NASH-WILLIAMS’ ARGUMENT IN INFINITE RAMSEY THEORY AS AN INDUCTION PRINCIPLE

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1. Introduction

In [2], Nash-Williams found an extension of the proof of Ramsey’s theorem [3] that turned out to be of utmost importance in infinite Ramsey theory. He employed his argument to carry over Ramsey’s theorem to certain families of finite subsets of \( \mathbb{N} \) (see also [4, Theorem 1.14]).\(^1\) Later, stimulated by work of Galvin and Prikry on an extension of Ramsey’s theorem to families of infinite subsets of \( \mathbb{N} \), Ellentuck [1] refined Nash-Willimas’ argument and connected it with topological ideas (see also [4, Theorem 1.54]). Further generalizations in multiple directions of Ellentuck’s method gave rise to rich developments in infinite dimensional Ramsey theory. One can learn about them from Todorcevic’s book [4].

So Nash-Williams’ argument is fundamental. In this note, I will try to convince the reader that it can be viewed as an instance of an even more fundamental principle—induction. In Proposition 1, I will formulate (and prove) an induction principle that encapsulates Nash-Williams’ argument. Later, I will tell the reader how to obtain Ellentuck’s theorem for closed sets from the induction principle.

The material in Section 2 is elementary and self-contained; Section 3 requires only some general topology.

2. The induction principle

By convention 0 is a natural number and each natural number \( n \) is identified with the set of its predecessors, that is, \( n = \{ i \in \mathbb{N} : i < n \} \). In particular, \( 0 = \emptyset \).

Let \( \mathbb{N}^{< \infty} \) be the family of all finite subsets of \( \mathbb{N} \), including the empty set. Let \( \mathbb{N}^{\infty} \) be the family of all infinite subsets of \( \mathbb{N} \). We use the letters \( s, t \) to denote elements of \( \mathbb{N}^{< \infty} \) and the letters \( M, N, P \) possibly with subscripts to denote elements of \( \mathbb{N}^{\infty} \).

Throughout the note, \( \mathcal{R} \) will be a subset of \( \mathbb{N}^{< \infty} \times \mathbb{N}^{\infty} \). We write \( \mathcal{R}(s, M) \) for \( (s, M) \in \mathcal{R} \). We assume that

\[
(1) \quad \mathcal{R}(s, M) \text{ and } M \subseteq N \implies \mathcal{R}(s, N)
\]

and

\[
(2) \quad \mathcal{R}(s, M) \implies \mathcal{R}(s, M \setminus (1 + \max s)),
\]

where, by convention, \( \max \emptyset = -1 \).

\(^1\)For a fixed natural number \( k > 0 \), the family of \( k \) element subsets of \( \mathbb{N} \) is one to which Nash-Williams’ result applies and gives Ramsey’s original theorem.
Proposition 1. Assume that for each \( s \) and \( M \)
- \( \mathcal{R}(\emptyset, M) \) and
- \( \mathcal{R}(s, M) \) implies that there exists \( n \in M \) with \( \mathcal{R}(s \cup \{n\}, M) \).

Then there exists \( M \) such that \( \mathcal{R}(s, N) \) for all \( s, N \subseteq M \).

The induction in Proposition 1 is parametrized by \( M \) varying over \( \mathbb{N}^{\infty} \). Fixing such an \( M \), the first assumption in the proposition is the 0-th step of the induction and the second assumption is the inductive step.

Our proof modifies and rearranges parts of Nash-Willimas’ argument. We start it with a lemma.

Lemma 2. There exists \( M \) such that for each \( s \subseteq M \), we have that if \( \mathcal{R}(s, N) \) for some \( N \subseteq M \), then \( \mathcal{R}(s, N) \) for all \( N \subseteq M \).

Proof. It suffices to find \( M \) such that for each \( s \subseteq M \) if \( \mathcal{R}(s, M \setminus (1 + \max s)) \), then \( \mathcal{R}(s, P) \) for all \( P \subseteq M \setminus (1 + \max s) \). Indeed, assume we have such \( M \). Given \( s \subseteq M \), let \( N_1 \subseteq M \) be such that \( \mathcal{R}(s, N_1) \). Let \( N_2 \subseteq M \). By (2) and (1), \( \mathcal{R}(s, N_1) \) implies that \( \mathcal{R}(s, M \setminus (1 + \max s)) \). Our assumption on \( M \) gives that \( \mathcal{R}(s, N_2 \setminus (1 + \max s)) \).

Applying again (2), we get \( \mathcal{R}(s, N_2) \).

Now we construct \( M \) as above. This is done by a diagonal argument as follows. For each \( l \), we find sets \( N_i \) and numbers \( k_1 < \cdots < k_{l+1} \) such that \( k_{i+1} = \min N_i \), \( N_{l+1} \subseteq N_l \), and for each \( s \subseteq \{k_1, \ldots, k_l\} \) if there exists \( P \subseteq N_l \) with \( \neg \mathcal{R}(s, P) \), then \( \neg \mathcal{R}(s, N_l) \). This will suffice since the set \( M = \{k_i : i \geq 1\} \) is as required. Indeed, let \( s \subseteq M \) and \( \mathcal{R}(s, M \setminus (1 + \max s)) \). Then there is \( l \) such that \( s \subseteq \{k_1, \ldots, k_l\} \) and \( M \setminus (1 + \max s) = \{k_i : i \geq l + 1\} \). Since \( \{k_i : i \geq l + 1\} \subseteq N_l \), by (1), we get that \( \mathcal{R}(s, N_l) \). Thus, by our construction we have that \( \mathcal{R}(s, P) \) for each \( P \subseteq N_l \), so also \( \mathcal{R}(s, P) \) for each \( P \subseteq M \setminus (1 + \max s) \).

Now we perform the construction. If there is \( P \subseteq N \) with \( \neg \mathcal{R}(\emptyset, P) \), let \( N_0 \) be such that \( \neg \mathcal{R}(\emptyset, N_0) \); if there is no such \( P \), let \( N_0 = \mathbb{N} \). Put \( k_1 = \min N_0 \). The desired conditions hold for \( l = 0 \).

Let \( l \geq 1 \), and assume we have \( N_{l-1} \) and \( k_1, \ldots, k_l \). Let \( s_i, i < 2^l \), be an enumeration of all subsets of \( \{k_1, \ldots, k_l\} \). Define \( P_i, i < 2^l \), as follows. Let \( P_0 = N_{l-1} \setminus \{k_l\} \). Given \( P_i \) for \( i < 2^l \), if there is \( P \subseteq P_i \) with \( \neg \mathcal{R}(s, P) \), let \( P_{i+1} \subseteq P_i \) be such that \( \neg \mathcal{R}(s_i, P_{i+1}) \), if there is no such \( P \subseteq P_i \), let \( P_{i+1} = P_i \). This construction leads to a set \( P_{2^l} \). Define \( N_l = P_{2^l} \) and \( k_{l+1} = \min N_l \). We only need to check that given \( s \subseteq \{k_1, \ldots, k_l\} \) if there is \( P \subseteq N_l \) with \( \neg \mathcal{R}(s, P) \), then \( \neg \mathcal{R}(s, N_l) \), the other conditions on \( N_l \) and \( k_{l+1} \) being obviously fulfilled. Fix \( i < 2^l \) such that \( s = s_i \). Fix \( P \subseteq N_l \) with \( \neg \mathcal{R}(s, P) \). Then \( P \subseteq P_i \), so, by our construction, \( \neg \mathcal{R}(s, P_{i+1}) \), which implies, by (1), that \( \neg \mathcal{R}(s, N_l) \) since \( P_{i+1} \supseteq N_l \).

The set \( M \) produced in Lemma 2 has the property that for \( s \subseteq M \), \( \mathcal{R}(s, N) \) does not depend on the second variable \( N \subseteq M \).

Lemma 3. Assume that for each \( s \) and \( M \)
- \( \mathcal{R}(\emptyset, M) \) and
- \( \mathcal{R}(s, M) \) implies that there exists \( N \subseteq M \) such that for each \( n \in N \), \( \mathcal{R}(s \cup \{n\}, M) \).
Then there exists \( M \) such that \( R(s, N) \) for all \( s, N \subseteq M \).

**Proof.** Let \( M_0 \) be as in Lemma 2. For simplicity of notation we assume that \( M_0 = \mathbb{N} \). Then \( R(s, M) \) does not depend on \( M \), so we write simply \( R(s) \). The two assumptions of the lemma can be restated as

- \( R(\emptyset) \)
- if \( R(s) \), then for each \( M \) there exists \( N \subseteq M \) such that \( R(s \cup \{n\}) \) for each \( n \in N \).

We aim to find \( M \) such that \( R(s) \) for each \( s \subseteq M \).

By recursion on \( l \in \mathbb{N} \), we find numbers \( k_1 < \cdots < k_l \) so that for each \( l, R(s) \) for each \( s \subseteq \{k_1, \ldots, k_l\} \). If this construction is performed, the set \( M = \{k_l : l \geq 1\} \) will be as needed.

For \( l = 0 \), only \( s = \emptyset \) needs to be considered. By our first assumption, we have \( R(\emptyset) \), as required. Assume we have \( k_1, \ldots, k_l \). Let \( s_1, \ldots, s_{2^l} \) list all subsets of \( \{k_1, \ldots, k_l\} \). By our recursive assumption, we have \( R(s_i) \) for each \( 1 \leq i \leq 2^l \). The second assumption of the lemma allows us to produce by recursion sets \( P_l \) for \( 0 \leq i \leq 2^l \) so that \( P_0 = \mathbb{N} \) and, for \( i < 2^l \), \( P_{i+1} \subseteq P_i \) and \( R(s_{i+1} \cup \{n\}) \) for each \( n \in P_{i+1} \). Let \( k_{l+1} \) be an element of \( P_{2^l} \) with \( k_l < k_{l+1} \). Since \( k_{l+1} \in P_i \) for each \( 1 \leq i \leq 2^l \), we have that \( R(s_i \cup \{k_{l+1}\}) \) for each such \( i \). Since we also have \( R(s_i) \), for \( 1 \leq i \leq 2^l \), we are done.

**Proof of Proposition 1.** Let \( M_0 \) be as in Lemma 2. For simplicity of notation, we assume that \( M_0 = \mathbb{N} \). Again \( R(s, M) \) does not depend on \( M \), and so we write \( R(s) \). We check that the second assumption of Lemma 3 holds, which will give the conclusion. Let \( s \) with \( R(s) \) be given. If the second assumption of Lemma 3 fails, then there exists \( N \) such that \( \neg R(s \cup \{n\}) \) for each \( n \in N \). Then the second assumption of the current proposition implies that \( \neg R(s) \), contradiction.

**3. Ellentuck’s theorem for Ellentuck closed sets**

We derive Ellentuck’s theorem for Ellentuck closed sets (see [4, Lemma 1.52, Theorem 1.54]) from Proposition 1. For \( s \in \mathbb{N}^{[<\infty]} \) and \( M \in \mathbb{N}^{[\infty]} \), \( [s, M] \) stands for the set of all \( N \in \mathbb{N}^{[\infty]} \) such that

\[
s = N \cap (1 + \max s) \quad \text{and} \quad N \setminus s \subseteq M.
\]

Sets of the form \([s, M] \) form a basis of a topology on \( \mathbb{N}^{[\infty]} \). This topology is called the Ellentuck topology. It refines the topology \( \mathbb{N}^{[\infty]} \) gets when viewed as a subset of the compact space of all subsets of \( \mathbb{N} \), that is, the space \( \{0, 1\}^{\mathbb{N}} \).

Let \( X \subseteq \mathbb{N}^{[\infty]} \) be Ellentuck closed. We aim to show that there is \( M \) with \([0, M] \cap X = \emptyset \) or \([0, M] \subseteq X \).

For \( s \in \mathbb{N}^{[<\infty]} \) and \( M \in \mathbb{N}^{[\infty]} \), define \( R(s, M) \) to hold precisely when \([s, M] \cap X \neq \emptyset \). It is clear that \( R \) fulfills (1) and (2) since \([s, M] \subseteq [s, N] \) if \( M \subseteq N \) and \([s, M] = [s, M \setminus (1 + \max s)] \). Note that if \( R(\emptyset, M) \) fails for some \( M \), then for this \( M \) we have \([0, M] \cap X = \emptyset \) and we are done. So we can assume that the first assumption in Proposition 1 holds. One easily checks that the second assumption holds since if \( N \in [s, M] \cap X \), then \( N \in [s \cup \{\min(N \setminus s)\}, M] \) and \( \min(N \setminus s) \in M \). Now it follows from Proposition 1 that there exists \( M \) such that \( R(s, N) \) holds for
each \( s, N \subseteq M \), that is, \([s, N] \cap X \neq \emptyset\) for each \( s, N \subseteq M \). Since \( X \) is closed, we immediately get that \([\emptyset, M] \subseteq X\).

It must be pointed out that with some additional work the statement above can be proved not only for sets closed with respect to the Ellentuck topology but also for sets that have the Baire property with respect to this topology. This is a large class of sets containing, for example, all subsets of \( \mathbb{N}^{[\infty]} \) that are analytic with respect to the topology inherited from \( \{0, 1\}^\mathbb{N} \). For proofs of these additional facts we refer the reader to [1] and [4].

References