

# Strong continuity of invariant probability charges

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## Abstract

Consider a semigroup action on a set. We derive conditions, in terms of the induced action of the semigroup on  $\{0, 1\}$ -valued probability charges, which ensure that all invariant probability charges are strongly continuous.

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## 1 Introduction

Let  $\mathcal{A}$  be an algebra of subsets of some set  $X$  and let  $G$  be a semigroup acting from the left on  $X$ . Assume that  $\mathcal{A}$  is  $G$ -invariant, that is,  $g^{-1}A = \{x \in X : gx \in A\} \in \mathcal{A}$  for every  $g \in G$ ,  $A \in \mathcal{A}$ . By a probability charge ( $p$ -charge) on  $\mathcal{A}$  we mean a probability finitely additive measure on  $\mathcal{A}$ . In this setting  $p$ -charge  $\mu$  on  $\mathcal{A}$  is said to be  $G$ -invariant if  $g\mu = \mu$  for every  $g \in G$ , where  $g\mu(A) = \mu(g^{-1}A)$ . The  $p$ -charge  $\mu$  is called strongly continuous if for every  $\varepsilon > 0$  there exists a finite partition  $\{A_1, \dots, A_n\}$  of  $X$  in  $\mathcal{A}$  with  $\sup_{1 \leq i \leq n} \mu(A_i) \leq \varepsilon$ .

In this paper, we are interested in conditions on  $\mathcal{A}$ ,  $G$ , and the action of  $G$  on  $X$  which ensure that invariance of a  $p$ -charge implies its strong continuity. Note that a  $p$ -charge  $\mu$  defined on a  $\sigma$ -algebra is strongly continuous if and only if  $\mu$  is strongly nonatomic, i.e.  $\{\mu(B) : B \in \mathcal{A}, B \subset A\} = [0, \mu(A)]$  for every  $A \in \mathcal{A}$  (cf. [3, 5.1.6, 11.4.5]). A further interesting feature of strongly continuous  $p$ -charges can be found in [10, Theorem 3]. For existence of invariant  $p$ -charges the reader is referred to Wagon [16] and Paterson [12].

## 2 Main results

For  $p$ -charges  $\lambda$  and  $\mu$  on  $\mathcal{A}$  we write  $\lambda \ll_w \mu$  if  $\lambda$  is weakly absolutely continuous with respect to  $\mu$ , i.e.  $\lambda(A) = 0$  whenever  $A \in \mathcal{A}$  and  $\mu(A) = 0$ . The following theorem is fundamental to the other results in the paper.

**Theorem 1** *Let  $\mu$  be a  $G$ -invariant  $p$ -charge on  $\mathcal{A}$ . If the orbit  $G\lambda = \{g\lambda : g \in G\}$  is infinite for every  $\{0, 1\}$ -valued  $p$ -charge  $\lambda$  on  $\mathcal{A}$  with  $\lambda \ll_w \mu$ , then  $\mu$  is strongly continuous.*

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We present two proofs of the above result: the first one based on representing a  $p$ -charge as the barycenter of a regular probability measure on the space of all  $\{0, 1\}$ -valued  $p$ -charges and the second one of an elementary nature.

The first proof requires two lemmas. Let  $K$  denote the set of all  $p$ -charges on  $\mathcal{A}$  equipped with the weak\* topology and  $H$  the set of all  $\{0, 1\}$ -valued  $p$ -charges on  $\mathcal{A}$ . Since  $K$  is a (compact) Bauer simplex and  $H$  is the extreme boundary of  $K$ , the map

$$K \rightarrow M^1(H), \mu \mapsto \rho_\mu$$

with

$$\mu(A) = \int_H \lambda(A) d\rho_\mu(\lambda), A \in \mathcal{A}$$

is an affine homeomorphism, where  $M^1(H)$  denotes the set of all regular probability measures on the Borel  $\sigma$ -algebra  $\mathcal{B}(H)$  of  $H$  equipped with the vague topology (cf. [2, II. 4.2]). The semigroup  $G$  acts from the left on  $H$  by  $\lambda \mapsto g\lambda$ . This action is (weak\*-)continuous and thus  $\mathcal{B}(H)$  is  $G$ -invariant. The induced left action of  $G$  on  $M^1(H)$  is given by  $\rho \mapsto g\rho$ ,  $g\rho(B) = \rho(g^{-1}B)$ ,  $B \in \mathcal{B}(H)$ , where regularity of  $g\rho$  can be seen as follows. For every  $B \in \mathcal{B}(H)$ , we have

$$\begin{aligned} g\rho(B) &= \sup\{\rho(F) : F \text{ compact, } F \subset g^{-1}B\} \\ &\leq \sup\{\rho(F) : F \text{ compact, } gF \subset B\} \\ &\leq \sup\{\rho(g^{-1}gF) : F \text{ compact, } gF \subset B\} \\ &\leq \sup\{g\rho(D) : D \text{ compact, } D \subset B\} \\ &\leq g\rho(B) \end{aligned}$$

which implies

$$g\rho(B) = \inf\{g\rho(O) : O \text{ open, } O \supset B\}.$$

**Lemma 1**  $K \rightarrow M^1(H)$ ,  $\mu \mapsto \rho_\mu$  is  $G$ -equivariant, that is,  $\rho_{g\mu} = g\rho_\mu$  for every  $g \in G, \mu \in K$ .

**Proof.** Let  $\mathcal{C}$  denote the algebra of subsets of  $H$  which are both compact and open. It is well known that  $\varphi : \mathcal{A} \rightarrow \mathcal{C}, \varphi(A) = \{\lambda \in H : \lambda(A) = 1\}$  is an isomorphism of algebras. (Note that  $\varphi(A)$  coincides with the weak\* closure of  $\{\varepsilon_x : x \in A\}$ ). We have  $\mu(\varphi^{-1}C) = \rho_\mu(C)$  and  $\varphi^{-1}(g^{-1}C) = g^{-1}(\varphi^{-1}C)$  which gives

$$\begin{aligned} g\rho_\mu(C) &= \rho_\mu(g^{-1}C) = \mu(g^{-1}(\varphi^{-1}C)) \\ &= g\mu(\varphi^{-1}C) = \rho_{g\mu}(C) \end{aligned}$$

for every  $C \in \mathcal{C}, g \in G$ . Since  $\mathcal{C}$  is a basis of the weak\* topology on  $H$  and  $g\rho_\mu$  and  $\rho_{g\mu}$  are regular, this yields  $g\rho_\mu = \rho_{g\mu}$ .  $\square$

For  $\rho \in M^1(H)$ , let  $\text{supp}(\rho)$  denote the topological support of  $\rho$ . Part (c) of the following lemma is well known.

**Lemma 2** Let  $\mu \in K$ .

- (a)  $\text{supp}(\rho_\mu) = \{\lambda \in H : \lambda \ll_w \mu\}$ .
- (b)  $\{\lambda \in H : \rho_\mu(\{\lambda\}) > 0\} = \{\lambda \in H : c_\lambda \lambda \leq \mu \text{ for some } c_\lambda > 0\}$ .
- (c)  $\mu$  is strongly continuous if and only if  $\rho_\mu$  is continuous.

**Proof.** Let  $\mathcal{C}$  and  $\varphi$  be defined as in the proof of Lemma 1.

(a) Let  $\lambda \in H$ . Then  $\lambda \in \text{supp}(\rho_\mu)$  if and only if  $\rho_\mu(C) > 0$  for every  $C \in \mathcal{C}$  with  $\lambda \in C$ . Since  $\rho_\mu(\varphi(A)) = \mu(A)$ ,  $A \in \mathcal{A}$ , the latter condition means  $\mu(A) > 0$  for every  $A \in \mathcal{A}$  with  $\lambda(A) = 1$  and this is equivalent to  $\lambda \ll_w \mu$ .

(b) For  $\lambda \in H$ , let  $\mathcal{A}_\lambda = \{A \in \mathcal{A} : \lambda(A) = 1\}$ . Then  $\bigcap_{A \in \mathcal{A}_\lambda} \varphi(A) = \{\lambda\}$  and regularity of  $\rho_\mu$  gives

$$\rho_\mu(\{\lambda\}) = \inf_{A \in \mathcal{A}_\lambda} \rho_\mu(\varphi(A)) = \inf_{A \in \mathcal{A}_\lambda} \mu(A).$$

If  $\rho_\mu(\{\lambda\}) > 0$ , choose  $c_\lambda = \inf_{A \in \mathcal{A}_\lambda} \mu(A)$ . Then  $c_\lambda \lambda \leq \mu$ . If  $c_\lambda \lambda \leq \mu$  for some  $c_\lambda > 0$ , then  $\mu(A) \geq c_\lambda$  for every  $A \in \mathcal{A}_\lambda$  and hence  $\rho_\mu(\{\lambda\}) > 0$ .

(c) follows e.g. from (b) and the fact that  $\mu$  is strongly continuous if and only if the right hand side of (b) is empty (cf. e.g. [11, Example (2)]).  $\square$

**First proof of Theorem 1.** By Lemma 1,  $\rho_\mu$  is  $G$ -invariant. Therefore

$$\rho_\mu(\{g\lambda\}) = \rho_\mu(g^{-1}\{g\lambda\}) \geq \rho_\mu(\{\lambda\})$$

for every  $\lambda \in H, g \in G$ . The assumption yields  $\rho_\mu(\{\lambda\}) = 0$  for every  $\lambda \in H$  with  $\lambda \ll_w \mu$ . Hence, according to Lemma 2(a),  $\rho_\mu$  is continuous. The assertion follows from Lemma 2(c).  $\square$

**Second proof of Theorem 1.** Fix a  $p$ -charge  $\mu$  on an algebra  $\mathcal{A}$  of subsets of  $X$ . For a  $p$ -charge  $\lambda$  on  $\mathcal{A}$ , we write  $\lambda \ll \mu$  if for any  $\epsilon > 0$  there is  $\delta > 0$  such that for  $A \in \mathcal{A}$ ,  $\mu(A) < \delta$  implies  $\lambda(A) < \epsilon$ . This notion is, of course, stronger than  $\lambda \ll_w \mu$ . Note that for a  $\{0, 1\}$ -valued  $p$ -charge  $\lambda$ ,  $\lambda \ll \mu$  holds precisely when  $b(\lambda) > 0$  where

$$b(\lambda) = \inf\{\mu(A) : A \in \mathcal{A} \text{ and } \lambda(A) = 1\}.$$

We will start with two claims which do not involve semigroup actions.

Claim 1. If  $\mu$  is not strongly continuous, then there exists a  $\{0, 1\}$ -valued  $p$ -charge  $\lambda \ll \mu$ .

Proof of Claim 1. We say that  $A \in \mathcal{A}$  is  $\epsilon$ -covered if  $A$  can be covered by finitely many sets from  $\mathcal{A}$  of  $\mu$ -measure  $< \epsilon$ . By assumption for some  $\epsilon > 0$ , which we fix,  $X$  is not  $\epsilon$ -covered so the infimum in the following definition makes sense. Let  $b = \inf\{\mu(A) : A \in \mathcal{A} \text{ is not } \epsilon\text{-covered}\}$ . Note that  $b \geq \epsilon > 0$ . Let  $A_0 \in \mathcal{A}$  be such that it cannot be  $\epsilon$ -covered and  $\mu(A_0) < 2b$ . Define

$$\mathcal{I} = \{A \in \mathcal{A} : A \cap A_0 \text{ is } \epsilon\text{-covered}\}.$$

Now  $X \notin \mathcal{I}$ ,  $\mathcal{I}$  is closed under taking subsets which are in  $\mathcal{A}$  and under finite unions. If  $A \in \mathcal{A}$ , then, since  $\mu(A_0) < 2b$ ,  $\mu(A \cap A_0) < b$  or  $\mu((X \setminus A) \cap A_0) < b$ . Thus, either  $A \cap A_0$  or  $(X \setminus A) \cap A_0$  is  $\epsilon$ -covered, that is, either  $A \in \mathcal{I}$  or  $X \setminus A \in \mathcal{I}$ . This shows that  $\lambda$  defined by  $\lambda(A) = 0$  if  $A \in \mathcal{I}$  and  $\lambda(A) = 1$  if  $A \in \mathcal{A} \setminus \mathcal{I}$  is a  $\{0, 1\}$ -valued  $p$ -charge. Since no set in  $\mathcal{A} \setminus \mathcal{I}$  is  $\epsilon$ -covered,  $b(\lambda) \geq b > 0$ .

Claim 2. For any  $b > 0$  there are at most finitely many  $\{0, 1\}$ -valued  $p$ -charges  $\lambda$  with  $b(\lambda) > b$ .

Proof of Claim 2. Given  $b > 0$ , let  $n_b$  be the smallest natural number with  $n_b b > 1$ . Then, obviously, there is a small enough  $\delta > 0$  such that  $1 < n_b b - (n_b(n_b + 1)/2)\delta$ . Therefore, using the formula

$$\mu\left(\bigcup_{i \leq n} A_i\right) \geq \sum_{i \leq n} \mu(A_i) - \sum_{i < j \leq n} \mu(A_i \cap A_j),$$

we can find a  $\delta > 0$  such that for any sequence  $A_n \in \mathcal{A}$ ,  $n \in \mathbb{N}$ , with  $\mu(A_n) > b$  there are  $i < j$  with  $\mu(A_i \cap A_j) \geq \delta$ .

Let  $\lambda_n$ ,  $n \in \mathbb{N}$ , be pairwise distinct  $\{0, 1\}$ -valued  $p$ -charges with  $b(\lambda_n) > b$ . We can assume that for some  $b' \geq b$  and all  $n$ ,  $b' < b(\lambda_n) < b' + \delta$ . Now, for each  $i$  and  $n$  with  $i < n$ , we can

find  $A_{i,n} \in \mathcal{A}$  with  $\lambda_i(A_{i,n}) = 0$  and  $\lambda_n(A_{i,n}) = 1$ . If we let  $A_n = \bigcap_{i < n} A_{i,n}$ , then we obtain sets  $A_n \in \mathcal{A}$  with  $\lambda_n(A_n) = 1$  and  $\lambda_i(A_n) = 0$  for all  $i < n$ . Since  $b(\lambda_n) < b' + \delta$ , we can assume that  $\mu(A_n) < b' + \delta$  by intersecting  $A_n$  with an  $A \in \mathcal{A}$  such that  $\lambda_n(A) = 1$  and  $\mu(A) < b' + \delta$ . Note that  $\mu(A_n) \geq b(\lambda_n) > b'$  for all  $n$  which, by the choice of  $\delta$ , gives two natural numbers  $i < j$  such that  $\mu(A_i \cap A_j) > \delta$ . Then

$$\mu(A_i \setminus (A_i \cap A_j)) < (b' + \delta) - \delta < b(\lambda_i).$$

Thus,  $\lambda_i(A_i \setminus (A_i \cap A_j)) = 0$  whence  $\lambda_i(A_i \cap A_j) = 1$  contradicting  $\lambda_i(A_j) = 0$ .

To prove Theorem 1 from the above claims, we proceed as follows. If a semigroup  $G$  acts on  $X$  with  $\mathcal{A}$  being  $G$ -invariant and  $\mu$  being  $G$ -invariant and not strongly continuous, then by Claim 1 there exists a  $\{0, 1\}$ -valued  $p$ -charge  $\lambda \ll \mu$ . Obviously, we then have  $\lambda \ll_w \mu$ . Moreover,  $b(\lambda) > 0$  and, by  $G$ -invariance of  $\mu$ ,  $b(g\lambda) \geq b(\lambda)$  for each  $g \in G$ . Thus,  $G\lambda$  is finite by Claim 2.  $\square$

Under a suitable condition on (the action of) the semigroup  $G$ , infinity of the orbits  $G\lambda$ ,  $\lambda \in H$ , is also a necessary condition for strong continuity of all  $G$ -invariant  $p$ -charges on  $\mathcal{A}$ . (See, however, Remark 1 (a) below.) The equivalence of the statements (i) and (ii) of the following theorem is due to Adler [1]. We give a different proof of it based on the measure  $\rho_\mu$ . The following condition will be used:

$$gG\lambda = G\lambda \text{ for every } \lambda \in H, g \in G. \tag{C}$$

**Theorem 2** *The following statements are equivalent.*

(i) *Every  $G$ -invariant  $p$ -charge on  $\mathcal{A}$  is strongly continuous.*

(ii) *There is no  $G$ -invariant  $p$ -charge on  $\mathcal{A}$  of the form  $\frac{1}{n} \sum_{i=1}^n \lambda_i$  with  $n \in \mathbb{N}$ ,  $\lambda_1, \dots, \lambda_n \in H$ .*

*If condition (C) is satisfied or if  $G$  is left amenable (as discrete semigroup), then (i) is also equivalent to*

(iii)  *$G\lambda$  is infinite for every  $\lambda \in H$ .*

**Proof** (i)  $\Rightarrow$  (ii) is obvious and (iii)  $\Rightarrow$  (ii) follows from Theorem 1.

(ii)  $\Rightarrow$  (i). Assume that there exists a  $G$ -invariant  $p$ -charge  $\mu$  on  $\mathcal{A}$  which is not strongly continuous. Since  $\rho_\mu$  is  $G$ -invariant by Lemma 1, we have

$$\rho_\mu(gH_0) = \rho_\mu(g^{-1}(gH_0)) \geq \rho_\mu(H_0)$$

for every finite subset  $H_0$  of  $H$  and  $g \in G$ . Let  $a = \max\{\rho_\mu(\{\lambda\}) : \lambda \in H\}$  and

$$H_0 = \{\lambda \in H : \rho_\mu(\{\lambda\}) = a\}.$$

By Lemma 2(c),  $a > 0$  and thus  $H_0$  is a nonempty finite subset of  $H$ . Let  $g \in G$ . Since

$$a \geq \rho_\mu(\{g\lambda\}) \geq \rho_\mu(\{\lambda\}) = a$$

for every  $\lambda \in H_0$ , one obtains  $gH_0 \subset H_0$ . In view of  $\rho_\mu(gH_0) \geq \rho_\mu(H_0)$ , this implies  $\rho_\mu(gH_0) = \rho_\mu(H_0)$  which gives  $gH_0 = H_0$ . It follows that the  $p$ -charge  $\frac{1}{|H_0|} \sum_{\lambda \in H_0} \lambda$  on  $\mathcal{A}$  is  $G$ -invariant.

(i)  $\Rightarrow$  (iii). Suppose that  $G\lambda$  is finite for some  $\lambda \in H$ . Under condition (C), the  $p$ -charge  $\frac{1}{|G\lambda|} \sum_{\nu \in G\lambda} \nu$  on  $\mathcal{A}$  is  $G$ -invariant and not strongly continuous. Now let  $G$  be a left amenable semigroup. The convex hull  $co(G\lambda)$  of the finite orbit  $G\lambda$  is a weak\*-compact subset of  $K$ . Furthermore,  $co(G\lambda)$  is  $G$ -invariant, i.e.  $gco(G\lambda) \subset co(G\lambda)$  for every  $g \in G$  and the left action of  $G$  on  $K$  is weak\*-continuous. Since  $G$  is left amenable, Day's fixed point theorem (cf. [12, 1.14]) yields a  $G$ -invariant charge in  $co(G\lambda)$ .  $\square$

**Remark 1.** (a) It is not possible to replace (iii) in Theorem 2 with the weaker condition saying that all orbits of the action of  $G$  on  $X$  are infinite. To see this, consider  $G = \mathbb{Z}$ , the integers with addition, acting on  $X = \mathbb{Z}$  by translations,  $\mathcal{A}$  consisting of all finite and all cofinite subsets of  $\mathbb{Z}$ , and  $\mu$  defined on  $\mathcal{A}$  to be 0 on finite sets and 1 on cofinite sets. Then orbits of the  $\mathbb{Z}$ -action on  $X$  are infinite and yet (i) fails as shown by  $\mu$ .

(b) Condition (C) or amenability cannot be dispensed with in the proof of (i)  $\Rightarrow$  (ii) in Theorem 2. This is shown by the semigroup  $G$  consisting of the two constant maps from  $\{0, 1\}$  to  $\{0, 1\}$  taken with composition and acting on  $X = \{0, 1\}$  with  $\mathcal{A} = \mathcal{P}(\{0, 1\})$  in the natural way. For this action (i) holds (as there are no  $G$ -invariant  $p$ -charges) and (iii) fails.

(c) If  $G$  is a right simple semigroup, i.e.  $gG = G$  for every  $g \in G$ , and in particular, if  $G$  is a group, then condition (C) is satisfied.

### 3 Applications

In case  $\mathcal{A} = \mathcal{P}(X)$ , the following lemma [4, Theorem 3.3] provides an important tool to verify that orbits  $G\lambda$  are infinite. For  $g \in G$ , let  $X_g$  denote the set of  $g$ -fixed points in  $X$ , i.e.  $X_g = \{x \in X : gx = x\}$ . (This lemma is proved in [4] for a countable set  $X$  but the argument there works for an arbitrary  $X$ .)

**Lemma 3** *Let  $\lambda$  be a  $\{0, 1\}$ -valued  $p$ -charge on  $\mathcal{P}(X)$  and  $g \in G$ . Then  $g\lambda = \lambda$  if and only if  $\lambda(X_g) = 1$ .*

A first application is as follows. Let  $X$  be a right cancellative semigroup (i.e.  $x_1y = x_2y$  implies  $x_1 = x_2$ ) and  $\mathcal{A} = \mathcal{P}(X)$ . Then left multiplication by  $g \in X$  has no fixed point if and only if  $g$  is not idempotent. Let  $g \in X$  be of infinite order and  $G = \{g^n : n \in \mathbb{N}\}$ .  $G$  acts on  $X$  by left multiplication. By Lemma 3,  $g^n\lambda = g^{n-m}g^m\lambda \neq g^m\lambda$  for every  $\lambda \in H, n > m \geq 1$ . Therefore,  $G\lambda$  is infinite for every  $\lambda \in H$ . It follows from Theorem 1 that every  $G$ -left invariant  $p$ -charge on  $\mathcal{P}(X)$  is strongly continuous (see Granirer [8, p 387] for a related result). This extends a corresponding result for groups by Francke et al. [7].

The following two corollaries provide extensions of a result of Chou [6]; see also Stroetman [15]. In case  $G$  is a group, a  $G$ -invariant  $p$ -charge  $\mu$  on  $\mathcal{P}(X)$  is called aperiodic if  $\mu(X_g) = 0$  for every  $g \in G, g \neq e$ .

**Corollary 1** *Let  $G$  be an infinite group and  $\mathcal{A} = \mathcal{P}(X)$ . Then every  $G$ -invariant aperiodic  $p$ -charge on  $\mathcal{P}(X)$  is strongly continuous.*

**Proof.** Let  $\mu$  be a  $G$ -invariant aperiodic  $p$ -charge on  $\mathcal{P}(X)$  and let  $\lambda \in H$  with  $\lambda \ll_w \mu$ . Then  $\lambda(X_g) = 0$  for every  $g \in G, g \neq e$  and therefore, by Lemma 3, the isotropy group of  $\lambda$  in  $G$  satisfies  $G_\lambda = \{e\}$ . Since  $G$  is infinite, this implies that the orbit  $G\lambda$  is infinite. It follows from Theorem 1 that  $\mu$  is strongly continuous.  $\square$

**Corollary 2** *Let  $G$  be a group and  $\mathcal{A} = \mathcal{P}(X)$ . Let  $F$  denote the subgroup of  $G$  generated by  $\bigcup_{x \in X} G_x$ , where  $G_x$  denotes the isotropy group of  $x$  in  $G$ . Consider the following statements:*

(i) Every  $G$ -invariant  $p$ -charge on  $\mathcal{P}(X)$  is strongly continuous.

(ii)  $G/F$  is infinite.

Then (ii) implies (i) and if  $F/G_x$  is finite for some  $x \in X$ , (i) and (ii) are equivalent.

**Proof.** (ii)  $\Rightarrow$  (i). Let  $g \in G, g \notin F$ . Then  $X_g = \emptyset$  and by Lemma 3,  $g \notin \bigcup_{\lambda \in H} G_\lambda$ . We obtain  $G_\lambda \subset F$  for every  $\lambda \in H$ . By (ii), this yields that  $G/G_\lambda$  is infinite for every  $\lambda \in H$ . The assertion follows from Theorem 1.

(i)  $\Rightarrow$  (ii). By Theorem 2,  $G\varepsilon_x$  is infinite for every  $x \in X$  where  $\varepsilon_x$  is the  $\{0, 1\}$ -valued  $p$ -charge concentrated on  $x$ . Hence  $G/G_x$  is infinite for every  $x \in X$ . Since  $|G/G_x| = |G/F| \cdot |F/G_x|$  and  $F/G_x$  is finite for some  $x \in X$ , this implies that  $G/F$  is infinite.  $\square$

**Remark 2.** (a) Let  $G$  be a group and let  $\mu$  be a  $G$ -invariant  $p$ -charge on  $\mathcal{A}$ . In case  $\mathcal{A} = \mathcal{P}(X)$ ,  $\mu$  is aperiodic if and only if  $\rho_\mu$  is aperiodic, i.e.,  $\rho_\mu(H_g) = 0$  for every  $g \in G, g \neq e$ , where  $H_g = \{\lambda \in H : g\lambda = \lambda\}$ . In fact, by Lemma 3

$$\mu(X_g) = \rho_\mu(\{\lambda \in H : \lambda(X_g) = 1\}) = \rho_\mu(H_g).$$

For  $\mathcal{A} \neq \mathcal{P}(X)$  the following version of Corollary 1 holds. Let  $G$  be an infinite group. Then every  $G$ -invariant  $p$ -charge  $\mu$  on  $\mathcal{A}$  with aperiodic measure  $\rho_\mu$  is strongly continuous. To see this assume  $\rho_\mu(\{\lambda\}) > 0$  for some  $\lambda \in H$ . Then  $G\lambda$  is finite and hence  $G_\lambda \neq \{e\}$ . So  $\lambda \in H_g$  for some  $g \neq e$  which gives  $\rho_\mu(H_g) > 0$ .

(b) The example in Remark 1 (a) shows that Corollaries 1 and 2 are not valid, in general, for  $\mathcal{A} \neq \mathcal{P}(X)$ . It is possible to relate the general case  $\mathcal{A} \neq \mathcal{P}(X)$  to the case  $\mathcal{A} = \mathcal{P}(X)$ . Let  $\mu$  be a  $p$ -charge on  $\mathcal{A}$ . Then  $\mu$  is strongly continuous if and only if every extension of  $\mu$  to a  $p$ -charge on  $\mathcal{P}(X)$  is strongly continuous (cf. [13]). However, for our purposes this result has the disadvantage that for invariant charges  $\mu$  it is not enough to consider only invariant extensions of  $\mu$  as is again illustrated by the example in Remark 1 (a): the group  $G = \mathbb{Z}$  is Abelian, therefore, there are many extensions of  $\mu$  to a  $G$ -invariant  $p$ -charge on  $\mathcal{P}(\mathbb{Z})$ ; by Corollary 1 (or Corollary 2), these extensions are strongly continuous and yet  $\mu$  is not.

The following corollary provides an extension of a result of Snell [14]. Recall that a group  $F$  equipped with a  $\sigma$ -algebra  $\mathcal{B}$  is called Lusin measurable group if  $F \times F \rightarrow F, (g, h) \mapsto gh^{-1}$  is  $(\mathcal{B} \otimes \mathcal{B}, \mathcal{B})$ -measurable and the measurable space  $(F, \mathcal{B})$  is isomorphic to a Polish space equipped with its Borel  $\sigma$ -algebra.

**Corollary 3** *Let  $G$  be an infinite subsemigroup of a Lusin measurable group  $(F, \mathcal{B})$  and let  $\mathcal{A} = G \cap \mathcal{B}$ . Assume that there exists a non-trivial  $\sigma$ -finite  $F$ -left quasi-invariant measure on  $\mathcal{B}$ . Then every  $G$ -left invariant  $p$ -charge on  $\mathcal{A}$  is strongly continuous.*

**Proof.** According to a result of Mackey, there is a topology on  $F$  such that  $F$  becomes a locally compact group and  $\mathcal{B}$  is the Borel  $\sigma$ -algebra (cf. [5]). Now it follows from [9, Theorem 3] that there is no  $G$ -left invariant  $p$ -charge in  $co(H)$ . By Theorem 2, this gives the assertion.  $\square$

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