E)$ ,  $n \in \mathbb{N}$ , such that  $F_n \rightarrow F$ . Let  $A \subseteq E$  be clopen. Then  $F_n \setminus A \rightarrow F \setminus A$ .*

*Proof.* Let  $F, F_n, A$  be as above and let  $\mathcal{U} \subseteq \mathcal{K}(E)$  be an open set of the form  $\{K \in \mathcal{K}(E) : K \subseteq V_0, K \cap V_i \neq \emptyset \forall 1 \leq i \leq N\}$  for open sets  $V_0, \dots, V_N$ . Suppose that  $F \setminus A \in \mathcal{U}$ . By replacing each  $U_i$ ,  $1 \leq i \leq N$  with the nonempty open set  $U_i \setminus A$ , we may assume that each  $U_i$  is disjoint from  $A$ . Now consider the open set  $\mathcal{U}' = \{K \in \mathcal{K}(E) : K \subseteq V_0 \cap A, K \cap V_i \neq \emptyset \forall 1 \leq i \leq N\}$ . There exists some  $n_0$  such that for any  $n \geq n_0$ ,  $F_n \in \mathcal{U}'$ . This means that for any  $n \geq n_0$ ,  $F_n \setminus A \in \mathcal{U}$ . □

## 3.2 A conjecture

Consider a  $G_\delta$  ideal  $I$  of compact sets, with property (\*), and the set  $\mathcal{F}$  (or sequence  $(\mathcal{F}_n)$ ) that represents it. The notion of smallness that defines  $I$  might also apply in a natural way to  $G_\delta$  sets — as familiar examples consider meagerness of a set or the condition of being a zero set for some measure. In such a case, while for a closed set  $K$  the meagerness of  $K^*$  in  $\mathcal{F}$  (or in each  $\mathcal{F}_n$ ) corresponds to the smallness of  $K$ , for a  $G_\delta$  set  $G$  this correspondence may break down. It is however clear that in all cases, if  $G$  can be covered by a countable number of closed sets in  $I$ , then  $G^*$  must be meager in  $\mathcal{F}$  (or in each  $\mathcal{F}_n$ ). We conjecture that any  $G_\delta$  ideal  $I$  with property (\*) has a representing sequence  $(\mathcal{F}_n)$  for which those  $G$  that can be covered by a countable number of sets in  $I$  are the *only*  $G_\delta$  sets for which  $G^*$  will be meager in each  $\mathcal{F}_n$ .

**Conjecture 1.** *Let  $I$  be a  $G_\delta$  ideal of compact sets, with property (\*), and suppose*

that  $I$  contains only meager sets. Then there exists a sequence of nonempty sets  $\mathcal{F}_n \subseteq \mathcal{K}(E)$ , closed and upward closed, such that

$$\forall G \in G_\delta(E)$$

$$(\forall n \ G^* \text{ meager in } \mathcal{F}_n) \Leftrightarrow (\exists (K_n)_{n \in \mathbb{N}} \subseteq I, G \subseteq \bigcup_{n \in \mathbb{N}} K_n). \quad (3.2)$$

Note that in the presence of property  $(*)$ , if (3.2) holds then the sets  $\mathcal{F}_n$  form a representing sequence for  $I$ : if  $K \subseteq E$  is closed then  $K^*$  is meager in each  $\mathcal{F}_n$  if and only if  $K$  is in  $I$ . Note also that the condition that  $I$  not contain any nonmeager sets is required. In fact, if  $I$  contains a nonmeager set, then  $I$  cannot be represented by upward closed sets at all. (This is because, as has been mentioned earlier, for any upward closed set  $\mathcal{F} \subseteq \mathcal{K}(E)$  and nonempty open set  $U \subseteq E$ , the set  $U^* \cap \mathcal{F}$  is a nonempty open subset of  $\mathcal{F}$ .)

We will approach the conjecture through an equivalent condition, as follows. Let  $G_\delta(E)$  denote the collection of  $G_\delta$  subsets of  $E$ . Let  $I$  be an ideal of compact sets with property  $(*)$ , and let  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  be a representing sequence for  $I$ , so that

$$\forall K \in \mathcal{K}(E) \ (K^* \text{ meager in each } \mathcal{F}_n \Leftrightarrow K \in I).$$

Then in the conjecture, the implication from right to left in (3.2) is always true, for if  $G \subseteq \bigcup_n K_n$ , then  $G^* \subseteq \bigcup_n K_n^*$ . The substance of the conjecture then lies in the implication from left to right, i.e., we would wish to show that

$$\forall G \in G_\delta(E) \ (G \text{ cannot be covered by countably many sets in } I$$

$$\Rightarrow \exists n \ G^* \text{ is nonmeager in } \mathcal{F}_n). \quad (3.3)$$

This condition is equivalent to the following:

$\forall G \in G_\delta(E)$

$$(\forall U \text{ open } U \cap G \neq \emptyset \Rightarrow \overline{U \cap G} \notin I) \Rightarrow \exists n \ G^* \text{ is nonmeager in } \mathcal{F}_n. \quad (3.4)$$

To see this equivalence, let (3.4) hold and suppose  $G$  is a  $G_\delta$  set not covered by the countable union of any sets in  $I$ . Set  $G_0 = G$  and recursively define for successor ordinals  $\alpha + 1$  and limit ordinals  $\lambda$  the sets

$$\begin{aligned} G_{\alpha+1} &= G_\alpha \setminus \bigcup \{U : U \subseteq G_\alpha \text{ relatively open in } G_\alpha \text{ and } \overline{U} \in I\}; \\ G_\lambda &= \bigcap_{\alpha < \lambda} G_\alpha. \end{aligned}$$

For each ordinal  $\alpha$ ,  $G_\alpha$  is a closed subset of  $G$ . So for some  $\alpha_0 < \omega_1$ ,  $G_{\alpha_0} = G_{\alpha_0+1}$ . If  $G_{\alpha_0}$  were empty, then  $G$  would be covered by a countable number of sets  $U$  with  $\overline{U} \in I$ , a contradiction. So  $G_{\alpha_0}$  is nonempty, and the closure of every relatively open nonempty subset of it is outside  $I$ . Now by (3.4),  $G_{\alpha_0}^*$  is nonmeager in some  $\mathcal{F}_n$  and therefore so is  $G^*$ .

Conversely, let (3.3) hold and suppose that  $G$  is a  $G_\delta$  set such that for every nonempty relatively open  $U \subseteq G$ , we have  $\overline{U} \notin I$ . Suppose  $G \subseteq \bigcup_n F_n$  for some closed sets  $F_n \in I$ .  $G$  is nonempty and Polish,  $G = \bigcup_n F_n \cap G$  and each  $F_n \cap G$  is relatively closed in  $G$ . So there exist  $n \in \mathbb{N}$  and a set  $U$  relatively open in  $G$  such that  $\emptyset \neq U \subseteq F_n \cap G$ . Now  $\overline{U} \subseteq \overline{F_n \cap G} \subseteq F_n \in I$ , a contradiction.

### 3.3 The conjecture for some natural examples

In this section we examine the following  $G_\delta$  ideals of compact sets with property (\*):  $MGR(\mathcal{A})$  for a countable family  $\mathcal{A}$  of closed sets (defined as the collection of compact sets meager in each member of  $\mathcal{A}$ ), the null ideal for an atomless finite Borel measure, the ideal of closed zero dimensional sets, and the null ideal

for certain submeasures, including Hausdorff measure. As described in Section 2.2, each of these ideals is associated with a natural sequence  $(\mathcal{F}_n)$  that works to characterize membership in the ideal for closed sets. In each case we try to establish the conjecture by establishing condition (3.4) for this particular sequence. In the case of  $MGR(\mathcal{A})$  and Borel measures, we prove the conjecture outright; for zero dimensional sets we prove it in the presence of an extra condition on  $E$ . In our investigation of Hausdorff measures we describe the difficulty presented by the natural representing sequence that leads us to construct an alternative sequence. This construction is described in the main theorem of Section 3.3.6, which applies more generally to a certain class of submeasures. Our approach to proving condition (3.4) uses a Banach-Mazur game that we set up along the same lines in each theorem. This game is described in Section 3.3.1 below.

### 3.3.1 The general approach

What follows is the common account of the steps we follow to prove the conjecture in various settings. Given the ideal  $I = \bigcap_{n \in \mathbb{N}} \mathcal{U}_n$ , where the sets  $\mathcal{U}_n \subseteq \mathcal{K}(E)$  are open and downward closed, it is appropriate to think of the sets in  $\mathcal{U}_n$  as ‘small’ in some sense (though obviously not in the sense of belonging to  $I$ ). Defining a sequence  $(\mathcal{A}_n)_{n \in \mathbb{N}}$  by the formula

$$\mathcal{A}_n = \{F \in \mathcal{K}(E) : \overline{E \setminus F} \in \mathcal{U}_n\},$$

or a suitable variant, we may think of the sets in  $\mathcal{A}_n$  as being ‘big’. We then set  $\mathcal{F}_n = \overline{\mathcal{A}_n}$  to obtain sets such that for any  $K \in \mathcal{K}(E)$ ,

$$K \in I \implies K^* \text{ is meager in } \mathcal{F}_n \forall n \in \mathbb{N}, \text{ and}$$

$$K \notin I \implies \exists n \in \mathbb{N}, \text{ such that } \mathcal{F}_n \subseteq K^*.$$

Once we establish these conditions, showing that the conjecture holds for this sequence is equivalent to showing that condition (3.4) holds, i.e., we wish to show that for any  $G_\delta$  set  $G$ , if  $\bar{U} \notin I$  for all nonempty relatively open sets  $U \subseteq G$ , then  $G^*$  is nonmeager in some  $\mathcal{F}_n$ . Let therefore  $G$  be such a set. Since  $\bar{G} \notin I$ , there exists some  $m \in \mathbb{N}$  such that  $\mathcal{F}_m \subseteq \bar{G}^*$ . We will fix such an  $m$  and play the Banach-Mazur game in  $\mathcal{F}_m$ , on the set  $G^*$ . In this game, Players I and II take turns playing nonempty open subsets of  $\mathcal{F}_m$  as follows:

$$\text{Player I } \quad \mathcal{U}_0 \quad \mathcal{U}_1 \quad \dots$$

$$\text{Player II } \quad \mathcal{V}_0 \quad \mathcal{V}_1 \quad \dots$$

satisfying  $\mathcal{U}_0 \supseteq \mathcal{V}_0 \supseteq \mathcal{U}_1 \supseteq \mathcal{V}_1 \supseteq \dots$ . Player II wins this run of the game if  $\bigcap_n \mathcal{V}_n (= \bigcap_n \mathcal{U}_n) \subseteq G^*$ . The key fact about this game is that Player II has a winning strategy if and only if  $G^*$  is comeager in  $\mathcal{F}_m$ ; see, for example, Section 8.H of [7] for a proof. We will describe a winning strategy for Player II. Our convention will be that the sets  $\mathcal{U}_i$  and  $\mathcal{V}_i$  will be open sets of  $\mathcal{K}(E)$  satisfying  $\mathcal{U}_i \cap \mathcal{F}_m \neq \emptyset$  and  $\mathcal{V}_i \cap \mathcal{F}_m \neq \emptyset$  for each  $i \in \mathbb{N}$ . Further, we may assume that the sets  $\mathcal{U}_i$  and  $\mathcal{V}_i$  are basic sets of the form  $\{F : F \subseteq U_0, F \cap U_i \neq \emptyset \forall 1 \leq i \leq k\}$ , for some  $k \in \mathbb{N}$  and open sets  $U_0, \dots, U_k \subseteq E$ .

Let  $(F_n)$  be an increasing sequence of closed relatively meager subsets of  $\bar{G}$  such that  $G = \bar{G} \setminus \bigcup_n F_n$ . Playing as Player II, we want to ensure that  $\bigcap_n \mathcal{V}_n \subseteq G^*$ , i.e., for any  $K$  in  $\bigcap_n \mathcal{V}_n$ ,  $K$  intersects  $\bar{G}$  in some point that avoids each of the  $F_n$ . The usefulness of this game lies in enabling us to deal with the sets  $F_n$  one at a time. At the  $n^{\text{th}}$  stage of the game we will construct a set  $D_n$ , a nonempty relatively open subset of  $\bar{G}$ , such that  $\bar{D}_n \cap F_n = \emptyset$ , and we will play a set  $\mathcal{V}_n$  such that  $\mathcal{V}_n \cap \mathcal{F}_m \subseteq \bar{D}_n^*$ . As long as we ensure at each stage that  $D_n \subseteq D_{n-1}$  for  $n \geq 1$ , the compactness of  $E$  makes this a winning strategy for II.

To illuminate somewhat how  $D_n$  and  $\mathcal{V}_n$  are obtained, let us look at the initial

move. Since  $m$  was chosen so as to ensure that  $\mathcal{F}_m \subseteq \overline{G}^*$ , the starting set  $\mathcal{U}_0$  played by Player I satisfies  $\emptyset \neq \mathcal{U}_0 \cap \mathcal{F}_m \subseteq \overline{G}^*$ . Suppose that

$$\mathcal{U}_0 = \{F : F \subseteq U_0, F \cap U_i \neq \emptyset \forall 1 \leq i \leq k\},$$

where the sets  $U_i \subseteq E$  are open. Note that since  $\emptyset \neq \mathcal{U}_0 \cap \mathcal{F}_m \subseteq \overline{G}^*$ , the set  $U_0 \cap \overline{G}$  is nonempty. Since  $F_0$  is meager in  $\overline{G}$ , we can certainly find within  $U_0$  some nonempty relatively open subset of  $\overline{G}$ , say  $D_0$ , whose closure avoids  $F_0$  — so far, so good. We also know that  $\overline{D_0} \notin I$ . The difficulty lies in the fact that, although  $\overline{D_0}$  is big in the sense of not belonging to  $I$ , we must expect it to be much smaller than  $\overline{G}$  — while we do have some  $m' \in \mathbb{N}$  such that  $\mathcal{F}_{m'} \subseteq \overline{D_0}^*$ , that  $m'$  is going to be different from our  $m$ . (In those settings where the sets  $\mathcal{F}_i$  are decreasing,  $m'$  can be expected to be much larger than  $m$ .) The solution we employ is to ‘tighten’ the set  $U_0$  about  $\overline{D_0}$ , leaving it just big enough to accomodate some member of  $\mathcal{F}_m$ , but just small enough so that, if the set  $\overline{D_0}$  were to be removed from it, it could no longer accomodate any member of  $\mathcal{F}_m$ . In this way we obtain a set  $V_0$ , say, contained in  $U_0$ , such that  $\emptyset \neq \mathcal{K}(V_0) \cap \mathcal{F}_m \subseteq \overline{D_0}^*$ . Of course, we also have to ensure that  $\mathcal{K}(V_0) \cap \mathcal{U}_0 \cap \mathcal{F}_m \neq \emptyset$ . Player II plays  $\mathcal{V}_0 = \mathcal{K}(V_0) \cap \mathcal{U}_0$  and repeats this process at all the subsequent stages. The details that enable the construction vary according to the context, but they have in common the idea of ‘tightening’ described here.

### 3.3.2 The ideal $MGR(\mathcal{A})$ , for $\mathcal{A}$ a countable family of closed sets

Let  $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$  be a countable collection of closed sets, and let

$$I = MGR(\mathcal{A}) = \{K \in \mathcal{K}(E) : \forall n K \cap A_n \text{ meager in } A_n\}.$$

Assume that  $\mathcal{A}$  is such that all sets in  $I$  have empty interiors, i.e.,  $\bigcup \mathcal{A}$  is dense in  $E$ . We define a sequence of sets characterizing membership of closed sets in  $I$ , and show that the conjecture holds of this sequence.

Before considering the sequence, we first consider a single closed set  $A \subseteq E$ . Fix a basis of  $E$  and let  $\{W_n\}_{n \in \mathbb{N}}$  be an enumeration of those basic sets that intersect  $A$ . For  $n \in \mathbb{N}$ , define

$$\mathcal{F}_n = \overline{W_n \cap A}^*.$$

For any  $K \in \mathcal{K}(E)$ , if  $K \cap A$  is nonmeager in  $A$  then  $\mathcal{F}_n \subseteq K^*$  for some  $n \in \mathbb{N}$ . (Just pick  $n$  such that  $\emptyset \neq W_n \cap A \subseteq K$ .) Conversely, suppose  $K$  is meager and  $K \cap A$  is meager in  $A$ , and fix  $n$ . Since  $K$  is meager and  $\mathcal{F}_n$  is upward closed, to show that  $K^*$  is meager in  $\mathcal{A}_n$  it suffices to show that for any open set  $U$  with  $\mathcal{K}(U) \cap \mathcal{F}_n \neq \emptyset$ , the set  $(\mathcal{K}(U) \cap \mathcal{F}_n) \setminus K^*$  is nonempty. Since  $\mathcal{K}(U) \cap \mathcal{F}_n \neq \emptyset$ ,  $U \cap W_n \cap A$  is a nonempty open subset of  $A$ . Since  $K$  is meager in  $A$ , the set  $(U \cap W_n \cap A) \setminus K$  is nonempty. For any  $x$  in this set, we have  $\{x\} \in (\mathcal{K}(U) \cap \mathcal{F}_n) \setminus K^*$ .

**Theorem 17.** *Let  $A \subseteq E$  be closed and let  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  be the sequence defined above, i.e.,  $\mathcal{F}_n = \overline{W_n \cap A}^*$ , where  $W_n$  is an enumeration of basic sets intersecting  $A$ . Let  $G \subseteq E$  be a  $G_\delta$  set such that  $\overline{G}$  is not meager in  $A$ . Then for some  $n \in \mathbb{N}$ ,  $G^*$  is comeager in  $\mathcal{F}_n$ .*

*Proof.* Let  $G = \overline{G} \setminus \bigcup_n F_n$ , where  $(F_n)$  is an increasing sequence of closed relatively meager subsets of  $\overline{G}$ . Since  $\overline{G}$  is nonmeager in  $A$ , there exists  $m \in \mathbb{N}$  such that  $\emptyset \neq W_m \cap A \subseteq \overline{G}$ . As described in Section 3.3.1, we will fix such an  $m$  and play the Banach-Mazur game in  $\mathcal{F}_m$  on  $G^*$ , following the usual rules. Player I will play sets  $\mathcal{U}_n$  and Player II will respond with  $\mathcal{V}_n$ ; we will show that Player II has a winning strategy, i.e., a strategy that ensures  $\bigcap_n \mathcal{V}_n \cap \mathcal{F}_m \subseteq G^*$ .

At stage  $n$ , Player II will construct the following:

- $D_n$ , a nonempty relatively open subset of  $\overline{G}$ ,
- $V_n$ , a nonempty open subset of  $E$ ,

satisfying the following conditions:

- $D_n \subseteq D_{n-1}$  for  $n \geq 1$ ,
- $\overline{D_n} \cap F_n = \emptyset$ ,
- $\mathcal{K}(V_n) \cap \mathcal{U}_n \cap \mathcal{F}_m \neq \emptyset$ ,
- $\mathcal{K}(V_n) \cap \mathcal{F}_m \subseteq D_n^*$ .

Player II will play  $\mathcal{V}_n = \mathcal{K}(V_n) \cap \mathcal{U}_n$ . We then have  $\mathcal{V}_n \cap \mathcal{F}_m \subseteq D_n^*$ . The conditions on the sets  $D_n$  now ensure that this is a winning strategy for II.

For this ideal the construction of the sets  $V_n$  and  $D_n$  is not difficult. Let  $D_{-1} = W_m \cap \overline{G}$ , a nonempty relatively open subset of  $\overline{G}$ . Suppose that the set  $\mathcal{U}_n$  played by Player I satisfies  $\emptyset \neq \mathcal{U}_n \cap \mathcal{F}_m \subseteq D_{n-1}^*$ . Let  $U_0 \subseteq E$  be open and let  $U_1, \dots, U_k$  be open subsets of  $U_0$  such that  $\mathcal{U}_n = \{F : F \subseteq U_0, F \cap U_i \neq \emptyset \forall 1 \leq i \leq k\}$ . For each  $i = 0, \dots, k$ , two cases arise:

Case(i):  $U_i \cap W_m \cap A \neq \emptyset$ . In this case, we have  $U_i \cap W_m \cap \overline{G} \neq \emptyset$ . We may therefore pick  $W_{j_i} \subseteq U_i \cap W_m$  such that  $W_{j_i} \cap \overline{G} \neq \emptyset$  and  $\overline{W_{j_i} \cap \overline{G}} \cap F_n = \emptyset$ . Set  $D^i = W_{j_i} \cap \overline{G}$ ,  $V^i = W_{j_i}$ . Note that since  $\mathcal{U}_n \cap \mathcal{F}_m \neq \emptyset$ , this case covers  $i = 0$ .

Case(ii):  $U_i \cap W_m \cap A = \emptyset$ . In this case, set  $D^i = \emptyset$ ,  $V^i = U_i$ .

Now set

$$D_n = \bigcup_{i=0}^k D^i, \quad V = \bigcup_{i=0}^k V^i.$$

It is clear that  $\mathcal{K}(V) \cap \mathcal{U}_n \cap \mathcal{F}_m \neq \emptyset$  and  $\overline{D_n} \cap F_n = \emptyset$ . To see that  $\mathcal{K}(V) \cap \mathcal{F}_m \subseteq D_n^*$ , let  $F \in \mathcal{K}(V) \cap \mathcal{F}_m$ . Let  $x \in F \cap \overline{W_m \cap A} \subseteq \overline{G}$ . Pick  $i$  such that  $x \in V^i \subseteq U_i$ . For this  $i$ ,  $U_i \cap \overline{W_m \cap A} \neq \emptyset$ , and so  $U_i \cap W_m \cap A \neq \emptyset$ , i.e., Case(i) must hold for  $i$ . This means that  $V^i = W_{j_i}$ . So  $x \in W_{j_i} \cap \overline{G} = D^i \subseteq D_n$ .  $\square$

Now consider the countable collection of closed sets  $\mathcal{A} = \{A_j\}$  and the ideal  $I = MGR(\mathcal{A})$ . Fix a basis of  $E$  and for each  $j$ , let  $\{W_n^j\}_{n \in \mathbb{N}}$  be an enumeration of those basic sets that intersect  $A_j$ . For  $j, n \in \mathbb{N}$ , define

$$\mathcal{F}_n^j = \overline{W_n^j \cap A_j}^*.$$

Then, if  $K \cap A_j$  is nonmeager in  $A_j$ , then  $\mathcal{F}_n^j \subseteq K^*$  for some  $n$ , and if  $K \in I$ , then  $K^*$  is meager in each  $\mathcal{F}_n^j$ . The theorem above implies the following.

**Corollary 18.** *Let  $\mathcal{A}, I, \mathcal{F}_n^j$  be as above. Let  $G \subseteq E$  be a  $G_\delta$  set such that  $\overline{G} \notin I$ . Then there exist  $j, m \in \mathbb{N}$  such that  $G^*$  is comeager in  $\mathcal{F}_m^j$ .*

**Corollary 19.** *Let  $\mathcal{A}, I, \mathcal{F}_n^j$  be as above. Let  $G \subseteq E$  be a  $G_\delta$  set such that for any nonempty relatively open subset  $U$  of  $G$ ,  $\overline{U} \notin I$ . Then there exist  $j, m \in \mathbb{N}$  such that  $G^*$  is comeager in  $\mathcal{F}_m^j$ .*

### 3.3.3 The ideal of null sets for a finite atomless Borel measure

Let  $\mu$  be a finite atomless Borel measure defined on the compact Polish space  $E$  and consider its null ideal

$$I = \{K \in \mathcal{K}(E) : \mu(K) = 0\}.$$

For  $n \in \mathbb{N}$ , let

$$\mathcal{A}_n = \{F \in \mathcal{K}(E) : \mu(E \setminus F) < 1/n\}.$$

Then any closed subset  $K$  of  $E$  is in  $I$  if and only if  $K^*$  has empty interior in  $\mathcal{A}_n$  for all  $n \in \mathbb{N}$ . To see this, let  $K \in I$  and fix  $n$ . To show that  $K^*$  has empty interior in  $\mathcal{A}_n$  let  $V \subseteq E$  be an open set such that  $\mathcal{K}(V) \cap \mathcal{A}_n \neq \emptyset$ . Let  $F \in \mathcal{K}(V) \cap \mathcal{A}_n$ . By removing from  $F$  an open superset of  $K$  of sufficiently small measure, we get a

set in  $(\mathcal{K}(V) \cap \mathcal{A}_n) \setminus K^*$ . Conversely, if  $K \notin I$  then  $\mathcal{A}_n \subseteq K^*$  for any  $n$  satisfying  $1/n \leq \mu(K)$ .

As usual, for each  $n \in \mathbb{N}$  we then define  $\mathcal{F}_n$  as  $\overline{\mathcal{A}_n}$ .

**Theorem 20.** *Let  $I \subseteq \mathcal{K}(E)$  be the null-ideal for  $\mu$  and let  $\mathcal{F}_n$ ,  $n \in \mathbb{N}$  be the sets defined above. Let  $G \subseteq E$  be a  $G_\delta$  set such that for any nonempty relatively open set  $U \subseteq G$ , the set  $\overline{U}$  is not in  $I$ . Then there exists  $m \in \mathbb{N}$  such that  $G^*$  is comeager in  $\mathcal{F}_m$ .*

*Proof.* Without loss of generality we assume that  $\mu(E) = 1$ , so that

$$\mathcal{A}_n = \{F \in \mathcal{K}(E) : \mu(F) > 1 - 1/n\}.$$

Let  $G = \overline{G} \setminus \bigcup_n F_n$ , where  $(F_n)$  is an increasing sequence of closed relatively meager subsets of  $\overline{G}$ . Since  $\overline{G} \notin I$ , we may pick  $m$  such that  $1/m < \mu(\overline{G})$ , and thus  $\mathcal{F}_m \subseteq \overline{G}^*$ . We will show that Player II has a winning strategy in the Banach-Mazur game in  $\mathcal{F}_m$  on  $G^*$  as described in Section 3.3.1, with Player I playing sets  $\mathcal{U}_n$  and Player II responding with  $\mathcal{V}_n$ .

As for the previous ideal, at stage  $n$  Player II will construct the following:

- $D_n$ , a nonempty relatively open subset of  $\overline{G}$ ,
- an open set  $V_n \subseteq E$ ,

satisfying the following conditions:

- $D_n \subseteq D_{n-1}$  for  $n \geq 1$ ,
- $\overline{D_n} \cap F_n = \emptyset$ ,
- $\mathcal{K}(V_n) \cap \mathcal{U}_n \cap \mathcal{F}_m \neq \emptyset$ ,
- $\mathcal{K}(V_n) \cap \mathcal{F}_m \subseteq \overline{D_n}^*$ .

Player II's winning strategy will of course consist in playing the set  $\mathcal{V}_n = \mathcal{K}(V_n) \cap \mathcal{U}_n$ .

We now describe the construction. Of all the ideals we considered, this is the one for which the sets  $D_n$  and  $V_n$  present themselves most readily. For, if  $U$  is an open set of measure greater than some fixed  $\alpha \geq 0$ , and  $A$  is a subset of  $U$  of positive measure, however small, then it is easy to obtain inside  $U$  an open set  $V$  that does contain closed sets of measure greater than  $\alpha$ , but so that no such set inside  $V$  can avoid  $A$ . All we have to do to make the set  $V$  "tight" around  $A$  is to control its measure. Here are the details. Set  $D_{-1} = \overline{G}$ . Suppose that at the  $n$ th stage, Player I has played  $\mathcal{U}_n = \{F \in \mathcal{K}(E) : F \subseteq U_0, F \cap U_i \neq \emptyset \forall i = 1, \dots, k\}$ , for open sets  $U_0, \dots, U_k$ , with  $U_i \subseteq U_0$  for  $i = 1, \dots, k$ . Since  $\emptyset \neq \mathcal{U}_n \cap \mathcal{F}_m \subseteq \overline{D_{n-1}^*}$ , we have  $\overline{D_{n-1}} \cap U_0 \neq \emptyset$ . Therefore  $D_{n-1} \cap U_0$  is a nonempty open subset of  $\overline{G}$ . Since  $F_n$  is meager in  $\overline{G}$ , we may find a nonempty set  $D_n$ , relatively open in  $\overline{G}$ , such that  $\overline{D_n} \subseteq (D_{n-1} \cap U_0) \setminus F_n$ . By the condition on  $G$ , we know that  $\overline{D_n} \notin I$ , i.e.,  $\mu(\overline{D_n}) > 0$ . For  $i = 1, \dots, k$ , let  $x_i \in U_i$ , and let  $V$  be an open set satisfying

- $\overline{D_n} \cup \{x_i : i = 1, \dots, k\} \subseteq V \subseteq U_0$ ,
- $1 - 1/m < \mu(V) < 1 - 1/m + \mu(\overline{D_n})$ .

It is clear that  $\mathcal{K}(V) \cap \mathcal{U}_n \cap \mathcal{F}_m \neq \emptyset$ . Also, since  $\mu(V \setminus \overline{D_n}) < 1 - 1/m$ , we have  $\mathcal{K}(V) \cap \mathcal{F}_m \subseteq \overline{D_n^*}$ . Therefore the set  $V_n = V$  satisfies all the required properties.  $\square$

**Remark.** For  $E, \mu, I$  as above, there is also a natural single set  $\mathcal{F}$  characterizing membership of closed sets in  $I$ . Fix a countable basis for  $E$  closed under finite union, and pick  $r > 0$  such that  $r$  is not the measure of any basic open set. Let  $\mathcal{F} = \{K \in \mathcal{K}(E) : \mu(K) \geq r\}$ . It is straightforward to see that for any  $K \in \mathcal{K}(E)$ , the set  $K^*$  is meager in  $\mathcal{F}$  if and only if  $K$  is in  $I$ .

The conjecture also holds for this single set. Let  $G \subseteq E$  be a  $G_\delta$  set such that for any nonempty relatively open  $U \subseteq G$ ,  $\overline{U} \notin I$ . The proof above may be modified to show that  $G^*$  is nonmeager in  $\mathcal{F}$ . In this case we play the Banach-Mazur game on  $\mathcal{F} \setminus G^*$ , played in  $\mathcal{F}$ . Players I and II play sets  $\mathcal{U}_n$  and  $\mathcal{V}_n$  respectively, following the usual rules. Recall that  $\mathcal{F} \setminus G^*$  is comeager in  $\mathcal{F}$  if and only if Player II has a winning strategy. It therefore suffices to show that Player I has a winning strategy, i.e., Player I can ensure that  $\bigcap_n \mathcal{U}_n \not\subseteq \mathcal{F} \setminus G^*$ . At stage 0 (the initial move), Player I plays  $\mathcal{K}(U_0)$ , where  $U_0$  is an open set such that  $\mathcal{K}(U_0) \cap \mathcal{F} \subseteq \overline{G^*}$ . (By Corollary 13, we may assume that  $G$  is meager in  $E$  and thus we can find a nonempty open subset of  $\mathcal{F}$  of this form contained in  $\overline{G^*}$ .) After this initial move, the strategy described in the theorem above, when deployed by Player I, in fact ensures that  $\emptyset \neq \bigcap_n \mathcal{U}_n \subseteq G^*$ .

### 3.3.4 The ideal of zero dimensional sets

Let  $E$  be a compact Polish space. A subset  $A$  of  $E$  is *zero dimensional* if the relative topology on  $A$  has a basis of sets with empty boundaries, i.e., a basis of clopen sets. Let

$$I = \{K \in \mathcal{K}(E) : K \text{ is zero dimensional}\}.$$

We first determine a sequence  $(\mathcal{F}_n)$  of closed subsets of  $\mathcal{K}(E)$  such that for any  $K \in \mathcal{K}(E)$ ,  $K$  is in  $I$  if and only if  $K^*$  has empty interior in each set  $\mathcal{F}_n$ . As before, we shall find sets  $\mathcal{A}_n$ , not necessarily closed, that satisfy this condition and then set  $\mathcal{F}_n = \overline{\mathcal{A}_n}$ .

For any set  $A$ , we shall use the notation  $|A|$  for the diameter of  $A$ , and  $\text{Bd}(A)$  for the boundary of  $A$ . By basic results in dimension theory (see [5] or [2]), we know that any compact set is zero dimensional if and only if it can be covered by

a finite disjoint union of open sets each of whose diameters is less than  $\epsilon$ , for any given  $\epsilon > 0$ . The natural candidates for the sets  $\mathcal{A}_n$  are therefore

$$\mathcal{A}_n = \left\{ F \in \mathcal{K}(E) : \exists k \in \mathbb{N} \exists \text{ open, pairwise disjoint sets } U_1, \dots, U_k, \right. \\ \left. |U_i| \leq \frac{1}{n}, \text{ such that } E \setminus F \subseteq \bigcup_{i=1}^k U_i \right\}. \quad (3.5)$$

To show that these sets work, let  $K$  be closed and suppose  $K$  is not zero dimensional. Then there exists  $n \in \mathbb{N}$  such that  $K$  cannot be covered by finitely many disjoint open sets of diameter  $\leq 1/n$ . Let  $F \in \mathcal{A}_n$ . Since  $E \setminus F$  cannot cover  $K$ ,  $K$  must intersect  $F$ . So  $\mathcal{A}_n \subseteq K^*$ .

Conversely, suppose  $K$  is zero dimensional and fix  $n$ . We want to show that  $K^*$  contains no nonempty open subset of  $\mathcal{A}_n$ . Since  $\mathcal{A}_n$  is upward closed and  $K$  is meager, by Lemma 8 it suffices to consider nonempty sets of the form  $\mathcal{K}(U) \cap \mathcal{A}_n$  for nonempty open sets  $U$ . So let  $U$  be open and suppose  $\mathcal{K}(U) \cap \mathcal{A}_n \neq \emptyset$ . Let  $F \in \mathcal{K}(U) \cap \mathcal{A}_n$ , i.e.,  $F \subseteq U$ , and  $E \setminus F \subseteq W_1 \cup \dots \cup W_l$  for some disjoint nonempty open sets  $W_i$  of diameter  $\leq 1/n$ .

Pick  $\delta < 1/n$  satisfying  $0 < \delta < d(F, E \setminus U)$ . Since  $K$  is zero dimensional, we may cover  $K$  with disjoint open sets  $V_1, \dots, V_k, V_{k+1}, \dots, V_m$ , each of diameter less than  $\delta/2$ , where we assume  $V_1, \dots, V_k$  intersect  $F \cap K$  and  $V_{k+1}, \dots, V_m$  do not. Note that for any  $i$ ,  $\text{Bd}(V_i)$  does not intersect  $K$ . Let

$$F_1 = \left( F \setminus \bigcup_{i=1}^k V_i \right) \cup \left( \bigcup_{i=1}^k \text{Bd}(V_i) \right).$$

We then have

$$E \setminus F_1 \subseteq \bigcup_{i=1}^k V_i \cup \bigcup_{j=1}^l \left( W_j \setminus \bigcup_{i=1}^k \overline{V_i} \right),$$

which is a disjoint union of open sets of diameter  $\leq 1/n$ . Thus  $F_1 \in \mathcal{K}(U) \cap \mathcal{A}_n \setminus K^*$ .

So we have shown that  $K^*$  has nonempty interior in  $\mathcal{A}_n$ . Setting  $\mathcal{F}_n = \overline{\mathcal{A}_n}$  for  $n \in \mathbb{N}$ , we obtain a sequence of nonempty, upward closed compact sets such that, for any closed  $K \subseteq E$ ,

$$K \text{ is zero dimensional} \iff K^* \text{ is meager in } \mathcal{F}_n \forall n \in \mathbb{N}.$$

It is easy to see that the equivalence above fails for  $G_\delta$  subsets of  $E$ . Let  $G$  be the set of irrational points on the unit interval and let  $E \subset \mathbb{R}^2$  be a compact set containing  $G$ .  $G$  is  $G_\delta$ , and the closure of every relatively open subset of  $G$  is not in  $I$ . Pick  $n$  such that  $\mathcal{F}_n \subseteq \overline{G^*}$ . Now,  $G$  is zero dimensional, but  $G^*$  is comeager in  $\mathcal{F}_n$ , because  $\mathcal{F}_n \setminus G^* \subseteq Q^*$ , where  $Q$  is the set of rational points on the unit interval. Since  $Q$  is the countable union of closed zero dimensional sets,  $Q^*$  is meager in  $\mathcal{F}_n$ .

We wish to show that the conjecture holds, i.e., the only  $G_\delta$  sets  $G$  such that  $G^*$  is meager in each  $\mathcal{F}_n$  are those that can be covered by a countable union of closed sets in  $I$ . But let us first explicate the reason why, if a set  $A \in \mathcal{K}(E)$  is not zero dimensional,  $A^*$  contains one of the sets  $\mathcal{F}_n$ . Note that the compact set  $A$  is zero dimensional if and only if it has no component of more than one point. If  $A$  has a component of positive diameter, and if we pick  $n \in \mathbb{N}$  such that  $1/n$  is less than this diameter, then this component, and hence  $A$  itself, cannot be contained in finitely many disjoint open sets each of diameter less than  $1/n$ . (For, the connected component would be too big to fit inside a single member of such a cover, and too connected to fit inside several members.) So by the definition of  $\mathcal{A}_n$ , it follows that  $\mathcal{A}_n \subseteq A^*$ .

Turn now to the proof of the conjecture. Just as in the previous cases, we will fix a  $G_\delta$  set  $G$  whose every non-empty relatively open subset has closure outside  $I$ . We will pick  $n \in \mathbb{N}$  such that  $\overline{G^*}$  contains  $\mathcal{F}_n$ ; as discussed above, the reason

for this containment will be that  $\overline{G}$  has a component that is sufficiently large, i.e., of diameter greater than  $1/n$ . And as usual, we will show that Player II has a winning strategy in the appropriate Banach-Mazur game in  $\mathcal{F}_n$ . Again, the key step in the proof will be as follows. We will be dealing with an open set of the usual form,  $\mathcal{U} = \{K : K \subseteq U_0, K \cap U_i \neq \emptyset\}$ , satisfying  $\mathcal{U} \cap \mathcal{A}_n \neq \emptyset$ . Inside  $U_0$  will lie a part of  $G$ , within which part everywhere we can find many little compact subsets  $D$  that are not zero dimensional, and that have the desirable property of avoiding the taboo set of that stage. Those compact sets are not zero dimensional by virtue of having some connected component whose diameter, though positive, is much smaller than  $1/n$ . We want to establish such a set  $D$  along with an open superset  $V$  that is “tight” around  $D$  without being too small itself — a set  $V$  whose complement can still be covered by finitely many disjoint open sets of diameter less than  $1/n$ , but that simultaneously has the property that the addition of  $D$  to  $E \setminus V$  creates a set that cannot be so covered. Unlike in the case of a finite measure, it is not immediately obvious how to do this. The way we proceed is to ensure that the tiny connected pieces of  $D$  “stitch together” the disjoint pieces of  $E \setminus V$ . The stitching together we carry out in the proof requires a technical condition on the set  $E$ ; towards describing this condition we start below with a definition and a lemma. We will use basic facts about connected and locally connected sets without proving them; see [9] for proofs. Note that when we say that a connected set  $C$  connects sets  $A$  and  $B$  (or connects  $A$  to  $B$ ) we simply mean that it intersects both  $A$  and  $B$ .

**Definition 21.** *We say that a set  $A \subseteq E$  is fat if the following condition holds: if  $U_1, U_2$  are open sets intersecting  $\text{Bd}(A)$  and  $W$  is an open set intersecting  $A$  that is disjoint from  $U_1 \cup U_2$ , then for any  $\delta > 0$ , there exist continua  $A_1, A_2 \subseteq A + \delta$  such that  $A_i$  connects  $U_i$  to  $W$ , and, if the sets  $U_i$  are disjoint then so are the sets  $A_i$ .*

As an illustrative example, consider a set that is not fat: let  $E$  be a subset of the plane obtained as the union of two triangles (including their insides) that intersect only at a single common vertex, so that  $E$  looks like a bow-tie. This set is not fat because if  $W$  is a small open set inside one of the triangles, then any two continua joining  $W$  to points in the other triangle must pass through the common vertex and thus cannot be disjoint. (Since  $E$  is the whole space, an expansion by some positive  $\delta$  has no role here.) Note however, that the space  $E$  can be written as the union of two fat subsets, the triangles.

**Lemma 22.** *Let  $E$  be compact, connected and locally connected. Let  $A \subseteq E$  be a compact connected fat set. Let  $U_1, U_2$  be disjoint open sets, both intersecting  $\text{Bd}(A)$ , and let  $S \subseteq A$  be a set with more than one point. Then for any  $\delta > 0$ , there exist disjoint arcs  $A_1, A_2 \subseteq A + \delta$  such that  $A_i$  connects  $U_i$  to  $S$ .*

*Proof.* Let  $E, A, U_1, U_2$  and  $S$  be as stated and let  $x, y$  be distinct points in  $S$ . Let  $W_x, W_y$  be open connected neighbourhoods of  $x, y$  respectively, with disjoint closures. Fix a positive  $\delta$  and let  $A_1, A_2$  be disjoint continua in  $A + \delta$  such that  $A_i$  connects  $U_i$  to  $W_x$ . Let  $V_1, V_2$  be disjoint regions in  $A + \delta$  containing  $A_1, A_2$  respectively. As regions in the locally connected complete space  $E$ , they are arcwise connected. (This fact is due to Hahn and Mazurkiewicz; for a proof see the chapter on local connectedness in [9].) This allows us to assume that  $A_1, A_2$  are in fact arcs. The point  $y$  can be contained in at most one  $A_i$ . Therefore by shrinking  $W_y$ , we may assume that at least one of the sets  $A_i$  does not intersect  $\overline{W_y}$ . Say  $A_1 \cap \overline{W_y} = \emptyset$ . Extend  $A_1$  inside  $W_x$  to get an arc joining  $U_1$  to  $x$ ; call this extended arc  $A_1$  again. We still have that  $A_1$  is disjoint from  $\overline{W_y}$ . Now let  $B_1, B_2$  be disjoint arcs in  $A + \delta$  such that  $B_i$  connects  $U_i$  to  $W_y$ . Take suitable sub-arcs, which we continue to call  $B_1, B_2$ , such that for each  $i$ ,  $B_i \cap \overline{W_y}$  is precisely an endpoint of  $B_i$ .

If  $A_1 \cap B_2 = \emptyset$ , then we may extend  $B_2$  inside a small connected neighborhood of  $W_y$  to get an arc from  $U_2$  to  $y$  that is disjoint from  $A_1$ , and we are done.

If  $A_1 \cap (B_1 \cup B_2) \neq \emptyset$ , then starting at  $x$ , follow  $A_1$  out until you hit the first point, say  $z$ , of  $B_1 \cup B_2$ . If  $z \in B_1$ , combine the appropriate pieces of  $B_1$  and  $A_1$  to get an arc from  $x$  to  $U_1$ . This new arc, call it  $C$ , is disjoint from  $\overline{W_y}$ . Now extend  $B_2$  inside a small connected neighborhood of  $W_y$  to get an arc from  $U_2$  to  $y$  that is disjoint from  $C$ , and we are done. A similar construction works for the case when  $z \in B_2$ .  $\square$

**Theorem 23.** *Let  $E$  be compact and suppose that for any  $\epsilon > 0$ ,  $E$  can be covered by finitely many fat sets of diameter less than  $\epsilon$ . Let  $\mathcal{A}_n$ ,  $n \in \mathbb{N}$ , be the sets defined in formula (3.5) and let  $\mathcal{F}_n = \overline{\mathcal{A}_n}$ . Then for any  $G_\delta$  set  $G \subseteq E$  such that  $\overline{U}$  is not zero dimensional for every nonempty relatively open subset  $U$  of  $G$ , there exists  $r \in \mathbb{N}$  such that  $G^*$  is comeager in  $\mathcal{F}_r$ .*

*Proof.* Note first that if  $A$  is fat, then so is  $\overline{A}$ ; this follows easily from the definition of fatness. It is also easily checked that if  $A$  is a compact fat set with nonempty boundary, then  $A$  is connected. So in fact, for any  $\epsilon > 0$ ,  $E$  can be covered by finitely many compact fat sets of diameter less than  $\epsilon$ , such that each set in this cover whose boundary is nonempty is connected.

We claim that, if a component  $C$  of  $E$  is not a singleton, then  $C$  is open and locally connected.

To see this, fix  $\epsilon < |C|$  and let  $A_i$ ,  $i = 1, \dots, k$ , be a finite  $\epsilon$ -cover of  $E$  consisting of compact fat sets. (We use the term  $\epsilon$ -cover for a cover consisting of sets of diameter less than  $\epsilon$ .) Let  $x \in C$  and let  $U$  be an open set containing  $x$ , such that  $U$  is disjoint from any set  $A_i$  not containing  $x$ . If  $x \in A_i$  then, by choice of  $\epsilon$ ,  $\text{Bd}(A_i) \neq \emptyset$  and so  $A_i$  is connected. Now,  $x \in U \subseteq \bigcup_{x \in A_i} A_i \subseteq C$ . This shows that  $C$  is open. Also, since  $\bigcup_{x \in A_i} A_i$  is a connected set of diameter

less than  $2\epsilon$  containing  $x$  in its interior, this shows that  $C$  is locally connected.

Let  $G$  be the  $G_\delta$  set mentioned in the statement of the theorem. Since  $\overline{G}$  is not zero dimensional, it has some connected component  $S$  consisting of more than one point. Let  $C$  be the component of  $E$  containing  $S$ . By the claim above,  $C$  is clopen in  $E$  and locally connected. By replacing  $E$  with  $C$  and  $G$  with  $G \cap C$ , we will assume that  $E$  is compact, connected and locally connected. Since  $E$  is complete, connected and locally connected, we have that every region of  $E$  is arcwise connected. We will also use the fact that, for every region  $R \subseteq E$ , there exists a sequence of regions  $R_n$  such that  $R = \bigcup_n R_n$  and  $\overline{R_n} \subseteq R_{n+1}$ .

Pick  $r \in \mathbb{N}$  such that  $|S| > 1/r$ . We will play the Banach-Mazur game in  $\mathcal{F}_r$  on  $G^*$ , in which Players I and II play sets  $\mathcal{U}_n$  and  $\mathcal{V}_n$  respectively, following the usual rules. We will show that Player II has a winning strategy, i.e., Player II can ensure that  $\bigcap_n \mathcal{V}_n \cap \mathcal{F}_r \subseteq G^*$ .

By a *chain* we will mean a finite sequence of sets  $B_0, \dots, B_k$ , such that  $B_i \cap B_{i+1} \neq \emptyset \forall i$ . We say the chain is *irreducible* if  $B_i \cap B_j \neq \emptyset$  if and only if  $|i - j| \leq 1$ .

Let  $(F_n)_{n \in \mathbb{N}}$  be an increasing sequence of closed meager subsets of  $\overline{G}$  such that  $G = \overline{G} \setminus \bigcup_n F_n$ . At the  $n^{\text{th}}$  stage of the game, Player I plays an open set  $\mathcal{U}_n$  such that  $\mathcal{U}_n \cap \mathcal{F}_r \neq \emptyset$ . In response, Player II will recursively construct the following objects:

- a compact set  $L \in \mathcal{U}_n \cap \mathcal{A}_r$ ,
- $k_n + 1$  open connected subsets of  $E$ :  $C_0^n, C_1^n, \dots, C_{k_n}^n$ , whose closures are disjoint from  $L$ ,
- $k_n$  relatively open subsets of  $\overline{G}$ :  $D_1^n, D_2^n, \dots, D_{k_n}^n$ , and the set  $D_n = \bigcup_i D_i^n$ ,
- $k_n$  disjoint closed connected subsets of  $\overline{G}$ :  $S_1^n, S_2^n, \dots, S_{k_n}^n$ , such that  $S_i^n \subseteq D_i^n$ , and the set  $S_n = \bigcup_i S_i^n$ ,

satisfying the following conditions:

- $C_0^n, D_1^n, C_1^n, D_2^n, C_2^n, \dots, D_{n_k}^n, C_{n_k}^n$  is an irreducible chain,
- $C_0^n, S_1^n, C_1^n, S_2^n, C_2^n, \dots, S_{n_k}^n, C_{n_k}^n$  is also an irreducible chain,
- $|\bigcup_i C_i^n| > 1/r$ ,
- if  $n > 0$ ,  $\overline{D_n} \cap F_n = \emptyset$ ,
- if  $n > 0$ ,  $D_n \subseteq D_{n-1}$ .

If the conditions above are satisfied, we may find an open set  $V_n \supseteq L$  such that  $V_n$  is disjoint from  $C_i^n$  for all  $i = 1, \dots, k_n$ . Player II's  $n^{\text{th}}$  move will be  $\mathcal{V}_n = \mathcal{K}(V_n) \cap \mathcal{U}_n$ . The conditions above imply that, if  $K \in \mathcal{K}(V_n) \cap \mathcal{A}_r$ , then for some  $i$ ,  $K \cap S_i^n \neq \emptyset$ , for otherwise  $E \setminus K$  would contain a connected set of diameter  $> 1/r$ . Therefore  $\mathcal{K}(V_n) \cap \mathcal{A}_r \subseteq S_n^*$ . Since  $S_n^*$  is closed, we have  $\mathcal{K}(V_n) \cap \mathcal{F}_r \subseteq S_n^*$  and therefore  $\mathcal{K}(V_n) \cap \mathcal{F}_r \subseteq D_n^*$ . The conditions on the sequence  $(D_n)$  then imply that  $\bigcap_n \mathcal{V}_n \cap \mathcal{F}_r \subseteq G^*$ .

We now describe the construction.

*The 0<sup>th</sup> move:* Player I plays an open set  $\mathcal{U}_0$  such that  $\mathcal{U}_0 \cap \mathcal{A}_r \neq \emptyset$ . Recall that  $\overline{G}$  has a connected subset,  $S$ , of diameter  $> 1/r$ . Pick  $a, b \in S$  such that  $d(a, b) > 1/r$ . Pick  $L \in \mathcal{U}_0 \cap \mathcal{A}_r$  such that  $a, b \notin L$ . (We use here the fact that, for any fixed  $x \in E$ , the set  $\{K \in \mathcal{A}_r : x \notin K\}$  is dense in  $\mathcal{A}_r$ .)

Let  $C_0^0, C_1^0$  be connected open sets whose closures are disjoint from  $L$ , such that  $a \in C_0^0, b \in C_1^0$ .

Set  $S_1^0 = S$  and  $D_1^0 = \overline{G}$ . Then the sets  $C_0^0, C_1^0, S_1^0, D_1^0$  and  $L$  satisfy the required conditions. Let  $V_0 \supseteq L$  be an open set disjoint from  $C_0^0, C_1^0$ . Player II plays  $\mathcal{V}_0 = \mathcal{K}(V_0) \cap \mathcal{U}_0$ .

The 1<sup>st</sup> move: Player I plays  $\mathcal{U}_1$ . Let  $K \in \mathcal{U}_1 \cap \mathcal{A}_r$ , and pick  $\epsilon > 0$  sufficiently small so that  $B(K, 3\epsilon) \subseteq \mathcal{U}_1$ . We will start by constructing an irreducible chain

$$W_0, S_1, W_1, S_2, \dots, W_{m-1}, S_m, W_m \quad (3.6)$$

such that:

- the sets  $W_i$  are open connected sets with disjoint closures,
- $|W_i| < 3\epsilon$  for each  $i$  such that  $K \cap \overline{W_i} \neq \emptyset$ ,
- $|\bigcup_i W_i| > 1/r$ ,
- the sets  $S_i$  are compact connected subsets of  $\overline{G} \setminus F_1$ ,
- for each  $i = 1, \dots, m$ , there exists  $D_i$ , an open subset of  $\overline{G}$ , such that  $S_i \subseteq D_i \subseteq \overline{D_i} \subseteq \overline{G} \setminus F_1$ .

Once we have this chain, we obtain the required objects as follows. Let  $J = \{i : K \cap \overline{W_i} \neq \emptyset\}$ , and let

$$L = (K \setminus \bigcup_{i \in J} W_i) \cup \bigcup_{i \in J} \text{Bd}(W_i).$$

The set  $L$  is within  $3\epsilon$  of  $K$ , so  $L \in \mathcal{U}_1 \cap \mathcal{A}_r$ . For  $i \in J$ , let  $C_i$  be a subregion of  $W_i$  such that  $\overline{C_i} \subseteq W_i$ ,  $C_i$  intersects both  $S_i$  and  $S_{i+1}$ , and  $|\bigcup_i C_i| > 1/r$ .

For  $i \notin J$ , let  $C_i = W_i$ .

For each  $i$  now let  $C_i^1 = C_i$ ,  $S_i^1 = S_i$ , and  $D_i^1 = D_i$ . The sets  $L$ ,  $C_i^1$ ,  $S_i^1$ ,  $D_i^1$  now satisfy the required conditions.

We now describe the construction of the chain (3.6).

Let  $A_i$  be a finite collection of compact fat sets of diameter less than  $\epsilon/2$  such that  $E = \bigcup_i A_i$ . Note that  $\overline{G} \subseteq \bigcup\{A_i : A_i \text{ contains an open subset of } \overline{G}\}$ . To see

this, let  $x \in \overline{G}$  and let  $U$  be an open neighbourhood of  $x$  that is disjoint from each  $A_i$  not containing  $x$ . Then  $\emptyset \neq U \cap \overline{G} \subseteq \bigcup \{A_i \cap \overline{G} : x \in A_i\}$ . One of these  $A_i$  has nonempty  $\overline{G}$ -interior and contains  $x$ .

Consider the sets  $C_0^0, C_1^0, S_1^0, D_1^0$  constructed at the previous move.  $S_1^0$  is a closed connected set intersecting  $C_0^0$  and  $C_1^0$ . Consider the sets  $A_i$  in which  $\overline{G}$  has nonempty interior. Since  $S_1^0$  is covered by these sets, we may find among them an irreducible chain  $A_{i_0}, \dots, A_{i_m}$  such that  $A_{i_0}$  intersects  $C_0^0$  and  $A_{i_m}$  intersects  $C_1^0$ . For convenience we may assume that the chain is simply  $A_0, \dots, A_m$ .

We will assume that  $m$  is even. (If  $m$  is not even, we may replace the last two sets in the chain with their union  $A_{m-1} \cup A_m$ . This union is connected but may not be fat, however, we will only be using fatness of the odd sets. We still have that each  $A_i$  has diameter  $< \epsilon$ .) Fix a positive  $\delta < \epsilon$  such that  $A_0 + \delta, \dots, A_m + \delta$  is still an irreducible chain and each  $A_i + \delta$  is still of diameter  $< \epsilon$ .

Step 1. For even  $i$  such that  $0 < i < m$ , construct an arc  $H_i$  in  $A_i + \delta$  that intersects both  $A_{i-1}$  and  $A_{i+1}$ . By taking a suitable subarc, we may assume that  $H_i$  intersects each of these sets, respectively, in a single point. In  $A_0 + \delta$  construct an arc  $H_0$  that joins  $C_0^0$  to  $A_1$  and in  $A_m + \delta$  construct an arc  $H_m$  that joins  $A_{m-1}$  to  $C_1^0$ . Again, we assume that  $H_0 \cap A_1$  and  $H_m \cap A_{m-1}$  are singletons.

Step 2. For odd  $i$ ,  $A_i$  contains an open subset  $D_i$  of  $\overline{G}$ . The conditions on  $G$  mean that  $\overline{G}$  has no finite open sets. So we may assume that  $D_i$  is disjoint from  $H_{i-1} \cup H_{i+1}$ . Since  $F_1$  is meager in  $\overline{G}$ , we may further assume that  $\overline{D_i}$  is disjoint from  $F_1$ . The conditions on  $G$  allow us to find a compact connected subset  $S_i$  of  $D_i$ .

Step 3. For even  $i$ , let  $U_i$  be an open superset of  $H_i$  such that the sets  $\overline{U_i}$  and  $S_i$  are still all pairwise disjoint.

Step 4. For odd  $i$ , the condition of fatness allows us to find disjoint continua in  $A_i + \delta$  that join  $S_i$  to  $U_{i-1}$  and  $U_{i+1}$ . By fattening these continua appropriately,

construct regions  $U_i^{left}$  and  $U_i^{right}$ , with disjoint closures, such that  $U_i^{left}$  connects  $U_{i-1}$  to  $S_i$  and  $U_i^{right}$  connects  $U_{i+1}$  to  $S_i$ . Also set  $U_{-1}^{right} = C_0^0$  and  $U_{m+1}^{left} = C_1^0$ .

Step 5. For even  $i$ , let

$$W_i = U_{i-1}^{right} \cup U_i \cup U_{i+1}^{left}.$$

Note that  $K \cap (A_0 \cup A_1) = \emptyset$ , for otherwise,  $K \cup A_0 \cup A_1$  would be a set in  $\mathcal{U}_1 \cap \mathcal{A}_r$  that also intersected  $C_0^0$  — a contradiction. Therefore  $K \cap \overline{W_0} = \emptyset$ . A similar argument applies to  $\overline{W_m}$ . For  $0 < i < m$  we have  $|W_i| \leq 3\epsilon$ .

This completes the construction of the chain (3.6).

*The  $n + 1^{st}$  move:* Player I plays  $\mathcal{U}_{n+1}$ . Let  $K \in \mathcal{U}_{n+1} \cap \mathcal{A}_r$ , and pick  $\epsilon > 0$  sufficiently small so that  $B(K, 3\epsilon) \subseteq \mathcal{U}_{n+1}$ . Again, it suffices to construct an irreducible chain

$$W_0, S_1, W_1, S_2, \dots, W_{m-1}, S_m, W_m$$

satisfying the same set of conditions as in the  $1^{st}$  move, with  $F_1$  replaced by  $F_{n+1}$ .

Consider the sets  $C_i^n, D_i^n, S_i^n$  constructed at the previous move. Each  $S_i^n$  intersects  $C_{i-1}^n$  and  $C_i^n$ . We repeat the construction described in the  $1^{st}$  move for each triple  $C_i^n, S_{i+1}^n, C_{i+1}^n$ .

Consider first just the chain  $C_0^n, S_1^n, C_1^n$  and the set  $D_1^n \supseteq S_1^n$ . As in the first move, construct

$$W_0, S_1, W_1, S_2, \dots, W_{m_0-1}, S_{m_0}, W_{m_0}$$

satisfying  $C_0^n \subseteq W_0$  and  $C_1^n \subseteq W_{m_0}$ .  $K$  does not intersect  $\overline{W_0} \cup \overline{W_{m_0}}$ .

For each triple  $C_i^n, S_{i+1}^n, C_{i+1}^n$  similarly construct

$$W_{m_{i-1}+1}, S_{m_{i-1}+2}, \dots, S_{m_i}, W_{m_i}$$

satisfying  $C_i^n \subseteq W_{m_{i-1}+1}$  and  $C_{i+1}^n \subseteq W_{m_i}$ . Again,  $K$  does not intersect  $\overline{W_{m_{i-1}+1}} \cup$

$\overline{W_{m_i}}$ .

The required chain is now the concatenation of these chains. To obtain the form described, in this long chain we combine the last element of the  $i^{\text{th}}$  subchain with the first element of the  $i + 1^{\text{st}}$  subchain, i.e., we replace the sets  $W_{m_i}$  and  $W_{m_{i+1}}$  with their union.

This completes the description of the strategy.

□

We conclude this section with some remarks indicating the wide applicability of the hypothesis of Theorem 23.

1. If  $E$  is compact, connected and locally connected, then for any  $\epsilon$ ,  $E$  may be written as the union of finitely many compact, connected and locally connected sets of diameter  $< \epsilon$ . If, in addition, these sets may be chosen to be fat, then  $E$  satisfies the hypothesis of the theorem.
2. If  $E$  is a compact and locally connected space such that for any region  $R$  of  $E$ ,  $R$  is not separated by any singleton, then  $E$  satisfies the hypothesis of the theorem.

### 3.3.5 The ideal of null sets for a Hausdorff measure

Let  $\mathcal{H}$  denote the class of all functions  $h : [0, \infty) \rightarrow [0, \infty]$  that are monotonically increasing, continuous on the right and non-zero for non-zero values. Let  $\mathcal{H}_0$  denote the subclass of  $\mathcal{H}$  consisting of those functions  $h$  for which  $h(0) = 0$ .

Let  $E$  be a compact metric space and let  $h \in \mathcal{H}$ . Again we shall use  $|A|$  to denote the diameter of  $A$  for any subset  $A$  of  $E$ . For any  $A \subseteq E$  and  $\delta > 0$ , we say that a countable collection of sets  $\{U_i : i \in \mathbb{N}\}$  is a  $\delta$ -cover of  $A$  if  $|U_i| < \delta$  for

each  $i \in \mathbb{N}$  and  $A \subseteq \bigcup_i U_i$ . Define

$$\mu_\delta^h(A) = \inf \left\{ \sum_{i=1}^{\infty} h(|U_i|) : U_i \text{ is a countable } \delta\text{-cover of } A \right\}.$$

The *Hausdorff measure corresponding to the function  $h$* , or  *$h$ -measure*, is then defined by setting

$$\mu^h(A) = \lim_{\delta \rightarrow 0} \mu_\delta^h(A).$$

For an extensive exposition on Hausdorff measures (as well as a proof that the formula above defines a measure) see [12]. If  $s \geq 0$  and  $h$  is the function  $x \mapsto x^s$ , then the corresponding  $h$ -measure is  *$s$ -dimensional Hausdorff measure*. These measures are used to define the Hausdorff dimension of a set; see [3] or [4] for details.

Here we are of course interested in the ideal of compact sets

$$I = \{K \in \mathcal{K}(E) : \mu^h(K) = 0\}.$$

$I$  is an ideal with property  $(*)$ . From this point on, we shall assume that  $h \in \mathcal{H}_0$  (indeed, if  $h \notin \mathcal{H}_0$  then  $I$  is empty) so that all singletons are in  $I$ . We also assume that  $E$  and  $h$  are such that  $I$  only contains meager sets; as observed before, this assumption is necessary if  $I$  is to be represented by a sequence of upward closed sets.

From the definition of  $\mu^h$  we have the following natural representing sequence for  $I$ : for  $n, m \in \mathbb{N}$ , define

$$\mathcal{A}_{n,m} = \left\{ F \in \mathcal{K}(E) : \mu_{\frac{1}{n}}^h(E \setminus F) < \frac{1}{m} \right\} \quad (3.7)$$

and set  $\mathcal{F}_{n,m} = \overline{\mathcal{A}_{n,m}}$ . Checking that the sets  $\mathcal{F}_{n,m}$  do in fact form a representing

sequence for  $I$  is straightforward. Here we are of course using the the fact that  $\mu^h$  can be obtained from the countable collection of submeasures  $\mu_{1/n}^h$ ,  $n \in \mathbb{N}$ . It is possible, however, to obtain the ideal  $I$  from a single submeasure as follows. For a subset  $A$  of a compact metric space  $E$ , define

$$r^h(A) = \inf \left\{ \sum_{i=1}^k h(|U_i|) : k \in \mathbb{N}, A \subseteq \bigcup_{i=1}^k U_i, U_i \text{ open} \right\}. \quad (3.8)$$

So  $r^h(A)$  is the least you can possibly get  $\sum_i h(|U_i|)$  to be for a cover of  $A$ , regardless of the diameters of the covering sets. (This value is sometimes called *h-content*.) We now claim that for  $K \in \mathcal{K}(E)$ ,  $\mu^h(K) = 0$  if and only if  $r^h(K) = 0$ . Before proving the claim it is convenient to define  $\delta_m = h(1/m)$  for  $m \in \mathbb{N}$ . Note that since  $h(0) = 0$  and  $h$  is continuous from the right,  $\delta_m \rightarrow 0$  as  $m \rightarrow \infty$ .

Now to prove the claim, first assume that  $\mu^h(K) = 0$ , i.e.,  $\mu_\delta^h(K) \rightarrow 0$  as  $\delta \rightarrow 0^+$ . Since this is an increasing limit we have  $\mu_\delta^h(K) = 0$  for all  $\delta > 0$ . It is thus obvious that  $r^h(K) = 0$ .

Conversely, suppose that  $r^h(K) = 0$ . Fix  $\delta > 0$ ; we want to show that  $\mu_\delta^h(K) < \delta$ . Pick  $m \in \mathbb{N}$  large enough to ensure that  $1/m < \delta$  and  $\delta_m < \delta$ . Since  $r^h(K) = 0$ , we may find a finite cover  $U_1, \dots, U_k$  of  $K$  such that  $\sum_i h(|U_i|) < \delta_m$ . Now for each  $i$  we have  $h(|U_i|) < \delta_m = h(1/m)$  and since  $h$  is monotonically increasing this means that  $|U_i| < 1/m < \delta$ . The sets  $U_i$  are thus a  $\delta$ -cover of  $K$  satisfying  $\sum h(|U_i|) < \delta$ , and we are done.

We can now use  $r^h$  to define for  $m \in \mathbb{N}$ ,

$$\mathcal{A}_m = \left\{ F \in \mathcal{K}(E) : r^h(E \setminus F) < \frac{1}{m} \right\}. \quad (3.9)$$

Set  $\mathcal{F}_m = \overline{\mathcal{A}_m}$ . Once again it is easy to confirm that  $(\mathcal{F}_m)$  is a representing sequence for the ideal. When we come to proving the conjecture, though, the

essential difficulty presented by the set  $\mathcal{A}_m$  is this: in general one can enlarge a set  $A$  substantially without increasing the value of  $r^h(A)$ . So, if we have a set  $F$  with  $r^h(E \setminus F) < 1/m$  and we find a set  $D$  contained in  $F$ , or even in the interior of  $F$ , such that  $r^h(D) > 0$ , we can enlarge  $E \setminus F$ , which is to say we can shrink  $F$ , to obtain a new set still in  $\mathcal{A}_m$  that now misses  $D$  completely. This gives Player I in the usual Banach-Mazur game that we play to prove this conjecture a lot of undesirable wiggle room. The same issue arises with the sequence in (3.7). The solution is to make membership in  $\mathcal{A}_m$  more restrictive. In the next section we show that this can be done successfully if the space  $E$  contains a sequence of pairwise disjoint compact sets whose  $r^h$ -values are bounded away from 0, and which are discrete in the sense that they can be contained in open sets with pairwise disjoint closures. As an illustration of what such a collection might look like, consider a compact rectangle  $E$  in  $\mathbb{R}^2$  and consider 1-dimensional Hausdorff measure on  $E$ . The function  $h$  in this case is just  $x \mapsto x$ . For a connected set  $A$  it is easy to see that  $r^h(A) = |A|$ . Now if we take a suitably discrete sequence of disjoint arcs all of diameter greater than some fixed positive value, we have such a collection.

### 3.3.6 The null ideals of certain submeasures

Let  $E$  be a compact metric space. We will call a function  $\mu : \mathcal{K}(E) \rightarrow [0, \infty)$  a *submeasure on compact sets* if it satisfies the conditions that for any  $A, B \in \mathcal{K}(E)$ ,

1.  $\mu(A \cup B) \leq \mu(A) + \mu(B)$ ; and
2. if  $A \subseteq B$  then  $\mu(A) \leq \mu(B)$ .

Assume that  $\mu$  also satisfies the condition that

$$\forall K \in \mathcal{K}(E) (\mu(K) = 0 \iff \inf\{\mu(\bar{U}) : K \subseteq U, U \text{ open}\} = 0). \quad (3.10)$$

Then the set  $I = \{K \in \mathcal{K}(E) : \mu(K) = 0\}$  is a  $G_\delta$  ideal of compact sets with property (\*). To see that it is a  $G_\delta$  set, note that

$$I = \bigcap_{n \in \mathbb{N}} \left\{ K \in \mathcal{K}(E) : K \subseteq U \text{ for some open } U \text{ with } \mu(\overline{U}) < \frac{1}{n} \right\}.$$

To establish property (\*), fix compact sets  $K_i \in I$  such that  $K = \bigcup_{i=1}^{\infty} K_i$  is compact, and let  $\epsilon > 0$ . For each  $i$ , let  $U_i \supseteq K_i$  be open such that  $\mu(\overline{U_i}) < \epsilon/2^i$ . Since  $K \subseteq \bigcup_{i=1}^{\infty} U_i$ , there exists  $m$  such that  $K \subseteq \bigcup_{i=1}^m U_i$ . Now  $\mu(K) \leq \sum_{i=1}^m \mu(\overline{U_i}) \leq 2\epsilon$ .

Assume that  $I$  contains only meager sets, as is required for the existence of upward-closed representing sets for  $I$ , and also that  $I$  contains all singletons. (It is worth noting that even if we allow submeasures to take on the value  $\infty$ , if  $I$  contains all singletons then  $\mu(E)$  is forced to be finite.) For such an  $I$  we wish to show that the conjecture holds, i.e., we wish to find a representing sequence  $(\mathcal{F}_n)$  of closed and upward closed subsets of  $\mathcal{K}(E)$  such that, for any  $G_\delta$  set  $G \subseteq E$ ,

$$(\forall n \ G^* \text{ is meager in } \mathcal{F}_n) \iff \exists (F_i)_{i \in \mathbb{N}} \subseteq I \text{ such that } G \subseteq \bigcup_{i \in \mathbb{N}} F_i.$$

The natural candidates for the representing sets  $\mathcal{F}_n$  would be the closures of the sets

$$\mathcal{A}_n = \{F \in \mathcal{K}(E) : \mu(\overline{E \setminus F}) < 1/n\}.$$

It is easy to see that this sequence is upward closed and does in fact represent the ideal. However, when we play the usual Banach-Mazur game in these sets in order to prove the conjecture, we find that we want to control not just the size of  $E \setminus F$  but also its location, as it were, and do so in a sufficiently restrictive way. In Theorem 24 we show that this may be done successfully if there exists a suitable sequence of sets in  $\mathcal{K}(E)$  that are uniformly big with respect to  $\mu$ . We

use this sequence to establish “codes” that determine *precisely* what basic sets are allowed in a cover of  $E \setminus F$ .

*Remark.* In the usual definition of a submeasure, the function  $\mu$  is defined on an algebra  $\mathcal{A}$  of subsets of  $E$ , closed under taking complements and finite unions. The conditions of monotonicity and subadditivity must hold for all sets in  $\mathcal{A}$ ; it is also required that  $\mu(\emptyset) = 0$ . Suppose that  $\mathcal{A}$  contains all closed (and therefore all open) subsets of  $E$ . It is straightforward to see that for any  $K \in \mathcal{K}(E)$ ,

$$\inf\{\mu(\overline{U}) : K \subseteq U, U \text{ open}\} = \inf\{\mu(U) : K \subseteq U, U \text{ open}\}.$$

So if you can approximate the  $\mu$ -measure of any compact set (or just any compact null-set) by the measure of an open superset, then the restriction of  $\mu$  to  $\mathcal{K}(E)$  is a submeasure on compact sets satisfying 3.10. Note that the function  $r^h$  defined in the previous section for  $h \in \mathcal{H}_0$  is a submeasure defined for all subsets of  $E$  that by definition allows approximation via open sets.

**Theorem 24.** *Let  $E$  be a compact metric space. Let  $\mu : \mathcal{K}(E) \rightarrow [0, \infty)$  be a submeasure on compact sets such that for any  $K \in \mathcal{K}(E)$ , if  $\mu(K) = 0$  then  $\inf\{\mu(\overline{U}) : K \subseteq U, U \text{ open}\} = 0$ . Suppose that the set*

$$I = \{K \in \mathcal{K}(E) : \mu(K) = 0\},$$

*which is a  $G_\delta$  ideal of compact sets with property (\*), contains all singletons and no nonmeager sets, and suppose that there exists a convergent sequence of sets,  $(F_i)_{i \in \mathbb{N}} \subseteq \mathcal{K}(E)$ , such that the sets  $F_i$  are pairwise disjoint as well as disjoint from their limit, and  $\inf_i \mu(F_i) > 0$ . Then there exists a sequence  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  of closed and*

upward closed subsets of  $\mathcal{K}(E)$  such that for any  $G_\delta$  set  $G \subseteq E$ ,

$$(\forall n \ G^* \text{ is meager in } \mathcal{F}_n) \iff \exists (F_i)_{i \in \mathbb{N}} \subseteq I \text{ such that } G \subseteq \bigcup_{i \in \mathbb{N}} F_i,$$

i.e., the conjecture holds for  $I$ .

The following lemma describes a construction used in the proof of the theorem. We will make repeated use of the following calculation: if  $F \subseteq E$  is closed and  $U \subseteq E$  is open, then

$$\mu(F) \leq \mu(F \cup \bar{U}) = \mu((F \setminus U) \cup \bar{U}) \leq \mu(F \setminus U) + \mu(\bar{U}),$$

so that  $\mu(F \setminus U) \geq \mu(F) - \mu(\bar{U})$ .

**Lemma 25.** *Let  $E, \mu, I$  be as in the statement of Theorem 24. Let  $\alpha > 0$ , let  $U$  be an open subset of  $E$  that contains some compact subset of  $\mu$ -measure greater than  $\alpha$ , and let  $\delta$  be positive. Then there exist a closed set  $F \subseteq U$  with  $\mu(F) > \alpha$  and there exist sets  $K_j \in \mathcal{K}(U)$ ,  $j \in \mathbb{N}$ , such that the following conditions all hold:*

1. *the sets  $K_j$  are pairwise disjoint and disjoint from  $F$ ;*
2. *the sequence  $(K_j)$  converges to a finite set  $\{x_1, \dots, x_k\} \subseteq U$ , disjoint again from each  $K_j$  and from  $F$ ;*
3. *each  $K_j$  is of the form  $\bigcup_{i=1}^k \bar{V}_{i,j}$  for some open sets  $V_{i,j}$  with disjoint closures; and*
4. *if  $K \in \mathcal{K}(E)$  and  $K \cap F \neq \emptyset$ , then there exists  $i_0 \in \{1, \dots, k\}$  such that for any  $j \in \mathbb{N}$ , the set  $K \cup \bar{V}_{i_0,j}$  is within  $\delta$  of  $K$  in the sense of the Hausdorff metric on  $\mathcal{K}(E)$ .*

*Proof.* We first construct the set  $F$  along with finitely many pairwise disjoint open sets  $V_i \subseteq E$ ,  $i = 1, \dots, k$ , such that  $|V_i| < \delta/2$  for each  $i$  and the set  $\bigcup_i \bar{V}_i$  is

contained in  $U$ , disjoint from  $F$  and within  $\delta/2$  of  $F$  in the sense of the Hausdorff metric on  $\mathcal{K}(E)$ .

Fix  $L \in \mathcal{K}(U)$  such that  $\mu(L) > \alpha$ . Let  $\{x_i : i = 1, \dots, k\}$ , be a finite set of distinct points in  $L$  such that any point of  $L$  is within  $\delta/4$  of some  $x_i$ . For  $i = 1, \dots, k$ , let  $W_i$  be an open neighbourhood of  $x_i$ , where the sets  $W_i$  are chosen small enough so that their closures are disjoint and contained in  $U$ , and for each  $i$ ,  $|W_i| < \delta/2$  and  $\mu(F \setminus \bigcup_i W_i) > \alpha$ . Since  $E$  has no finite open sets (by the conditions on  $I$ ) we may pick points  $y_i \in W_i \setminus \{x_i\}$ . Set  $F = (L \setminus \bigcup_i W_i) \cup \bigcup_i \{y_i\}$ . Now let  $V_i$  be open sets such that  $x_i \in V_i \subseteq \overline{V_i} \subseteq W_i \setminus \{y_i\}$ . Note that expanding  $\bigcup_i \{x_i\}$  by  $\delta/2$  captures all of  $L$  as well as the points  $y_i$ , and expanding  $\{y_i\}$  by  $\delta/2$  captures all of  $\overline{V_i}$ . Therefore the sets  $F, V_i$  are as required.

Now fix  $1 \leq i \leq k$ . Using again the fact that  $E$  has no finite open sets, inside  $V_i$  we may find countably many open sets  $V_{i,j}$ ,  $j \in \mathbb{N}$ , whose closures are pairwise disjoint and contained in  $V_i$  by simply taking a convergent sequence of distinct points and putting a small ball about each one. For each  $j \in \mathbb{N}$ , set  $K_j = \bigcup_{i=1}^k \overline{V_{i,j}}$ . The sets  $F$  and  $K_j$  are as required; conditions 1, 2 and 3 are immediate and 4 is easily checked as well.  $\square$

We now proceed with the proof of Theorem 24.

*Proof.* We will first define a sequence  $(\mathcal{F}_n)$  of closed and upward closed subsets  $\mathcal{K}(E)$  that determine membership in  $I$  of closed sets  $K$  in the usual way, i.e., via the meagerness of  $K^*$  in the sets  $\mathcal{F}_n$ . To show that the conjecture holds, we will then let  $G$  be a  $G_\delta$  set such that the closure of any nonempty relatively open subset of  $G$  is not in  $I$ . We will pick an  $\mathcal{F}_n$  such that  $\mathcal{F}_n \subseteq \overline{G}^*$  and show that Player II has a winning strategy in the Banach-Mazur game on  $G^*$  played in  $\mathcal{F}_n$ . The sets  $\mathcal{F}_n$  are naturally formulated with this game in mind. As it happens, our representing sequence will be indexed by pairs of natural numbers; we will define

sets  $\mathcal{F}_{n,m}$ .

First fix a countable basis  $\mathcal{B}$  for  $E$ , closed under finite unions. It is easy to see that for any  $K \in \mathcal{K}(E)$ ,

$$K \in I \iff \inf\{\mu(\overline{B}) : K \subseteq B \in \mathcal{B}\} = \inf\{\mu(\overline{U}) : K \subseteq U, U \text{ open}\} = 0.$$

For the sequence  $(F_i)$  given in the hypothesis of the theorem we have  $\inf_i \mu(F_i) > 0$ ; fix some  $\alpha > 0$  such that  $\mu(F_i) > 2\alpha$  for each  $i \in \mathbb{N}$ .

Now we construct sets that will serve as “codes”. Since  $I$  contains all singletons, for each  $x \in E$  we may pick an open set  $W_x$  such that  $x \in W_x$  and  $\mu(\overline{W_x}) < \alpha$ . Let  $V_x$  be an open set such that  $x \in V_x \subseteq \overline{V_x} \subseteq W_x$ . The compactness of  $E$  allows us to pick finitely many  $x$ , say  $x_1, \dots, x_N$  such that  $E = \bigcup_{n=1}^N V_{x_n} = \bigcup_{n=1}^N \overline{V_{x_n}}$ . For  $n = 1, \dots, N$  set

$$A_n = V_{x_n}; \quad U_n = E \setminus \overline{V_{x_n}}; \quad L_i^n = F_i \setminus W_{x_n} \text{ for } i \in \mathbb{N}.$$

We thus have  $E = \bigcup_{n=1}^N A_n = \bigcup_{n=1}^N \overline{A_n}$  and for each  $n$  and  $i$ ,  $U_n \cap \overline{A_n} = \emptyset$ ,  $F_i^n \subseteq U_n$ , and  $\mu(L_i^n) > \alpha$ .

Fix  $n \leq N$ . By going to a subsequence we may now assume that the sequence  $(L_i^n)_{i \in \mathbb{N}}$  converges. Since the limit must necessarily be a subset of the set  $\lim_i F_i$ , the pairwise disjointness of  $\lim_i F_i$  and the sets  $F_i$  now allows us to find open sets  $U_i^n$ ,  $i \in \mathbb{N}$  such that  $L_i^n \subseteq U_i^n$  and the sets  $\overline{U_i^n}$  are disjoint and contained in  $U_n$ .

Now for each fixed  $n$  and  $i$ , since  $U = U_i^n$  has a subset (namely  $L_i^n$ ) of  $\mu$ -measure greater than  $\alpha$ , we may apply Lemma 25 with  $U = U_i^n$  and  $\delta = 1/i$  to obtain a set  $F_i^n$  and a sequence  $(K_{i,j}^n)_{j \in \mathbb{N}}$  satisfying the conclusions of the lemma. Further, by fixing a bijection between  $\{K_{i,j}^n : j \in \mathbb{N}\}$  and the basis  $\mathcal{B}$ , we re-index the set  $\{K_{i,j}^n : j \in \mathbb{N}\}$  as  $\{K_{i,B}^n : B \in \mathcal{B}\}$ .

For each fixed  $n, i$  and  $B \in \mathcal{B}$ , the set  $K_{i,B}^n$  as constructed in the lemma is the finite disjoint union of the closure of some open sets; if those sets are  $V_1, \dots, V_r$  then we shall refer to the sets  $\overline{V_k}$  as the “pieces of  $K_{i,B}^n$ ”. The set  $K_{i,B}^n$  thus consists of a finite number of disjoint closed pieces, each of which has non-empty interior and thus a positive  $\mu$ -measure. With this terminology, the lemma implies the following significant facts: for fixed  $n$  and  $i$ ,

- if  $K \in \mathcal{K}(E)$  and  $K \cap F_i^n \neq \phi$ , then for any  $B \in \mathcal{B}$ , we may add a piece of  $K_{i,B}^n$  to  $K$  while staying within  $1/i$  of  $K$  in the sense of the Hausdorff metric on  $\mathcal{K}(E)$ ;
- the collection  $\{K_{i,B}^n : B \in \mathcal{B}\}$  has a single limit set, which is a finite set contained in  $U_i^n$ ;
- if  $L$  is the limit of the collection  $\{K_{i,B}^n : B \in \mathcal{B}\}$ , then for any one piece  $P$  of some fixed set  $K_{i,B_0}^n$ , the set

$$\left( L \cup \bigcup_{B \in \mathcal{B}} K_{i,B}^n \right) \setminus P$$

is closed. (This follows immediately from points 1, 2 and 3 in the conclusion of Lemma 25.)

Now we define sets  $\mathcal{A}_{n,m}$  for  $n \leq N$  and  $m \in \mathbb{N}$ ; the closures of these sets will form the representing sequence for  $I$ . For  $F \in \mathcal{K}(E)$ , membership of  $F$  in  $\mathcal{A}_{n,m}$  will be determined by whether the complement of  $F$  has a specific cover; via the code sets  $F$  itself will determine what sets may appear in this cover. Before proceeding with the definition, we establish some shorthand that we will use in the context of a fixed  $n \leq N$ .

1. For any  $k \in \mathbb{N}$ , recall that  $\mu(F_k^n) > \alpha$ . If  $\mu(F \cap F_k^n) > \alpha/2$ , we will say that “ $F$  is big in  $F_k^n$ ”.

2. For any  $k \in \mathbb{N}$ , any  $B \in \mathcal{B}$  and any piece  $P$  of the set  $K_{k,B}^n$ , recall that  $\mu(P) > 0$ . If  $\mu(F \cap P) > \mu(P)/2$ , we will say that “ $F$  is big in a piece of  $K_{k,B}^n$ ”.
3. For any set  $B \in \mathcal{B}$ , recall that  $\mu(\overline{B})$  is positive (again due to the non-empty interior of  $\overline{B}$ ). Let  $k \in \mathbb{N}$  be such that  $\mu(\overline{B}) \in [\frac{1}{2^{k+1}}, \frac{1}{2^k})$ . We will say that “ $F$  allows  $B$ ” if  $F$  is big in  $F_k^n$  and  $F$  is big in a piece of  $K_{k,B}^n$ . Here we can think of the bigness of  $F$  in  $F_k^n$  as “allowing” a set of the size of  $B$ , and the bigness of  $F$  in some piece of  $K_{k,B}^n$  as then allowing the set  $B$  itself.

Now fix  $n \leq N$ ,  $m \in \mathbb{N}$ . We define  $\mathcal{A}_{n,m}$  thus: for any  $F \in \mathcal{K}(E)$ ,  $F \in \mathcal{A}_{n,m}$  if and only if, first,

$$\overline{E \setminus F} \setminus U_n \subseteq \bigcup_{j=1}^p B_j$$

for some  $p \in \mathbb{N}$  and some basic sets  $B_1, B_2, \dots, B_p \in \mathcal{B}$  satisfying all of the following:

- $\mu(\overline{B_j}) < \frac{1}{2^{m+1}}$  for each  $j = 1, \dots, p$ ,
- no two of the numbers  $\mu(\overline{B_j})$  for  $j = 1, \dots, p$  fall in the same binary interval of the form  $[\frac{1}{2^{k+1}}, \frac{1}{2^k})$ , and
- for each  $j = 1, \dots, p$ ,  $F$  allows  $B_j$ ;

and second,

$$\exists \delta > 0 \exists r \in \mathbb{N} \text{ such that } \forall i \geq r, \mu(F \cap F_i^n) > \alpha/2 + \delta.$$

This concludes the definition of  $\mathcal{A}_{n,m}$ .

From now on, for a set  $F \in \mathcal{A}_{n,m}$  satisfying the conditions above, we will say that the tuple  $\langle B_1, B_2, \dots, B_p, \delta, r \rangle$  witnesses the membership of  $F$  in  $\mathcal{A}_{n,m}$ . Note

that for such an  $F$ ,

$$\mu(\overline{E \setminus F} \setminus U_n) \leq \sum_{j=1}^p \mu(\overline{B_j}) < \sum_{k=m+1}^{\infty} \frac{1}{2^k} < \frac{1}{2^m}.$$

Set  $\mathcal{F}_{n,m} = \overline{\mathcal{A}_{n,m}}$ . Since any superset  $F'$  of a set  $F$  will be big in any set that  $F$  was big in, and its complement will only be smaller, if  $F$  is in  $\mathcal{A}_{n,m}$  then so will  $F'$  be. (The same witness will work.) It is thus immediate that each  $\mathcal{A}_{n,m}$ , and thus  $\mathcal{F}_{n,m}$ , is upward closed.

We now show that for any  $K \in \mathcal{K}(E)$ , if  $K$  is not in  $I$  then  $K^*$  contains some  $\mathcal{A}_{n,m}$  (and thus  $\mathcal{F}_{n,m}$ ); conversely, if  $K$  is in  $I$  then  $K^*$  has empty interior in each  $\mathcal{A}_{n,m}$  (and thus in each  $\mathcal{F}_{n,m}$ ). First assume that  $K$  is not in  $I$ . Since  $E = \bigcup_n \overline{A_n}$ , we have  $K = \bigcup_n K \cap \overline{A_n}$ . Since  $I$  is a  $\sigma$ -ideal, one of the sets in this union is not in  $I$ , say  $K \cap \overline{A_{n_0}}$ . Let  $m_0$  be such that  $\mu(K \cap \overline{A_{n_0}}) > \frac{1}{2^{m_0}}$ , in other words  $\mu(K \setminus U_{n_0}) > \frac{1}{2^{m_0}}$ . Let  $F \in \mathcal{A}_{n_0, m_0}$ ; since  $\mu(\overline{E \setminus F} \setminus U_{n_0}) < \frac{1}{2^{m_0}}$ , it cannot be the case that  $K \subseteq E \setminus F$ . Thus  $K^*$  contains  $\mathcal{A}_{n_0, m_0}$ .

Conversely, suppose that  $K$  is in  $I$  and fix  $n \leq N$ ,  $m \in \mathbb{N}$  and  $\epsilon > 0$ . Let  $F \in \mathcal{A}_{n,m}$  and let the tuple  $\langle B_1, B_2, \dots, B_p, \delta, r \rangle$  witness the membership of  $F$  in  $\mathcal{A}_{n,m}$ , exactly as in the definition above. For each  $j = 1, \dots, p$ ,  $F$  allows the set  $B_j$  by being big in the appropriate set  $F_{k_j}^n$  ( $k_j$  is determined by  $\mu(\overline{B_j})$ ) and by also being big in a piece of  $K_{k_j, B_j}^n$ . We denote this piece  $P_j$ . By replacing  $r$  with a larger number, we may assume that  $r > \max\{k_1, \dots, k_p\}$ ,  $1/r < \epsilon/2$  and  $1/2^r < \delta/2$ . Also assume that  $r$  is sufficiently large so that the removal from  $F$  of a basic open set  $V$  with  $\mu(\overline{V}) < 1/2^r$  will leave behind a set that is still big in each of the finitely many sets  $F_{k_j}^n$  and  $P_j$  for  $j = 1, \dots, p$ .

We wish to find a set in  $\mathcal{A}_{n,m} \setminus K^*$  within  $\epsilon$  of  $F$ . Now, since  $\mu(K) = 0$ , we can cover  $K$  with a basic set  $B$  in  $\mathcal{B}$  such that  $\mu(\overline{B}) < 1/2^r$ . Suppose that  $\mu(\overline{B}) \in [\frac{1}{2^{k+1}}, \frac{1}{2^k})$ . Note that  $k \geq r$ , so that  $1/k < \epsilon/2$  and  $1/2^k < \delta/2$ .

Since  $k \geq r$ ,  $F$  is big in  $F_k^n$ , in fact  $\mu(F \cap F_k^n) > \alpha/2 + \delta$ . Since in particular  $F \cap F_k^n \neq \emptyset$ , there exists a piece  $P$  of  $K_{k,B}^n$  such that  $P \cup F$  is within  $\epsilon/2$  of  $F$  in the sense of the Hausdorff metric. Let  $F' = (F \setminus B) \cup P$ . Since  $\mu(\overline{B}) < \delta/2$ , we have  $\mu(F' \cap F_k^n) > \alpha/2 + \delta/2$ . Outside of  $P$ , we have that  $F'$  is disjoint from  $K$ . Now cover  $K \cap P$  with an open set  $W$  such that  $W \subseteq B$  and  $\mu(\overline{W})$  is small enough to ensure the bigness of  $P \setminus W$  in  $P$ . Let  $F'' = F' \setminus W$ . We still have  $\mu(F'' \cap F_k^n) > \alpha/2 + \delta/2$ , and the tuple  $\langle B_1, B_2, \dots, B_p, B, \delta/2, r \rangle$  witnesses the membership of  $F''$  in  $\mathcal{A}_{n,m}$ . It is clear that  $F'' \notin K^*$ . The set  $F''$  is contained in  $F \cup P$  and  $F \cup P$  is within  $\epsilon/2$  of  $F$ ; to obtain a set in  $\mathcal{A}_{n,m} \setminus K^*$  within  $\epsilon$  of  $F$  we may now simply add to  $F''$  finitely many points not in  $K$  (making use of the facts that  $\mathcal{A}_{n,m}$  is upward closed and  $K$  is meager.)

Having established that the countable family  $\{\mathcal{F}_{n,m}\}$  determines membership in  $I$  for closed sets, now let  $G$  be a  $G_\delta$  set such that the closure of any nonempty relatively open subset of  $G$  is not in  $I$ . Let  $(F_n)$  be an increasing sequence of closed sets that are relatively meager in  $\overline{G}$  such that  $G = \overline{G} \setminus \bigcup_n F_n$ . Since  $G = \bigcup_{n=1}^N (G \cap A_n)$ , there exists an  $n_0$  such that  $G \cap A_{n_0}$  is nonempty. By considering the set  $G \cap A_{n_0}$  in place of  $G$ , we can assume that  $\overline{G}$  is disjoint from  $U_{n_0}$ . Now fix  $m_0$  such that  $\mu(\overline{G}) > 1/2^{m_0}$ , so that  $\mathcal{F}_{n_0,m_0} \subseteq \overline{G}^*$ . We show that Player II has a winning strategy in the Banach-Mazur game on  $G^*$  played in  $\mathcal{F}_{n_0,m_0}$ . Recall again what this means: Players I and II take turns playing basic open sets  $\mathcal{U}_n$  and  $\mathcal{V}_n$  respectively, satisfying  $\mathcal{U}_0 \supseteq \mathcal{V}_0 \supseteq \mathcal{U}_1 \supseteq \mathcal{V}_1 \supseteq \dots$ . (These are supposed to be basic open subsets of  $\mathcal{F}_{n_0,m_0}$ ; as usual we will simply use basic open subsets of  $\mathcal{K}(E)$  with the understanding that their intersection with  $\mathcal{F}_{n_0,m_0}$  is nonempty.) Our moves will be the sets  $\mathcal{V}_n$  played by II, and a winning strategy is one that ensures  $\bigcap_n \mathcal{V}_n \subseteq G^*$ .

Let  $F \in \mathcal{A}_{n_0,m_0}$  and let  $\langle B_1, B_2, \dots, B_p, \delta, r \rangle$  witness the membership of  $F$  in

$\mathcal{A}_{n_0, m_0}$ . As observed already, from the definition of  $\mathcal{A}_{n_0, m_0}$  we have

$$\mu(\overline{E \setminus F} \setminus U_{n_0}) \leq \mu\left(\bigcup_{j=1}^p \overline{B_j}\right) < 1/2^{m_0}$$

and since  $\mu(\overline{G}) > 1/2^{m_0}$ , the set

$$\overline{G} \setminus \bigcup_{j=1}^p \overline{B_j}$$

is a nonempty relatively open subset of  $\overline{G}$ . Thus we have rather more than the simple fact that  $F$  intersects  $\overline{G}$ .

Set  $D_{-1} = \overline{G}$  and  $F_{-1} = \emptyset$ . At the  $n^{\text{th}}$  stage of the game, Player II will construct  $D_n$ , a nonempty relatively open subset of  $\overline{G}$  and an open set  $V_n \subseteq E$  satisfying the following conditions:

- $D_n \subseteq D_{n-1}$ ,
- $\overline{D_n} \cap F_n = \emptyset$ ,
- $\mathcal{K}(V_n) \cap \mathcal{U}_n \cap \mathcal{F}_{n_0, m_0} \neq \emptyset$ ,
- $\mathcal{K}(V_n) \cap \mathcal{F}_{n_0, m_0} \subseteq \overline{D_n}^*$ , and in fact:
- for any  $F \in \mathcal{K}(V_n) \cap \mathcal{A}_{n_0, m_0}$ , if  $\langle B_1, B_2, \dots, B_p, \delta, r \rangle$  witnesses the membership of  $F$  in  $\mathcal{A}_{n_0, m_0}$ , then

$$\overline{D_n} \setminus \bigcup_{j=1}^p \overline{B_j} \text{ is nonempty.}$$

Player II's  $n^{\text{th}}$  move will then be  $\mathcal{V}_n = \mathcal{K}(V_n) \cap \mathcal{U}_n$ , and this will clearly result in a win for II.

Now we describe the construction of  $D_n$  and  $V_n$ . Suppose that Player I has

played  $\mathcal{U}_n$ , to which Player II must respond. At this point each of the following conditions holds:

- $\overline{D_{n-1}} \cap F_{n-1} = \emptyset$ ,
- $\emptyset \neq \mathcal{U}_n \cap \mathcal{F}_{n_0, m_0} \subseteq \overline{D_{n-1}}^*$ , and in fact:
- for any  $F \in \mathcal{U}_n \cap \mathcal{A}_{n_0, m_0}$ , if  $\langle B_1, B_2, \dots, B_p, \delta, r \rangle$  witnesses the membership of  $F$  in  $\mathcal{A}_{n_0, m_0}$  then

$$\overline{D_{n-1}} \setminus \bigcup_{j=1}^p \overline{B_j} \text{ is nonempty.}$$

For  $n = 0$  we have already shown that these conditions hold; for  $n > 0$  they are forced by Player II's previous move. Suppose that

$$\mathcal{U}_n = \{K \in \mathcal{K}(E) : K \subseteq W_0, K \cap W_i \neq \emptyset \forall i = 1, \dots, l\},$$

where the sets  $W_0, \dots, W_l$  are nonempty and open in  $E$  and  $\mathcal{U}_n \cap \mathcal{F}_{n_0, m_0} \neq \emptyset$ . Fix  $F \in \mathcal{U}_n \cap \mathcal{A}_{n_0, m_0}$ . Let  $\langle B_1, B_2, \dots, B_p, \delta, r \rangle$  witness the membership of  $F$  in  $\mathcal{A}_{n_0, m_0}$ . Since the open set

$$U = E \setminus \bigcup_{j=1}^p \overline{B_j}$$

intersects  $\overline{D_{n-1}}$ , it intersects  $D_{n-1}$ . Therefore, within  $D_{n-1}$  we can find a further open subset of  $\overline{G}$ , say  $D_n$ , such that  $\overline{D_n} \subseteq U$  and  $\overline{D_n} \cap F_n = \emptyset$ . (Recall that  $F_n$  is meager in  $\overline{G}$ .) By the original condition on  $\overline{G}$ , we know that  $\mu(\overline{D_n}) > 0$ . Fix a positive  $\delta < \mu(\overline{D_n})$ .

Now the dénouement. For each  $j = 1, \dots, p$ , we have that  $F$  is big in  $F_{k_j}^{n_0}$  for the appropriate  $k_j$  (determined by  $\mu(\overline{B_j})$ ); further,  $F$  is big in some piece of  $K_{k_j, B_j}^{n_0}$ , a piece that we shall denote  $P_j$ . Fix  $s \in \mathbb{N}$  such that  $s > r$ ,  $s > \max\{k_j : j = 1, \dots, p\}$  and  $\sum_{i=s}^{\infty} 1/2^i < \delta$ .

Let  $W$  be the set obtained by removing from  $W_0$  all of the following:

- remove all the sets  $F_i^{n_0}$  for  $i < s$ , other than the sets  $F_{k_j}^{n_0}$  for  $j = 1, \dots, p$ ;
- for each  $j = 1, \dots, p$ , remove  $\bigcup_{B \in \mathcal{B}} K_{k_j, B}^{n_0} \setminus P_j$ , along with the single finite limit set of the collection  $\{K_{k_j, B}^{n_0} : B \in \mathcal{B}\}$ .

By construction, the set being removed is closed, so that  $W$  is in fact open. We now establish that  $\mathcal{K}(W) \cap \mathcal{A}_{n_0, m_0} \neq \emptyset$ . Let

$$F' = \left( F \setminus \bigcup_{i < s} U_i^{n_0} \right) \cup \left( F \cap \left( \bigcup_{j=1}^p F_{k_j}^{n_0} \cup \bigcup_{j=1}^p P_j \right) \right).$$

Recall that for each  $k \in \mathbb{N}$ , the set  $U_k^{n_0}$  contains  $F_k^{n_0}$  as well as each set of the collection  $\{K_{k, B}^{n_0} : B \in \mathcal{B}\}$ , and  $U_k^{n_0}$  also contains the limit of this collection. It is thus clear that  $F'$  is in  $\mathcal{K}(W)$ ; the fact that it is in  $\mathcal{A}_{n_0, m_0}$  is witnessed by  $\langle B_1, B_2, \dots, B_p, \delta, s \rangle$ . To confirm that this witness works, we note that

$$E \setminus F' \subseteq E \setminus F \cup \bigcup_{i < s} U_i^{n_0},$$

and therefore

$$\overline{E \setminus F'} \subseteq \overline{E \setminus F} \cup \bigcup_{i < s} \overline{U_i^{n_0}}.$$

Recalling that for each  $i \in \mathbb{N}$ ,  $\overline{U_i^{n_0}} \subseteq U_{n_0}$ , we thus have

$$\overline{E \setminus F'} \setminus U_{n_0} \subseteq \overline{E \setminus F} \setminus U_{n_0} \subseteq \bigcup_{j=1}^p B_j,$$

and  $F'$  allows the sets  $B_1, B_2, \dots, B_p$  by being big in all the relevant codes. Thus  $F' \in \mathcal{K}(W) \cap \mathcal{A}_{n_0, m_0}$ .

Note that although  $F' \subseteq W_0$ , it may be the case that  $F' \notin \mathcal{U}_n$ . By adding to  $F'$  a finite set, say  $\{x_1, \dots, x_l\}$ , we obtain a set  $F''$  in  $\mathcal{U}_n \cap \mathcal{A}_{n_0, m_0}$ . We can put an open set  $W'$  about this finite set that is not big in any code set  $F_k^{n_0}$  for  $k < s$ .

If we have chosen the  $x_i$  carefully, we can also make sure that for  $k < s$ ,  $W'$  is not big in any piece of any set  $K_{k,B}^{n_0}$  for  $B \in \mathcal{B}$ . Let us explain what we mean by a careful choice. For  $k < s$  consider the finitely many collections  $\{K_{k,B}^{n_0} : B \in \mathcal{B}\}$ . Each of these collections has a finite and hence meager limit set. As long as we make sure that  $\{x_i : i = 1, \dots, l\}$  is disjoint from the union of these limit sets, the set  $W'$  can be chosen as described.

Note that if  $H$  is a compact subset of  $W \cup W'$  and  $k < s$ , then  $H$  can be big in  $F_k^{n_0}$  only for  $k = k_1, \dots, k_j$ . Further, for each  $k_1, \dots, k_j$ ,  $H$  can be big in  $K_{k_j,B}^{n_0}$  only for  $B = B_j$ .

Now set  $V_n = W \cup W'$ . We claim that the set  $V_n$  is as required. First note that since  $F'' \subseteq V_n$  by design,  $\mathcal{K}(V_n) \cap \mathcal{U}_n \cap \mathcal{F}_{n_0, m_0} \neq \emptyset$ . Now let  $H \in \mathcal{K}(V_n) \cap \mathcal{A}_{n_0, m_0}$ , where  $\langle B'_1, B'_2, \dots, B'_{p'}, \delta', r' \rangle$  witnesses the membership of  $H$  in  $\mathcal{A}_{n_0, m_0}$ .

We wish to show simply that

$$\overline{D_n} \setminus \bigcup_{j=1}^{p'} \overline{B'_j} \text{ is nonempty.}$$

But what is the worst that can happen, i.e., what is the biggest that  $\bigcup_j \overline{B'_j}$  can be? Each set  $B'_j$  has to have been allowed by  $H$ . Suppose that  $\mu(\overline{B'_j}) \in [\frac{1}{2^{k'_j+1}}, \frac{1}{2^{k'_j}})$ . Since  $H \subseteq V$ , if  $k'_j < s$  the set  $B'_j$  must be one of the sets  $B_j$  for  $j = 1, \dots, p$ . We already know that the whole of  $\overline{D_n}$  lies outside  $\bigcup_{j=1}^p \overline{B_j}$ . So

$$\overline{D_n} \setminus \bigcup_{j=1}^{p'} \overline{B'_j} = \overline{D_n} \setminus \bigcup_{\substack{j=1 \\ k'_j \geq s}}^{p'} \overline{B'_j},$$

and, by the choice of  $s$ , the latter union is too small to contain  $\overline{D_n}$ .

The sets  $V_n$  and  $D_n$  thus satisfy all the required conditions, and we are done.  $\square$

We conclude with a condition that implies the existence of a discrete sequence of uniformly big closed sets as in the hypothesis of Theorem 24. We will use the following theorem from [8].

**Theorem 26** (Kechris–Louveau–Woodin). *Let  $I$  be a  $\mathbf{\Pi}_1^1$   $\sigma$ -ideal of compact sets in a compact metrizable space  $E$ . Let  $A \subseteq E$  be  $\mathbf{\Pi}_2^0$ . If  $\mathcal{K}(A)$  contains uncountably many pairwise disjoint sets not in  $I$ , then there is a continuous function  $\phi : 2^\omega \rightarrow \mathcal{K}(A)$  such that*

1.  $\forall \alpha \in 2^\omega \quad \phi(\alpha) \notin I$ , and
2.  $\forall \alpha, \beta \in 2^\omega \quad \alpha \neq \beta \Rightarrow \phi(\alpha) \cap \phi(\beta) = \emptyset$ .

**Proposition 27.** *Let  $E$  be a compact metric space and let  $\mu$  be a real valued function defined on  $\mathcal{K}(E)$  such that the set  $I = \{K \in \mathcal{K}(E) : \mu(K) = 0\}$  is a  $\mathbf{\Pi}_1^1$   $\sigma$ -ideal of compact sets. If there exist uncountably many pairwise disjoint closed sets not in  $I$ , then there exist pairwise disjoint meager sets  $F, F_n, n \in \mathbb{N}$  such that  $F_n \rightarrow F$  and  $\inf_n \mu(F_n) > 0$ .*

*Proof.* Taking  $A = E$  in the hypothesis of Theorem 26, let  $\phi$  be the function given by that theorem. For  $k \in \mathbb{N}$ , clearly the sets  $\mathcal{A}_k = \{\alpha \in 2^\omega : \mu(\phi(\alpha)) > 1/k\}$  cannot all be finite. Fix  $k$  such that  $\mathcal{A}_k$  is infinite and in  $\mathcal{A}_k$  find a convergent sequence of distinct elements  $(\alpha_n)_{n \in \mathbb{N}}$ . Let  $\lim_n \alpha_n = \alpha$ , say. We may assume that for all  $n \in \mathbb{N}$ ,  $\alpha \neq \alpha_n$ . Set  $F_n = \phi(\alpha_n)$  and  $F = \phi(\alpha)$ .  $\square$

**Proposition 28.** *Let  $I \subseteq \mathcal{K}(E)$  contain only meager sets and suppose that for any dense  $G_\delta$  set  $G \subseteq E$ ,  $\mathcal{K}(G) \cap I \neq MGR(G)$  (i.e.,  $\mathcal{K}(G) \cap I \subsetneq MGR(G)$ ). Then  $E$  has uncountably many pairwise disjoint closed subsets not in  $I$ .*

*Proof.* For each ordinal  $\alpha < \omega_1$ , suppose we have defined pairwise disjoint compact meager sets  $F_\beta \notin I$  for all  $\beta < \alpha$ . The set  $E \setminus \bigcup_{\beta < \alpha} F_\beta$  is a nonempty dense  $G_\delta$  subset of  $E$ , so within it we may pick a meager  $F_\alpha \notin I$ . The conclusion follows.  $\square$

### 3.4 A second conjecture

Let  $E$  be a compact metric space and  $I \subseteq \mathcal{K}(E)$ .

**Definition 29.**  $I$  is calibrated if, for any  $K \in \mathcal{K}(E)$  and any sequence of sets  $(K_n)_{n \in \mathbb{N}} \subseteq I$ , if  $\mathcal{K}(K \setminus \bigcup_n K_n) \subseteq I$ , then  $K \in I$ .

For  $A \subseteq E$ , recall the definition of  $A^+$  on page 19:

$$A^+ = \{K \in \mathcal{K}(E) : K \cap A \text{ cannot be covered by countably many sets in } I\}.$$

Solecki shows in [13] that an ideal which has property  $(*)$  and is also calibrated can be represented through the operation  $A^*$  as well as through the operation  $A^+$ :

**Theorem 30** (Solecki). *Suppose  $I \subseteq \mathcal{K}(E)$  is co-analytic and nonempty. Then  $I$  has property  $(*)$  and is calibrated if and only if there exists a closed subset  $\mathcal{F}$  of  $\mathcal{K}(E)$ , such that for any  $K \in \mathcal{K}(E)$ ,*

$$K \in I \iff K^* \cap \mathcal{F} \text{ is meager in } \mathcal{F} \iff K^+ \cap \mathcal{F} \text{ is meager in } \mathcal{F}.$$

(In fact, this condition holds for any  $\mathcal{F}$  as in Theorem 2, i.e., if the first of these equivalences holds for  $\mathcal{F}$  for all closed  $K$ , then so does the second.)

The relation of the following conjecture to this theorem is analogous to that of the previous conjecture to Theorem 2.

**Conjecture 2.** *Suppose  $I$  is a calibrated  $G_\delta$  ideal of compact sets with property  $(*)$  containing only meager sets. Then there exists a sequence  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  of closed and upward closed subsets of  $\mathcal{K}(E)$ , such that for all  $G_\delta$  sets  $G \subseteq E$ ,*

$$G^* \text{ meager in each } \mathcal{F}_n \iff G \subseteq \bigcup_n K_n \text{ for some } K_n \in I;$$

$$G^+ \text{ meager in each } \mathcal{F}_n \iff \forall K \subseteq G, K \in I.$$

While we have not addressed this conjecture here, a natural starting point would be to investigate the second of the two displayed conditions for the sequences  $(\mathcal{F}_n)$  constructed for the examples of this dissertation, for which the first condition has been shown.

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